Interpolating Values in Code Space

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Abstract

A method is described for interpolating un-sampled values attributed to points in code space. A metric is used which counts the number of non-equal corresponding indices shared by two given points. A generalised interpolation equation is derived for nets, which is then applied specifically to code space. This interpolation equation is then solved in general for a set of given known sampled values in the space.

Assigning values to net nodes

On a general net, with a total of M node labels $m: 1 \leq m \leq M$, we can assign the respective node values ρ_m. The net edges can be described by the *adjacency* matrix (Weisstein 2019), G, with the elements taking binary values such that,

$$G_{mm'} = \begin{cases} 0, & \text{nodes } m, m' \text{ share no edge or } m = m' \\ 1, & \text{nodes } m, m' \text{ share one edge} \end{cases}$$

Derivation of Interpolation Equation on a Net

Suppose the value of node m, ρ_m is unknown (un-sampled) but all its nearest neighbour node values, $\{\rho_{m':G_{m'm}=1}\}$ are known. A good estimate for the value of ρ_m can be taken to be one which minimises the modulus of the difference vector $\boldsymbol{\delta\rho} \equiv (\{\delta\rho_{m'} \equiv \rho_{m'} - \rho_m\} \, \forall \, m')$ i.e.

$$\delta \sum_{m'} G_{m'm} \, (\rho_{m'} - \rho_m)^2 = 0$$

$$\frac{\partial}{\partial \rho_m} \sum_{m'} G_{m'm} \, (\rho_{m'} - \rho_m)^2 = 0$$

$$\Rightarrow \quad \sum_{m'} G_{m'm} \, (\rho_{m'} - \rho_m) = 0$$

Notice that if the nearest neighbours of m now also have un-sampled values, and their nearest neighbours likewise, and so on and so forth, then the above equation can be solved to yield an estimate for *all* the unknown node values provided *at least one* node value is known (sampled). The above equation is then solvable for determining estimated node values for *all* the unknown (un-sampled) nodes.

NB the interpolation equation can be re-arranged to reveal that a given un-sampled node value is estimated to be the mean of those of its nearest neighbours,

$$\sum_{m'=1}^{M} G_{mm'}(\rho_{m'} - \rho_m) = 0$$

$$-\rho_m \sum_{m'=1}^{M} G_{mm'} + \sum_{m'=1}^{M} G_{mm'}\,\rho_{m'} = 0$$

$$-\rho_m N_m + \sum_{m'=1}^{M} G_{mm'}\,\rho_{m'} = 0$$

$$\rho_m^{\text{unsampled}} \to \frac{1}{N_m} \sum_{m'=1}^{M} G_{mm'}\,\rho_{m'}$$

where $N_m \equiv \sum_{m'=1}^{M} G_{mm'}$ is the order of the node at m, i.e. the number of edges emanating from node m, i.e. the total number of its nearest neighbours.

Generalising to include 'sources' or 'charges' for the subset of the N sampled nodes $\{m_n\}$, we have,

$$\sum_{m'=1}^{M} G_{mm'}(\rho_{m'} - \rho_m) = Q_m$$

where $Q_m \equiv \sum_{n=1}^{N} Q_{m_n} \delta_{m,m_n}$ and the sampled value are $\{\rho_{m_n}\}$ respectively.

Linearity of the State Equation

That the state equation is linear is easily proven,

$$\sum_{m'=1}^{M} G_{mm'}\left(\rho_{m'}^{(1)} - \rho_m^{(1)}\right) = Q_m^{(1)}$$

$$\sum_{m'=1}^{M} G_{mm'}\left(\rho_{m'}^{(2)} - \rho_m^{(2)}\right) = Q_m^{(2)}$$

$$a \sum_{m'=1}^{M} G_{mm'}\left(\rho_{m'}^{(1)} - \rho_m^{(1)}\right) + b \sum_{m'=1}^{M} G_{mm'}\left(\rho_{m'}^{(2)} - \rho_m^{(2)}\right) = a\,Q_m^{(1)} + b\,Q_m^{(2)}$$

$$\Rightarrow \quad \sum_{m'=1}^{M} G_{mm'}\left(\left(a\,\rho_{m'}^{(1)} + b\,\rho_{m'}^{(2)}\right) - \left(a\,\rho_m^{(1)} + b\,\rho_m^{(2)}\right)\right) = a\,Q_m^{(1)} + b\,Q_m^{(2)}$$

'Charge' neutrality

Summing over the pseudo charges (sources) associated with the sampled node values in the state equation,

$$\sum_{m=1}^{M} Q_m = \sum_{m=1}^{M} \sum_{m'=1}^{M} G_{mm'} (\rho_{m'} - \rho_m)$$

$$= \sum_{m'=1}^{M} \left(\sum_{m=1}^{M} G_{mm'} \right) \rho_{m'} - \sum_{m=1}^{M} \rho_m \left(\sum_{m'=1}^{M} G_{mm'} \right)$$

$$= \sum_{m'=1}^{M} N_{m'} \rho_{m'} - \sum_{m=1}^{M} \rho_m N_m$$

$$= 0$$

Hence all solutions to the state equation are overall charge neutral.

Application to a Generalised Code Space, Σ

Consider the special case of the net and nodes forming a code space (Barnsley 1988 pp.125-133), Σ, with each node being a point labelled by a total of D integer indices such that,

$$m \rightarrow \boldsymbol{m} \equiv \{m_1, m_2, \cdots, m_D\} \quad ; \quad 0 \leq m_q \leq \mu_q - 1$$

Formulating in terms of a code space we let $S_q = \{0, 1, 2, \cdots \mu_q - 1\}$ be the set of μ_q first natural numbers. We define $m \in \Sigma$ if $m = m_1 m_2 m_3 \cdots m_\alpha \cdots m_D$. Notice that in general, the number of permissible values a given m_α can take can be set differently independently, i.e. $0 \leq m_q \leq \mu_q - 1$ where μ_q depends on q.

