The Inconsistency of Arithmetic

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Abstract. Based on a strengthened form of the strong Goldbach conjecture, this paper presents an antinomy within the Peano arithmetic (PA). We derive two contradictory statements by using the same main instrument as in the proof\(^2\) of the conjecture, i.e. a set that is a structuring of the natural numbers starting from 3.

Notations. Let \(\mathbb{N}\) denote the natural numbers starting from 1, let \(\mathbb{N}_n\) denote the natural numbers starting from \(n > 1\) and let \(\mathbb{P}_3\) denote the prime numbers starting from 3.

Theorem. The Peano arithmetic (PA) is inconsistent.

Proof. We define the set
\[ S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \]
and we consider the following two cases.

\((G)\) The numbers \(m\) in the components \(mk\) take all integer values \(x \geq 4\).
\(\neg(G)\) The numbers \(m\) in the components \(mk\) do not take all integer values \(x \geq 4\).

For each \(k \geq 1\), let \(S_g(m,k)\) denote the set of the middle components \(mk\) of the \(S_g\) triples. Then, by definition

\[ (G) \iff S_g(m,k) = k\mathbb{N}_4 \text{ for every } k \geq 1 \]
\[ \neg(G) \iff S_g(m,k) \neq k\mathbb{N}_4 \text{ for every } k \geq 1. \]

This implies that \(S_g\) does not contain the same triples in the cases \((G)\) and \(\neg(G)\):

\((I)\) \(\exists\) sets \(S, S'\) such that \(S \neq S'\) and ( \(((G) \Rightarrow S_g = S)\) and \(\neg(G) \Rightarrow S_g = S'\) ).

On the other hand, the case \(\neg(G)\) means that there is at least one \(n \geq 4\) such that \(nk\) is different from all the \(mk\) for each \(k \geq 1\), where all pairs \((p, q)\) of odd primes, that determine the numbers \(m\), are used in \(S_g\). For each \(k \geq 1\), such an \(nk\) can be written as some \(p k'\) when \(n\) is prime, as some \(p k'\) when \(n\) is composite and not a power of 2, or as \(4k'\) when \(n\) is a power of 2; \(p \in \mathbb{P}_3; k, k' \in \mathbb{N}\).

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The expression $p'_{k'}$ for $n_k$ with $k' = k$ or $k' \neq k$ is a first component of $S_g$ triples and the expression $4k'$ for $n_k$ is component of the triple $(3k', 4k', 5k')$. So, since $n_k$ equals some triple component $p'_{k'}$ or $4k'$ that exists by definition of $S_g$ and since the $S_g$ triples are generated by the first and third components, the triples are the same in the case $n_k$ exists and in the case $n_k$ does not exist.

In other words, the $S_g$ triples are always the same, regardless of whether $n_k$ as a component of them exists or not. Therefore, we obtain the contradiction that $S_g$ contains the same triples in the cases $(G)$ and $\neg(G)$:

\[
\forall \text{ sets } S, S' \quad ( ((G) \Rightarrow S_g = S) \text{ and } (\neg(G) \Rightarrow S_g = S') ) \Rightarrow S = S'
\]

\[
\iff
\]

(II) \quad $\nexists$ sets $S, S'$ such that $S \neq S'$ and $((G) \Rightarrow S_g = S)$ and $(\neg(G) \Rightarrow S_g = S')$.

\[\square\]

The statement (II) is built on two properties of $S_g$, namely that $n_k$, given by the case $\neg(G)$, for each $k \geq 1$ can be expressed by a $S_g$ triple component and that $n_k, k = 1$, cannot be the arithmetic mean of a pair of odd primes not used in $S_g$. We call these two properties of $S_g$ 'covering' and 'maximality'. Without them, we could establish only the statement (I) and there would be no contradiction.

The proof uses a strengthened form of the strong Goldbach conjecture:

**Strengthened strong Goldbach conjecture (SSGB):** *Every even integer greater than 6 can be expressed as the sum of two different primes.*

SSGB is equivalent to saying that all integers $x \geq 4$ appear as $m$ in a component $m_k$ of $S_g$. Therefore, SSGB is equivalent to the case $(G)$ and the negation $\neg$SSGB is equivalent to the case $\neg(G)$. We have seen above that the $S_g$ triples are the same in these two cases. This means that both SSGB and $\neg$SSGB hold.