About the Integrals :

_ Let's consider that a runner wants to run a race :
* The chronometer starts if and only if he overrides the starting line A.
* The chronometer starts if and only if he reaches the finishing line B.
_ Let's consider that A and B are two points of the line of the real numbers, so the distance between A and B is the distance where the chronometer isn't stopped which is the length l of the interval $]A, B]$.
We conclude that $l = B - A$.
And consequently, the length l_1 of $]A, B[$ is $l_1 = B - A - \varepsilon$
and the length l_2 of $[A, B]$ is $l_2 = B - A + \varepsilon$
with ε as follows :
$\forall A, B \in \mathbb{R}$ with $B > A$ we have,
$\exists M \in \mathbb{R} \setminus \mathbb{Q}$ with $M = \max\{x \,/\, x \in [A, B[\}$
hence $\exists \varepsilon \in \mathbb{R} \setminus \mathbb{Q}$ with $\varepsilon > 0$ and $M + \varepsilon = B$
and we have :

$$B - \frac{C}{n} \leqslant B - \varepsilon < B \qquad \forall n \in \mathbb{N}^* \quad \forall C \in \mathbb{R}_+^*$$

So we can consider that $\varepsilon = 0^+$
Now let's make a definition to the integral $\int_A^B f(x) \cdot dx$:

* The correct subdivision of the interval $[A, B]$ shoud be made with intervals which are $[a_{i-1}, a_i[$ since there length is exactly $(a_i - a_{i-1})$ and since they are separated (make separated surfaces)
Also: $\bigcup_{i=1}^{n} [a_{i-1} ; a_i[\iff [A, B[$ by considering :

$a_i > a_{i-1}$ $\forall n \in \mathbb{N}^*$ and $a_0 = A$ and $a_n = B$

and $\forall i \in \mathbb{N}^*$: $a_i = A + i\,\frac{B - A}{n}$ and $a_i - a_{i-1} = h = \frac{B - A}{n}$
we define also: $x_i = A + (i - 1 + \gamma) \cdot \frac{B - A}{n} = a_{i-1} + \gamma \cdot h$
with: $\gamma \in [0, 1[$

Finally we have: $S_f = \sum_{i=1}^{n} \left(f(x_i) \cdot (a_i - a_{i-1}) \right)$

$$= \frac{B-A}{n} \cdot \sum_{i=1}^{n} \left(f\left(A + (i-1+\gamma) \cdot \frac{B-A}{n}\right) \right)$$

* We also have: $[A, B] \Longleftrightarrow \bigcup_{i=1}^{n} [a_{i-1}, a_i[\ \cup \{B\}$

by considering $M = \max \{ x / x \in [a_{n-1}, B[\}$

we have $\{B\} \Longleftrightarrow]M, B]$

Hence $[A, B] \Longleftrightarrow \bigcup_{i=1}^{n} [a_{i-1}, a_i[\ \cup \ ]M, B]$

and we have $B - M = \varepsilon$ so we conclude that:

$$S_f + (B-M) \cdot f(B) = S_f + \varepsilon \cdot f(B) = F_A^B$$

Important: $\xrightarrow{\quad}$ we considered that $\frac{B-A}{n} \gg \varepsilon > 0$ from the beginning.

with F_A^B is the correct Riemann Sum to $\int_A^B f(x) \cdot dx$

which respects: $\lim_{n \to +\infty} F_A^B = \int_A^B f(x) \cdot dx$

which means: $\int_A^B f(x) \cdot dx - \varepsilon f(B) = \lim_{n \to +\infty} S_f$

$$\Longleftrightarrow \boxed{\int_A^B f(x) \cdot dx - \varepsilon \cdot f(B) = \lim_{n \to +\infty} \frac{B-A}{n} \cdot \sum_{i=1}^{+\infty} \left(f\left(A + (i-1+\gamma) \cdot \frac{B-A}{n}\right) \right)}$$

* Now if we use a subdivision with the intervals: $]a_{i-1}, a_i]$ and $\gamma \in]0, 1]$ and ε is the same for all the real numbers since the line of the real numbers is homogeneous: S_f stays the same but:

$$\boxed{\int_A^B f(x) \cdot dx - \varepsilon \cdot f(A) = \lim_{n \to +\infty} \frac{B-A}{n} \cdot \sum_{i=1}^{n} \left(f\left(A + (i-1+\gamma) \cdot \frac{B-A}{n}\right) \right)}$$

We conclude that: $\boxed{\int_A^B f(x) \cdot dx = \varepsilon \left(f(A) + f(B) \right)}$