Nodes in the net now inhabit this D-dimensional code space. Traversing an edge from one node (point) to another involves changing exactly one index $m_j \rightarrow m_j' \neq m_j$ for a given j. A metric will be used that counts the minimum number of edges traversed in traversing between two nodes, namely,

$$d_{\boldsymbol{m}'\boldsymbol{m}} \equiv \sum_{q=1}^{D} \overline{\delta_{m_q, m_q'}}$$

The gradient expression in the state equation can now be written with set of all nearest neighbours to (say) $\boldsymbol{m}$ being $\bigcup_{q=1}^{D} \bigcup_{m_q' \neq m_q} \{\boldsymbol{m} : m_q \rightarrow m_q'\}$,

$$\Delta_{\boldsymbol{m}} = \sum_{q=1}^{D} \sum_{\substack{m=0 \\ m \neq m_q}}^{\mu_q - 1} \left(\rho_{\boldsymbol{m}: m_q \rightarrow m} - \rho_{\boldsymbol{m}} \right) \equiv \sum_{q=1}^{D} \sum_{\alpha=1}^{\mu_q - 1} \left(\rho_{\boldsymbol{m}: m_q \rightarrow m_q - \alpha \bmod \mu_q} - \rho_{\boldsymbol{m}} \right)$$

where in the third expression the incremental index α sums over 'rotations' in each m_q in succession. **NB** note $\alpha \neq 0$ as $\boldsymbol{m}: m_q \rightarrow m_q - 0 \bmod \mu_q \equiv \boldsymbol{m}$ is not a nearest neighbour to $\boldsymbol{m}$.

Eigenfunction and truncated Green's function of rotation operator

We can define the rotation operator R, write down its eigenfunctions and reframe the gradient operator in terms of it,

$$R\, f(m) = f(m + 1 \bmod \mu)$$

$$\phi_\alpha^\mu(m) \equiv e^{\frac{2\pi i \alpha m}{\mu}}$$

$$\Delta \equiv \sum_{\beta=1}^{\mu-1} \left(R^\beta - 1\right)$$

Check $\phi_\alpha(m)$ is eigenfunction of R,

$$R\, \phi_\alpha^\mu(m) \equiv e^{\frac{2\pi i \alpha(m+1)}{\mu}} \equiv e^{\frac{2\pi i \alpha}{\mu}} e^{\frac{2\pi i \alpha m}{\mu}} \equiv e^{\frac{2\pi i \alpha}{\mu}} \phi_\alpha^\mu(m)$$

So $\phi_\alpha(m)$ also eigenfunction of Δ with eigenvalues L_α given by,

$$\Delta\, \phi_\alpha^\mu(m) \equiv \sum_{\beta=1}^{\mu-1} \left(R^\beta - 1\right)\, \phi_\alpha^\mu(m) \equiv \phi_\alpha^\mu(m) \sum_{\beta=1}^{\mu-1} \left(e^{\frac{2\pi i \alpha \beta}{\mu}} - 1\right)$$

$$\Rightarrow L_\alpha = \sum_{\beta=1}^{\mu-1} \left(e^{\frac{2\pi i \alpha \beta}{\mu}} - 1\right) = \sum_{\beta=0}^{\mu-1} e^{\frac{2\pi i \alpha \beta}{\mu}} - 1 - (\mu - 1) = \mu\, \delta_{\alpha,0} - \mu = \mu\, (\delta_{\alpha,0} - 1)$$

NB notice $L_0 = 0$. Now we can construct the Green's function for the gradient operator (Duffy 2001). The *truncated Green's function* centred on $m = 0$, of Δ, in terms of the eigenfunctions is then,

$$g(m) = \sum_{\alpha \neq 0}^{\mu-1} \frac{\phi_\alpha^\mu(m)}{L_\alpha}$$

where the sum does not include the singularity producing term $\alpha = 0$, thus why we have called it a *truncated* Green's function. To check this works one writes,

$$\Delta\, g(m) \equiv \sum_{\alpha\neq 0}^{\mu-1} \frac{\Delta\, \phi_\alpha^\mu(m)}{L_\alpha} = \sum_{\alpha\neq 0}^{\mu-1} \frac{L_\alpha\, \phi_\alpha^\mu(m)}{L_\alpha} = \sum_{\alpha\neq 0}^{\mu-1} e^{\frac{2\pi i \alpha m}{\mu}} = \sum_{\alpha=0}^{\mu-1} e^{\frac{2\pi i \alpha m}{\mu}} - 1 = \mu\, \delta_{m,0} - 1$$

$$\Delta\, g(m) = \mu \left(\underbrace{\delta_{m,0}}_{\substack{\text{unit}\\ \text{point}\\ \text{charge}}} \underbrace{-\frac{1}{\mu}}_{\substack{\text{uniform}\\ \text{neutralising}\\ \text{background}}} \right)$$

