The number : $\alpha=\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\ldots\right)\right)$

Edgar Valdebenito

22-04-2019 14:51:30

abstract

In this note we give some formulas related with the number:

$$\alpha=\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\ldots\right)\right)\right).$$

1. Introducción

El número

$$\alpha=\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{2}\ldots\right)\right)\right)=0.514933264661\ldots \tag{1}$$

Es la única raíz real de la ecuación: $x=\cos(2x)$. En esta nota mostramos algunas fórmulas relacionadas con el número α .

2. Fórmulas

2.1. Representaciones.

$$\alpha=\sin^{-1}\sqrt{\frac{1}{2}-\frac{1}{2}\sin^{-1}\sqrt{\frac{1}{2}-\frac{1}{2}\sin^{-1}\sqrt{\frac{1}{2}-\ldots}}} \tag{2}$$

$$\alpha=\cos^{-1}\sqrt{\frac{1}{2}+\frac{1}{2}\cos^{-1}\sqrt{\frac{1}{2}+\frac{1}{2}\cos^{-1}\sqrt{\frac{1}{2}+\ldots}}} \tag{3}$$

2.2. Sucesiones.

$$x_{n+1} = \frac{1}{2}\cos^{-1} x_n \quad , x_0 = 0 \Rightarrow x_n \to \alpha \tag{4}$$

$$x_{n+1} = \sin^{-1}\sqrt{\frac{1-x_n}{2}} \quad , x_0 = 0 \Rightarrow x_n \to \alpha \tag{5}$$

$$x_{n+1} = \cos^{-1}\sqrt{\frac{1+x_n}{2}} \quad , x_0 = 0 \Rightarrow x_n \to \alpha \tag{6}$$

$$x_{n+1} = \frac{x_n + \cos(2x_n)}{2} \quad , x_0 = 0 \Rightarrow x_n \to \alpha \tag{7}$$

$$x_{n+1} = \frac{2x_n + \cos(2x_n)}{3} \quad , x_0 = 0 \Rightarrow x_n \to \alpha \tag{8}$$

$$x_{n+1} = \frac{3x_n + 2\cos(2x_n)}{5} \quad , x_0 = 0 \Rightarrow x_n \to \alpha \tag{9}$$

2.3. Fórmulas con el número Pi.

$$\pi = 4\alpha + 2\sin^{-1}\alpha \tag{10}$$

$$\pi = 4\alpha + 2\tan^{-1}\frac{\alpha}{\sqrt{1-\alpha^2}} \tag{11}$$

$$\pi = 2\alpha + 2\sin^{-1}\sqrt{\frac{1+\alpha}{2}} \tag{12}$$

$$\pi = 4\alpha + 4\tan^{-1}\left((\tan\alpha)^2\right) \tag{13}$$

$$\pi = 4\alpha + 4\tan^{-1}\left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\right) \tag{14}$$

$$\pi = 4\alpha + 4\sin^{-1}\left(\frac{\alpha}{\sqrt{2+2\sqrt{1-\alpha^2}}} \right) \tag{15}$$

$$\pi = 8\alpha - 4\sin^{-1}\left(\frac{\sqrt{1-\alpha^2}-\alpha}{\sqrt{2}} \right) \tag{16}$$

$$\pi = 12\alpha - 2\sin^{-1}\left(3\alpha - 4\alpha^3 \right) \tag{17}$$

2.4. Una sucesión de números racionales .

$$u_0 = u_1 = 1 \quad , \quad u_n = u_{n-1} - \sum_{k=1}^{[n/2]} \frac{(-1)^k \, 2^{2k}}{(2k)!} u_{n-2k} \tag{18}$$

$$\alpha = \lim_{n\to\infty} \frac{u_n}{u_{n+1}} \tag{19}$$

2.5. Una sucesión de números enteros .

$$v_0 = v_1 = 1 \quad , \quad v_n = n\, v_{n-1} - \sum_{k=1}^{[n/2]} (-1)^k \, 2^{2k} \binom{n}{2k} v_{n-2k} \tag{20}$$

$$\alpha = \lim_{n\to\infty} \frac{(n+1)v_n}{v_{n+1}} \tag{21}$$

2.6. Una integral .

$$\int_0^{2\pi} \frac{e^{xi}}{e^{xi} - \cos\left(2e^{xi}\right)}\, dx = \frac{2\pi}{1+2\sin(2\alpha)} \tag{22}$$

$$\alpha = \frac{1}{2}\sin^{-1}\left(-\frac{1}{2} + \pi \left(\int_0^{2\pi} \frac{e^{xi}}{e^{xi} - \cos\left(2e^{xi}\right)}\, dx \right)^{-1} \right) \tag{23}$$

Observación: $i = \sqrt{-1}$.

