Surprising integral definition of the number *e*.

MSC2010: 58C05.

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Abstract

A new definition of the number *e* is presented by the integral of a function that involves an *infinite product of nested radicals* whose indexes form the sequence 1, 2, 3, ...

Theorem:

$$e \equiv \frac{1}{\int_0^1 x\sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\cdots}}}}\,dx}.$$

Proof.

Let

$$\boxed{I \equiv \int_0^1 f(x)dx} \tag{1}$$

with

$$\boxed{f(x) \equiv x\sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\cdots}}}}} \tag{2}$$

Applying Neperian logarithms to both members of (2),

$$\ln f(x) = \ln x + \frac{1}{2}\left(\ln x + \frac{1}{3}\left(\ln x + \frac{1}{4}\left(\ln x + \ldots\right)\right)\right) = \left(1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4}\ldots\right)\ln x$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\cdot\ln x = \left(\sum_{n=0}^{\infty}\frac{1}{n!} - \frac{1}{0!}\right)\cdot\ln x = (e-1)\cdot\ln x,$$

$$\boxed{\ln f(x) = (e-1)\ln x} \tag{3}$$

whence,

$$f(x) = e^{(e-1)\ln x} = \left(e^{\ln x}\right)^{e-1} = x^{e-1},$$

$$\boxed{f(x) = x^{e-1}} \tag{4}$$

From (1) and (4),

$$I = \int_0^1 x^{e-1} dx = \left. \frac{x^e}{e} \right|_0^1 = \frac{1}{e}.$$

$$\boxed{I = \frac{1}{e}} \tag{5}$$

Therefore,

$$\boxed{e \equiv \frac{1}{\int_0^1 x\sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\cdots}}}}\,dx}} \tag{6}$$