The effect of the omitted $\alpha = 0$ term is to contribute a uniform neutralising background charge, which is required to ensure a vanishing net charge. The truncated Green's function is then,

$$g(m) = \sum_{\alpha\neq 0}^{\mu-1} \frac{\phi_\alpha^\mu(m)}{L_\alpha} = \sum_{\alpha\neq 0}^{\mu-1} \frac{e^{\frac{2\pi i \alpha m}{\mu}}}{\mu\,(\delta_{\alpha,0} - 1)} = -\frac{1}{\mu} \sum_{\alpha\neq 0}^{\mu-1} e^{\frac{2\pi i \alpha m}{\mu}} = -\frac{1}{\mu} \left(\sum_{\alpha=0}^{\mu-1} e^{\frac{2\pi i \alpha m}{\mu}} - 1 \right) = -\frac{1}{\mu} \left(\mu\, \delta_{m,0} - 1\right)$$

$$g(m) = \frac{1}{\mu} - \delta_{m,0}$$

NB this means the Green's function for Δ is simultaneously the eigenvalue of Δ, with eigenvalue μ,

$$\Delta\, g(m) \equiv -\mu\, g(m)$$

Checking this is the truncated Green's function of the gradient operator,

$$\Delta\, g(m) = \sum_{\beta=1}^{\mu-1} (R^\beta - 1) \left(\frac{1}{\mu} - \delta_{m,0}\right) = -\sum_{\beta=1}^{\mu-1} (R^\beta - 1)\delta_{m,0} = -\sum_{\beta=1}^{\mu-1} (\delta_{m+\beta,0} - \delta_{m,0})$$

$$= \begin{Bmatrix} -(1-0) = -1, & m \neq 0 \\ -(0-\mu) = +\mu, & m = 0 \end{Bmatrix} = \mu \left(\delta_{m,0} - \frac{1}{\mu}\right)$$

as required.

Multidimensional space

Generalising to D dimensions we write,

The rotation operators $\{R_q\}$

$$R_q\, f\big(\boldsymbol{m} = \{m_j\}\big) = f(\boldsymbol{m}\colon m_q \to m_q + 1\; mod\; \mu_q)$$

The corresponding eigenfunctions,

$$\phi_{\boldsymbol{\alpha}}^{\boldsymbol{\mu}}(\boldsymbol{m}) \equiv \prod_{q=1}^{D} \phi_{\alpha_q}^{\mu_q}(m_q) = \prod_{q=1}^{D} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}}$$

The gradient operator,

$$\boldsymbol{\Delta} \equiv \sum_{q=1}^{D} \Delta_q = \sum_{q=1}^{D} \sum_{\beta=1}^{\mu_q - 1} \left(R_q^{\beta} - 1\right)$$

Calculating the eigenvalues of $\boldsymbol{\Delta}$ from the single dimension eigenvalues $\left\{L_{\alpha_q;q} \equiv \mu_q \left(\delta_{\alpha_q,0} - 1\right)\right\}$,

$$\boldsymbol{\Delta}\, \phi_{\boldsymbol{\alpha}}^{\boldsymbol{\mu}}(\boldsymbol{m}) \equiv \sum_{q=1}^{D} \Delta_q \prod_{k=1}^{D} \phi_{\alpha_k}^{\mu_k}(m_k) = \prod_{k\neq q}^{D} \phi_{\alpha_k}^{\mu_k}(m_k) \sum_{q=1}^{D} L_{\alpha_q;q}\, \phi_{\alpha_q}^{\mu_q}(m_q) = \phi_{\boldsymbol{\alpha}}^{\boldsymbol{\mu}}(\boldsymbol{m}) \sum_{q=1}^{D} L_{\alpha_q;q}$$

$$L_{\boldsymbol{\alpha}} = \sum_{q=1}^{D} L_{\alpha_q;q} = \sum_{q=1}^{D} \mu_q \left(\delta_{\alpha_q,0} - 1\right)$$

Noting that $L_{\boldsymbol{\alpha}} = 0 \Rightarrow \boldsymbol{\alpha} = \boldsymbol{0}$, the multidimensional truncated Green's function for $\boldsymbol{\Delta}$ is then,

$$g(\boldsymbol{m}) = \sum_{\substack{\{\alpha_q=0\} \\ \boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \frac{\phi_{\boldsymbol{\alpha}}^{\boldsymbol{\mu}}(\boldsymbol{m})}{L_{\boldsymbol{\alpha}}}$$

and checking this,

$$\boldsymbol{\Delta}\, g(\boldsymbol{m}) \equiv \sum_{\substack{\{\alpha_q=0\} \\ \boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \frac{L_{\boldsymbol{\alpha}}\, \phi_{\boldsymbol{\alpha}}^{\boldsymbol{\mu}}(\boldsymbol{m})}{L_{\boldsymbol{\alpha}}} = \sum_{\substack{\{\alpha_q=0\} \\ \boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \prod_{q=1}^{D} \phi_{\alpha_q}^{\mu_q}(m_q) = \sum_{\substack{\{\alpha_q=0\} \\ \boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \prod_{q=1}^{D} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}}$$

$$= \sum_{\{\alpha_q=0\}}^{\{\mu_q-1\}} \prod_{q=1}^{D} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}} - 1 = \prod_{q=1}^{D} \sum_{\alpha_q=0}^{\mu_q-1} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}} - 1 = \prod_{q=1}^{D} \mu_q\, \delta_{m_q,0} - 1 = \delta_{\boldsymbol{m},\boldsymbol{0}}\, V - 1$$

$$\mathbf{\Delta}\, g(\boldsymbol{m}) = V\left(\underbrace{\delta_{\boldsymbol{m},\boldsymbol{0}}}_{\substack{\text{unit}\\\text{point}\\\text{charge}}} \underbrace{-\frac{1}{V}}_{\substack{\text{uniform}\\\text{neutralising}\\\text{background}}}\right)$$

where the total 'volume', $V \equiv \prod_{q=1}^{D} \mu_q$ is the total number of microstates for the multidimensional system.