2.7. Series .

$$\pi = 6\sin^{-1}\alpha - 4\sqrt{3}\sum_{n=0}^{\infty}\left(\alpha-\frac{1}{2}\right)^{n+1}\frac{3^{-n}}{n+1}\sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}3^{k} \qquad (24)$$

$$\pi = 6\alpha + 2\sqrt{3}\sum_{n=0}^{\infty}\left(\alpha-\frac{1}{2}\right)^{n+1}\frac{3^{-n}}{n+1}\sum_{k=0}^{[n/2]}\binom{2n-2k}{n-k}\binom{n-k}{k}3^{k} \qquad (25)$$

$$\pi = 8\sum_{n=0}^{\infty}\frac{\sin\left((2n+1)\alpha\right)}{2n+1}\left(\frac{\sqrt{2}}{\sqrt{3-\alpha}+\sqrt{1-\alpha}}\right)^{2n+1} \qquad (26)$$

$$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^{n}\cos\left((2n+1)\alpha\right)}{2n+1}\left(\frac{\sqrt{2}}{\sqrt{3+\alpha}+\sqrt{1+\alpha}}\right)^{2n+1} \qquad (27)$$

$$\pi = 8\sum_{n=0}^{\infty}\frac{(-1)^{n}\cos\left((4n+2)\alpha\right)}{2n+1}\left(\frac{1}{\alpha+\sqrt{1+\alpha^{2}}}\right)^{2n+1} \qquad (28)$$

$$\pi = 8\sum_{n=0}^{\infty}\frac{\sin\left((4n+2)\alpha\right)}{2n+1}\left(\frac{1}{\sqrt{2-\alpha^{2}}+\sqrt{1-\alpha^{2}}}\right)^{2n+1} \qquad (29)$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{\sin\left(2n\alpha\right)}{n}\left(\frac{1}{\alpha+\sqrt{1-\alpha^{2}}}\right)^{n} \qquad (30)$$

$$\pi = 6\alpha + \sqrt{2\left(1-3\alpha+4\alpha^{3}\right)}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-3n}\left(1-3\alpha+4\alpha^{3}\right)^{n}}{2n+1} \qquad (31)$$

$$\pi\sqrt{\frac{1-\alpha}{2}} = 2 - 4\sum_{n=1}^{\infty}\frac{T_{n}(\alpha)}{4n^{2}-1} \qquad (32)$$

En la fórmula (26) , $T_n(\alpha)$ son los polinomios de Chebyshev :

$$T_{n}(\alpha) = \sum_{k=0}^{[n/2]}\binom{n}{2k}(-1)^{k}\alpha^{n-2k}\left(1-\alpha^{2}\right)^{k} \qquad (33)$$

$$T_{n+1}(\alpha)=2\alpha T_n(\alpha)-T_{n-1}(\alpha) \quad ,T_0(\alpha)=1, T_1(\alpha)=\alpha \tag{34}$$

2.8. Serie para α .

$$\alpha=\frac{\pi}{6}-\frac{\pi-3}{6\left(1+\sqrt{3}\right)}-\frac{(\pi-3)^2}{36\left(1+\sqrt{3}\right)^3}-\frac{\left(6+\sqrt{3}\right)(\pi-3)^3}{324\left(1+\sqrt{3}\right)^5}$$
$$-\frac{\left(41+8\sqrt{3}\right)(\pi-3)^4}{3888\left(1+\sqrt{3}\right)^7}-\frac{\left(462+125\sqrt{3}\right)(\pi-3)^5}{58320\left(1+\sqrt{3}\right)^9}$$
$$-\frac{\left(6265+1976\sqrt{3}\right)(\pi-3)^6}{1049760\left(1+\sqrt{3}\right)^{11}}-\frac{\left(108198+39241\sqrt{3}\right)(\pi-3)^6}{22044960\left(1+\sqrt{3}\right)^{13}}-\ldots$$

(35)

Observación: $\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=3.141592\ldots$

Referencias

1. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. : NIST Handbook of Mathematical Functions. Cambridge University Press , 2010.
2. Spanier, J., and Oldham, K.B. : An Atlas of Functions. Hemisphere Publishing, 1987.
3. Whittaker, E.T., and Watson, G.N. : A Course in Modern Analysis, 4th ed. Cambridge University Press , 1961.