The Inconsistency of Arithmetic

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Abstract. Based on a strengthened form of the strong Goldbach conjecture, this paper presents an antinomy within the Peano arithmetic (PA). We derive two contradictory statements by using the same main instrument as in the proof \(^2\) of the conjecture, i.e. a set that is a structuring of the natural numbers starting from 3.

Notations. Let \(\mathbb{N}\) denote the natural numbers starting from 1, let \(\mathbb{N}_n\) denote the natural numbers starting from \(n > 1\) and let \(\mathbb{P}_3\) denote the prime numbers starting from 3.

Theorem. The Peano arithmetic (PA) is inconsistent.

Proof. We define the set

\[ S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \]

and we consider the following two cases.

(G) The numbers \(m\) in the components \(mk\) take all integer values \(x \geq 4\).

\(\neg\)G) The numbers \(m\) in the components \(mk\) do not take all integer values \(x \geq 4\).

For each \(k \geq 1\), let \(S_g(m,k)\) denote the set of the middle components \(mk\) of the \(S_g\) triples. Then, by definition

\[ (G) \quad S_g(m,k) = k\mathbb{N}_4 \]

\[ \neg(G) \quad S_g(m,k) \neq k\mathbb{N}_4. \]

This implies that \(S_g\) does not contain the same triples in the cases \((G)\) and \(\neg(G)\):

(I) \(\exists\) sets \(S, S'\) such that \(S \neq S'\) and \(((G) \Rightarrow S_g = S)\) and \((\neg(G) \Rightarrow S_g = S')\).

On the other hand, the case \(\neg(G)\) means that for each \(k \geq 1\) there is an \(n_k, n \geq 4\), different from all the \(mk\), where all pairs \((p, q)\) of odd primes, that determine the numbers \(m\), are used in \(S_g\). For each \(k \geq 1\), such an \(n_k\) can be written as some \(pk\) when \(n\) is prime, as some \(pk'\) when \(n\) is composite and not a power of 2, or as \(4k'\) when \(n\) is a power of 2; \(p \in \mathbb{P}_3; k, k' \in \mathbb{N}\).

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\(^2\) https://vixra.org/abs/1702.0300
The expression $pk'$ for $nk$ with $k' = k$ or $k' \neq k$ is a first component of $S_g$ triples and the expression $4k'$ for $nk$ is component of the triple $(3k', 4k', 5k')$. So, since $nk$ equals some triple component $pk'$ or $4k'$ that exists by definition of $S_g$, the $S_g$ triples are the same in the case $nk$ exists and in the case $nk$ does not exist.

In other words, the $S_g$ triples are always the same, regardless of whether $nk$ as a component of them exists or not. Therefore, we obtain the contradiction that $S_g$ contains the same triples in the cases (G) and $\neg$(G):

( $\exists$ sets $S, S'$ such that ( $((G) \Rightarrow S_g = S)$ and $(\neg(G) \Rightarrow S_g = S')$ ) ) $\Rightarrow$ $S = S'$

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(II) $\nexists$ sets $S, S'$ such that $S \neq S'$ and ( $((G) \Rightarrow S_g = S)$ and $(\neg(G) \Rightarrow S_g = S')$ ). □

The proof above uses a strengthened form of the strong Goldbach conjecture:

**Strengthened strong Goldbach conjecture (SSGB):** *Every even integer greater than 6 can be expressed as the sum of two different primes.*

SSGB is equivalent to saying that all integers $x \geq 4$ appear as $m$ in a component $mk$ of $S_g$. Therefore, SSGB is equivalent to the case (G) and the negation $\neg$SSGB is equivalent to the case $\neg$(G). We have seen above that the $S_g$ triples are the same in these two cases. This means that both SSGB and $\neg$SSGB hold.