The multidimensional Green's function,

$$g(\boldsymbol{m}) = \sum_{\substack{\{\alpha_q=0\}\\\boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \frac{\phi_{\boldsymbol{\alpha}}(\boldsymbol{m})}{L_{\boldsymbol{\alpha}}} = \sum_{\substack{\{\alpha_q=0\}\\\boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \frac{\prod_{q=1}^{D} \phi_{\alpha_q}^{\mu_q}(m_q)}{\sum_{q=1}^{D} \mu_q\,(\delta_{\alpha_q,0}-1)}$$

The sum over the integers $\{\alpha_q\}$ can be separated into cases where each index is either vanishing or non-vanishing. Note the case for all vanishing is excluded by $L_{\boldsymbol{\alpha}} = \sum_{q=1}^{D} \mu_q\,(\delta_{\alpha_q,0}-1) \neq 0 \Rightarrow$ $\boldsymbol{\alpha} \neq \boldsymbol{0}$. If the number of integers for which $\alpha_q \neq 0$ is denoted by n then,

$$\sum_{\substack{\{\alpha_q=0\}\\\boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \times \equiv \sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \sum_{\{\alpha_q=0\}}^{\{\mu_q-1\}} \prod_{j\in\boldsymbol{j}}^{n} \overline{\delta_{\alpha_j,0}} \prod_{j\notin\boldsymbol{j}}^{n} \delta_{\alpha_j,0} \times$$

where the case of $\boldsymbol{\alpha} = \boldsymbol{0}$, is precluded by disallowing $n = 0$ (which would correspond to none of the indices $\{\alpha_q\}$ being non-zero),

$$g(\boldsymbol{m}) = \sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \sum_{\substack{\{\alpha_q=0\}\\\boldsymbol{\alpha}\neq\boldsymbol{0}}}^{\{\mu_q-1\}} \left(\prod_{\tau=1}^{n} \overline{\delta_{\alpha_{j_\tau},0}} \prod_{j\notin\boldsymbol{j}}^{n} \delta_{\alpha_j,0}\right) \frac{\prod_{q=1}^{D} \phi_{\alpha_q}^{\mu_q}(m_q)}{\sum_{q=1}^{D} \mu_q\,(\delta_{\alpha_q,0}-1)}$$

$$= \sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \sum_{\{\alpha_q=0\}}^{\{\mu_q-1\}} \left(\prod_{\tau=1}^{n} \overline{\delta_{\alpha_{j_\tau},0}} \prod_{j\notin\boldsymbol{j}}^{n} \delta_{\alpha_j,0}\right) \frac{\prod_{q\in\boldsymbol{j}}^{D} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}} \prod_{q\notin\boldsymbol{j}}^{D} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}}}{\sum_{q\in\boldsymbol{j}}^{D} \mu_q\,(\delta_{\alpha_q,0}-1) + \sum_{q\notin\boldsymbol{j}}^{D} \mu_q\,(\delta_{\alpha_q,0}-1)}$$

$$= \sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \sum_{\{\alpha_q=0\}}^{\{\mu_q-1\}} \left(\prod_{\tau=1}^{n} \overline{\delta_{\alpha_{j_\tau},0}} \prod_{j\notin\boldsymbol{j}}^{n} \delta_{\alpha_j,0}\right) \frac{\prod_{q\in\boldsymbol{j}}^{D} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}} \cdot 1}{\sum_{q\in\boldsymbol{j}}^{D} \mu_q\,(0-1) + \sum_{q\notin\boldsymbol{j}}^{D} \mu_q\,(1-1)}$$

$$= -\sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \sum_{\{\alpha_q=0\}}^{\{\mu_q-1\}} \left(\prod_{\tau=1}^{n} \overline{\delta_{\alpha_{j_\tau},0}} \prod_{j\notin\boldsymbol{j}}^{n} \delta_{\alpha_j,0}\right) \frac{\prod_{q\in\boldsymbol{j}}^{D} e^{\frac{2\pi i \alpha_q m_q}{\mu_q}}}{\sum_{q\in\boldsymbol{j}}^{D} \mu_q}$$

$$= -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\sum_{\{\alpha_q=0\}}^{\{\mu_q-1\}}\left(\prod_{\tau=1}^{n}\overline{\delta_{\alpha_{j_\tau},0}}\prod_{j\notin\boldsymbol{j}}^{n}\delta_{\alpha_j,0}\right)\prod_{q\in\boldsymbol{j}}^{D}e^{\frac{2\pi i\alpha_q m_q}{\mu_q}}$$

$$= -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\sum_{\{\alpha_q=0\}}^{\{\mu_q-1\}}\prod_{j\in\boldsymbol{j}}\overline{\delta_{\alpha_j,0}}\,e^{\frac{2\pi i\alpha_j m_j}{\mu_j}}\prod_{j\notin\boldsymbol{j}}^{n}\delta_{\alpha_j,0}$$

$$= -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\sum_{\{\alpha_{q\in\boldsymbol{j}}=0\}}^{\{\mu_q-1\}}\prod_{j\in\boldsymbol{j}}\overline{\delta_{\alpha_j,0}}\,e^{\frac{2\pi i\alpha_j m_j}{\mu_j}}\sum_{\{\alpha_{q\notin\boldsymbol{j}}=0\}}^{\{0\}}\prod_{j\notin\boldsymbol{j}}^{n}\delta_{\alpha_j,0}$$

$$= -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\sum_{\alpha_j=1}^{\mu_j-1}e^{\frac{2\pi i\alpha_j m_j}{\mu_j}} = -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\left(\sum_{\alpha_j=0}^{\mu_j-1}e^{\frac{2\pi i\alpha_j m_j}{\mu_j}}-1\right)$$

$$g(\boldsymbol{m}) = -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\left(\mu_j\delta_{m_j,0}-1\right) = -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\mu_j\left(\delta_{m_j,0}-\frac{1}{\mu_j}\right)$$

and using previous single dimension truncated Green's function $g(m) = \frac{1}{\mu} - \delta_{m,0} \Rightarrow \delta_{m,0} - \frac{1}{\mu} = -\,g_\mu(m)$, we have,

$$g(\boldsymbol{m}) = -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\left(-\mu_j\, g_{\mu_j}(m_j)\right)$$

This can be checked to be the correct truncated Green's function by substitution,

$$\boldsymbol{\Delta}\, g(\boldsymbol{m}) = -\sum_{q=1}^{D}\Delta_q\sum_{n=0}^{D-1}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\left(-\mu_j\, g_{\mu_j}(m_j)\right)$$

$$= -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\sum_{q=1}^{D}\Delta_q\prod_{j\in\boldsymbol{j}}\left(-\mu_j\, g_{\mu_j}(m_j)\right)$$

$$= -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\sum_{q\in\boldsymbol{j}}\mu_q\prod_{j\in\boldsymbol{j}}\left(-\mu_j\, g_{\mu_j}(m_j)\right) = -\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\prod_{j\in\boldsymbol{j}}\left(-\mu_j\, g_{\mu_j}(m_j)\right)$$

$$= -\sum_{n=1}^{D}e_n^D\left(\left\{-\mu_j\, g_{\mu_j}(m_j)\right\}\right)$$

where $e_n^D(\{z_j\}) \equiv \sum_{j_1<\cdots<j_n=0}^{D}\prod_{j\in\boldsymbol{j}}z_j$ are the elementary symmetric polynomials (MacDonald 1995) in the D variables $\{z_{j:1\le j\le D}\}$, in terms of which there is the identity for expanding the product,

$$\prod_{j=1}^{D}(z_j+\lambda)\equiv\sum_{n=0}^{D}\lambda^n\ e_n^D(\{z_j\})$$

So, setting $\left\{z_j=-\mu_j\ g_{\mu_j}(m_j)\right\}$ and $\lambda=1$ yields the identity,

$$\sum_{n=0}^{D}e_n^D\left(\left\{-\mu_j\ g_{\mu_j}(m_j)\right\}\right)\equiv 1+\sum_{n=1}^{D}e_n^D\left(\left\{-\mu_j\ g_{\mu_j}(m_j)\right\}\right)\equiv\prod_{j=1}^{D}\left(-\mu_j\ g_{\mu_j}(m_j)+1\right)$$

$$\sum_{n=1}^{D}e_n^D\left(\left\{-\mu_j\ g_{\mu_j}(m_j)\right\}\right)\equiv\prod_{j=1}^{D}\left(-\mu_j\ g_{\mu_j}(m_j)+1\right)-1$$

$$\equiv\prod_{j=1}^{D}\left(\mu_j\ \delta_{m_j,0}-1+1\right)-1\equiv V\ \delta_{\boldsymbol{m},\boldsymbol{0}}-1=V\left(\underbrace{\delta_{\boldsymbol{m},\boldsymbol{0}}}_{\substack{\text{unit}\\\text{point}\\\text{charge}}}\quad\underbrace{-\frac{1}{V}}_{\substack{\text{uniform}\\\text{neutralising}\\\text{background}}}\right)$$

where the multi-dimensional volume, $V\equiv\prod_{j=1}^{D}\mu_j$.

Subspaces

Defining the subspace σ_μ, as the set of elements $\{q\colon 1\le q\le D\}$, which have $\mu_q=\mu$ possible values,

$$\sigma_\mu\colon q\in\sigma_\mu\Leftrightarrow\ \mu_q=\mu$$

$$g(\boldsymbol{m})=-\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\left(-\mu_j\ g_{\mu_j}(m_j)\right)\ =-\sum_{n=1}^{D}\sum_{j_1<\cdots<j_n=0}^{D}\frac{1}{\sum_{q\in\boldsymbol{j}}^{D}\mu_q}\prod_{j\in\boldsymbol{j}}\left(\mu_j\ \delta_{m_j,0}-1\right)$$

NB can write $\mu_j\ \delta_{m_j,0}-1=-\left(1-\mu_j\right)^{\delta_{m_j,0}}$

$$\prod_{j\in\boldsymbol{j}}\left(\mu_j\ \delta_{m_j,0}-1\right)\equiv\prod_{j\in\boldsymbol{j}}(-1)\left(1-\mu_j\right)^{\delta_{m_j,0}}=\prod_{\mu=2}^{\mu_{\max}}\prod_{\substack{j\in\boldsymbol{j}\\\mu_j=\mu}}(-1)\left(1-\mu_j\right)^{\delta_{m_j,0}}=\prod_{\mu=2}^{\mu_{\max}}\prod_{\substack{j\in\boldsymbol{j}\\j\in\sigma_\mu}}(-1)(1-\mu)^{\delta_{m_j,0}}$$

$$=\prod_{\mu=2}^{\mu_{\max}}(-1)^{\sum_{\substack{j\in\boldsymbol{j}\\j\in\sigma_\mu}}1}\ (1-\mu)^{\sum_{\substack{j\in\boldsymbol{j}\\j\in\sigma_\mu}}\delta_{m_j,0}}=\prod_{\mu=2}^{\mu_{\max}}(-1)^{n_\mu}\ (1-\mu)^{\sum_{\substack{j\in\boldsymbol{j}\\j\in\sigma_\mu}}\left(1-\overline{\delta_{m_j,0}}\right)}$$

$$= \prod_{\mu=2}^{\mu_{\max}} (-1)^{n_\mu(\boldsymbol{j})} \, (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j})}$$

where $n_\mu(\boldsymbol{j}) \equiv \sum_{j\in\sigma_\mu,\boldsymbol{j}} 1$ is the number of the chosen elements $\boldsymbol{j}$ that are in σ_μ and $q_\mu(\boldsymbol{j},\boldsymbol{m}) \equiv \sum_{j\in\sigma_\mu,\boldsymbol{j}} \overline{\delta_{m_j,0}}$ is the number of chosen elements that are both in σ_μ AND have $m_j \neq 0$. From this it is obvious that $q_\mu(\boldsymbol{j}) \leq n_\mu(\boldsymbol{j})$ and $\sum_\mu n_\mu(\boldsymbol{j}) = n$. This gives,

$$\prod_{j\in\boldsymbol{j}} \left(\mu_j \, \delta_{m_j,0} - 1\right) = (-1)^n \prod_{\mu=2}^{\mu_{\max}} (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

and with $\sum_{q\in\boldsymbol{j}}^{D} \mu_q \equiv \sum_\mu \sum_{q\in\boldsymbol{j}}^{D} \delta_{\mu,\mu_q} \mu_q \equiv \sum_\mu \mu \, \sum_{q\in\boldsymbol{j}}^{D} \delta_{\mu,\mu_q} \equiv \sum_\mu \mu \, \sum_{j\in\sigma_\mu,\boldsymbol{j}} 1 \equiv \sum_\mu \mu \, n_\mu(\boldsymbol{j})$,

$$g(\boldsymbol{m}) = -\sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \frac{(-1)^n}{\sum_\mu \mu \, n_\mu(\boldsymbol{j})} \prod_{\mu=2}^{\mu_{\max}} (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

That this is the truncated Green's function can be checked,

The gradient operator is,

$$\boldsymbol{\Delta} = \sum_{j=1}^{D} \Delta_j \equiv \sum_\mu \sum_{j\in\sigma_\mu} \Delta_j$$

$$\Rightarrow \quad \boldsymbol{\Delta} \prod_{\nu=2}^{\mu_{\max}} (1-\nu)^{n_\nu(\boldsymbol{j}) - q_\nu(\boldsymbol{j},\boldsymbol{m})} \equiv \sum_\mu \sum_{j\in\sigma_\mu} \Delta_j \prod_{\nu=2}^{\mu_{\max}} (1-\nu)^{n_\nu(\boldsymbol{j}) - q_\nu(\boldsymbol{j},\boldsymbol{m})}$$

$$\equiv \sum_\mu \sum_{j\in\sigma_\mu} \left(\prod_{\nu\neq\mu} (1-\nu)^{n_\nu(\boldsymbol{j}) - q_\nu(\boldsymbol{j},\boldsymbol{m})} \right) \Delta_j \, (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

$$\Delta_{j\in\sigma_\mu,\boldsymbol{j}} \, (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m})} \equiv \sum_{\beta=1}^{\mu_j - 1} \left(R_j^\beta - 1\right) (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

NB $n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m}) \equiv \sum_{j\in\sigma_\mu,\boldsymbol{j}} 1 - \sum_{j\in\sigma_\mu,\boldsymbol{j}} \overline{\delta_{m_j,0}} \equiv \sum_{j\in\sigma_\mu,\boldsymbol{j}} \delta_{m_j,0}$

$$\Delta_{j\in\sigma_\mu,\boldsymbol{j}} \, (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m})} \equiv \Delta_j \, (1-\mu)^{\sum_{j\in\sigma_\mu,\boldsymbol{j}} \delta_{m_k,0}} \equiv \Delta_j \prod_{k\in\sigma_\mu,\boldsymbol{j}} (1-\mu)^{\delta_{m_k,0}}$$

$$\equiv \left(\Delta_{\mathrm{j}} \, (1-\mu)^{\delta_{m_j,0}}\right) \prod_{\substack{k\in\sigma_\mu,\boldsymbol{j} \\ k\neq j}} (1-\mu)^{\delta_{m_k,0}}$$

$$\Delta_j\,(1-\mu)^{\delta_{m_j,0}} \equiv \sum_{\beta=1}^{\mu_j-1}\left(R_j^{\beta}-1\right)(1-\mu)^{\delta_{m_j,0}} \equiv \sum_{\beta=1}^{\mu_j-1}\left(\,(1-\mu)^{\delta_{m_j+\beta,0}}-(1-\mu)^{\delta_{m_j,0}}\right)$$

$$(1-\mu)^{\delta_{m_j+\beta,0}}-(1-\mu)^{\delta_{m_j,0}} = \begin{Bmatrix} 1-(1-\mu)=+\mu & m_j=0 \\ (1-\mu)-1=-\mu & m_j+\beta=0 \Rightarrow m_j\neq 0 \\ 1-1=0 & m_j+\beta\neq 0, m_j\neq 0 \end{Bmatrix}$$

$$\sum_{\beta=1}^{\mu_j-1}\left(\,(1-\mu)^{\delta_{m_j+\beta,0}}-(1-\mu)^{\delta_{m_j,0}}\right) = \begin{Bmatrix} (\mu_j-1)\,\mu & m_j=0 \\ -\mu & m_j\neq 0 \end{Bmatrix} = -\mu\left(1-\mu_j\right)^{\delta_{m_j,0}}$$

$$\Rightarrow \quad \Delta_j\,(1-\mu)^{\delta_{m_j,0}} \equiv -\mu\left(1-\mu_j\right)^{\delta_{m_j,0}}$$

$$\Rightarrow \quad \Delta_{j\in\sigma_\mu,\boldsymbol{j}}\,(1-\mu)^{n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m})} \equiv \left(-\mu\left(1-\mu_j\right)^{\delta_{m_j,0}}\right)\prod_{\substack{k\in\sigma_\mu,\boldsymbol{j}\\ k\neq j}}(1-\mu)^{\delta_{m_k,0}} \equiv -\mu\,(1-\mu)^{n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

$$\boldsymbol{\Delta}\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu(\boldsymbol{j})-q_\nu(\boldsymbol{j},\boldsymbol{m})} \equiv \sum_{\mu}\sum_{j\in\sigma_\mu}\Delta_j\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu(\boldsymbol{j})-q_\nu(\boldsymbol{j},\boldsymbol{m})} \equiv -\sum_{\mu}\sum_{j\in\sigma_\mu,\boldsymbol{j}}\mu\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu(\boldsymbol{j})-q_\nu(\boldsymbol{j},\boldsymbol{m})}$$

$$\equiv -\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu(\boldsymbol{j})-q_\nu(\boldsymbol{j},\boldsymbol{m})}\right)\sum_{\mu}\mu\sum_{j\in\sigma_\mu,\boldsymbol{j}}1$$

$$\boldsymbol{\Delta}\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu(\boldsymbol{j})-q_\nu(\boldsymbol{j},\boldsymbol{m})} \equiv \left(-\sum_{\mu}\mu\,n_\mu(\boldsymbol{j})\right)\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu(\boldsymbol{j})-q_\nu(\boldsymbol{j},\boldsymbol{m})}$$

$$\boldsymbol{\Delta}\,g(\boldsymbol{m}) = -\sum_{n=1}^{D}(-1)^n\sum_{j_1<\cdots<j_n=0}^{D}\frac{-\sum_\mu \mu\,n_\mu(\boldsymbol{j})}{\sum_\mu \mu\,n_\mu(\boldsymbol{j})}\prod_{\mu=2}^{\mu_{\max}}(1-\mu)^{n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

$$\boldsymbol{\Delta}\,g(\boldsymbol{m}) = \sum_{n=1}^{D}(-1)^n\sum_{j_1<\cdots<j_n=0}^{D}\prod_{\mu=2}^{\mu_{\max}}(1-\mu)^{n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

NB again, $n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m}) \equiv \sum_{j\in\sigma_\mu,\boldsymbol{j}}1-\sum_{j\in\sigma_\mu,\boldsymbol{j}}\overline{\delta_{m_j,0}} \equiv \sum_{j\in\sigma_\mu,\boldsymbol{j}}\delta_{m_j,0}$

$$\prod_{\mu=2}^{\mu_{\max}}(1-\mu)^{n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m})} \equiv \prod_{\mu=2}^{\mu_{\max}}(1-\mu)^{\sum_{j\in\sigma_\mu,\boldsymbol{j}}\delta_{m_j,0}} \equiv \prod_{\mu=2}^{\mu_{\max}}\prod_{j\in\sigma_\mu,\boldsymbol{j}}(1-\mu)^{\delta_{m_j,0}} \equiv \prod_{j\in\boldsymbol{j}}\left(1-\mu_j\right)^{\delta_{m_j,0}}$$

$$\equiv \prod_{\tau=1}^{n}\left(1-\mu_{j_\tau}\right)^{\delta_{m_{j_\tau},0}}$$

$$\boldsymbol{\Delta}\, g(\boldsymbol{m}) = \sum_{n=1}^{D} (-1)^n \sum_{j_1<\cdots<j_n=0}^{D} \prod_{\tau=1}^{n} \left(1-\mu_{j_\tau}\right)^{\delta_{m_{j_\tau},0}} = \sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \prod_{\tau=1}^{n} \left(-\left(1-\mu_{j_\tau}\right)^{\delta_{m_{j_\tau},0}}\right)$$

$$= \sum_{n=1}^{D} e_n^D \left(\left\{-\left(1-\mu_{j_\tau}\right)^{\delta_{m_{j_\tau},0}}\right\}\right)$$

And again, recalling,

$$\prod_{j=1}^{D} (z_j + \lambda) \equiv \sum_{n=0}^{D} \lambda^n \ e_n^D(\{z_j\})$$

$$\Rightarrow \quad \sum_{n=0}^{D} e_n^D \left(\left\{-\left(1-\mu_j\right)^{\delta_{m_j,0}}\right\}\right) \equiv \prod_{j=1}^{D} \left(-\left(1-\mu_j\right)^{\delta_{m_j,0}} + 1\right)$$

$$-\left(1-\mu_j\right)^{\delta_{m_j,0}} + 1 = \begin{Bmatrix} \mu_j, & m_j = 0 \\ 0, & m_j \neq 0 \end{Bmatrix} = \mu_j \ \delta_{m_j,0}$$

$$\sum_{n=1}^{D} e_n^D \left(\left\{-\left(1-\mu_{j_\tau}\right)^{\delta_{m_{j_\tau},0}}\right\}\right) \equiv -1 + \sum_{n=0}^{D} e_n^D \left(\left\{-\left(1-\mu_{j_\tau}\right)^{\delta_{m_{j_\tau},0}}\right\}\right) = -1 + \prod_{j=1}^{D} \mu_j \ \delta_{m_j,0} = \delta_{\boldsymbol{m},\boldsymbol{0}}\, V - 1$$

$$\boldsymbol{\Delta}\, g(\boldsymbol{m}) = V \left(\delta_{\boldsymbol{m},\boldsymbol{0}} - \frac{1}{V}\right)$$

as required.

Projection of the truncated Green's function onto subspace metrics

In the truncated Green's function,

$$g(\boldsymbol{m}) = -\sum_{n=1}^{D} (-1)^n \sum_{j_1<\cdots<j_n=0}^{D} \frac{1}{\sum_\mu \mu\, n_\mu(\boldsymbol{j})} \prod_{\mu=2}^{\mu_{\max}} (1-\mu)^{n_\mu(\boldsymbol{j}) - q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

and the summation operator that picks out all combinations of more than zero elements from $j: 1 \le j \le D$ (corresponding to terms with $\alpha_q \neq 0$) is,

$$\sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \times \equiv \sum_{\{n_\mu\}} \sum_{\{q_\mu\}} \sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \delta_{n, \sum_\mu n_\mu(\boldsymbol{j})} \prod_{\mu} \left(\delta_{n_\mu, n_\mu(\boldsymbol{j})}\ \delta_{q_\mu, q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) \times$$

NB $q_\mu(\boldsymbol{j}) \le n_\mu(\boldsymbol{j}), d_\mu(\boldsymbol{m})$ and $n = \sum_\mu n_\mu(\boldsymbol{j})$. All this gives,

$$g(\boldsymbol{m}) = -\sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \frac{(-1)^n}{\sum_\mu \mu\, n_\mu(\boldsymbol{j})} \prod_{\mu=2}^{\mu_{\max}} (1-\mu)^{n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

$$= -\sum_{\{n_\mu\}} \sum_{\{q_\mu\}} \sum_{n=1}^{D} \sum_{j_1<\cdots<j_n=0}^{D} \delta_{n,\sum_\mu n_\mu(\boldsymbol{j})} \prod_\mu \left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\, \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) \frac{(-1)^n}{\sum_\mu \mu\, n_\mu(\boldsymbol{j})} \prod_{\mu=2}^{\mu_{\max}} (1-\mu)^{n_\mu(\boldsymbol{j})-q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

$$= -\sum_{\{n_\mu\}} \sum_{\{q_\mu\}} \sum_{n=1}^{D} \frac{(-1)^n\, \delta_{n,\sum_\mu n_\mu}}{\sum_\mu \mu\, n_\mu} \left(\prod_{\mu=2}^{\mu_{\max}} (1-\mu)^{n_\mu - q_\mu} \right) \sum_{j_1<\cdots<j_n=0}^{D} \prod_\mu \left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\, \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)$$

$$g(\boldsymbol{m}) = -\sum_{\{n_\mu\}} \sum_{\{q_\mu\}} \sum_{n=1}^{D} \frac{(-1)^n\, \delta_{n,\sum_\mu n_\mu}}{\sum_\mu \mu\, n_\mu} \left(\prod_{\mu=2}^{\mu_{\max}} (1-\mu)^{n_\mu - q_\mu} \right) w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m})$$

where $w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m}) \equiv \sum_{j_1<\cdots<j_n=0}^{D} \prod_\mu \left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\, \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)$.

Checking this is indeed still the truncated Green's function,

$$\boldsymbol{\Delta}\, w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m}) \equiv \sum_{q=1}^{D} \Delta_q \sum_{j_1<\cdots<j_n=0}^{D} \prod_\nu \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}$$

$$\equiv \sum_\mu \sum_{q\in\sigma_\mu} \Delta_q \sum_{j_1<\cdots<j_n=0}^{D} \prod_\nu \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}$$

$$\equiv \sum_{j_1<\cdots<j_n=0}^{D} \sum_\mu \sum_{q\in\sigma_\mu} \Delta_q \prod_\nu \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}$$

$$\Delta_{q\in\sigma_\mu} \prod_\nu \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})} \equiv \left(\prod_{\nu\neq\mu} \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})} \right) \Delta_{q\in\sigma_\mu} \left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\, \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)$$

$$\Delta_{q\in\sigma_\mu} \left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\, \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) \equiv \delta_{n_\mu,n_\mu(\boldsymbol{j})}\, \Delta_{q\in\sigma_\mu}\, \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

$$\equiv \delta_{n_\mu,n_\mu(\boldsymbol{j})} \sum_{\beta=1}^{\mu_q-1} \left(R^\beta_{q\in\sigma_\mu} - 1\right) \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})} \equiv \delta_{n_\mu,n_\mu(\boldsymbol{j})} \sum_{\beta=1}^{\mu_q-1} \left(\delta_{q_\mu, R^\beta_{q\in\sigma_\mu} q_\mu(\boldsymbol{j},\boldsymbol{m})} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})} \right)$$

$$R^\beta_q\, q_\mu(\boldsymbol{j},\boldsymbol{m}) \equiv R^\beta_q \sum_{j\in\sigma_\mu,\boldsymbol{j}} \overline{\delta_{m_j,0}}$$

$$\equiv R^{\beta}_{q\in\sigma_\mu}\left(\begin{Bmatrix}\overline{\delta_{m_q,0}}, & q\in\boldsymbol{j}\\ 0, & q\notin\boldsymbol{j}\end{Bmatrix}+\sum_{\substack{j\in\sigma_\mu,\boldsymbol{j}\\ j\neq q}}\overline{\delta_{m_j,0}}\right)\equiv R^{\beta}_{q\in\sigma_\mu}\left(\delta_{q\in\boldsymbol{j}}\,\overline{\delta_{m_q,0}}+\sum_{\substack{j\in\sigma_\mu,\boldsymbol{j}\\ j\neq q}}\overline{\delta_{m_j,0}}\right)$$

$$\equiv \delta_{q\in\boldsymbol{j}}\,\overline{\delta_{m_q+\beta \bmod \mu_q,0}}+\sum_{\substack{j\in\sigma_\mu,\boldsymbol{j}\\ j\neq q}}\overline{\delta_{m_j,0}}$$

$$\equiv \delta_{q\in\boldsymbol{j}}\,\left(\overline{\delta_{m_q+\beta \bmod \mu_q,0}}-\overline{\delta_{m_q,0}}\right)+\sum_{j\in\sigma_\mu,\boldsymbol{j}}\overline{\delta_{m_j,0}}\equiv \delta_{q\in\boldsymbol{j}}\,\left(\overline{\delta_{m_q+\beta \bmod \mu_q,0}}-\overline{\delta_{m_q,0}}\right)+q_\mu(\boldsymbol{j},\boldsymbol{m})$$

$$\overline{\delta_{m_q+\beta \bmod \mu_q,0}}-\overline{\delta_{m_q,0}}=\begin{Bmatrix}\bar{0}-\bar{1}=+1 & m_q=0\\ \bar{1}-\bar{0}=-1 & m_q+\beta \bmod \mu_q=0\Rightarrow\ m_q\neq 0\\ \bar{0}-\bar{0}=0 & m_q+\beta \bmod \mu_q\neq 0, m_q\neq 0\end{Bmatrix}$$

$$R^{\beta}_{q}\,q_\mu(\boldsymbol{j},\boldsymbol{m})\equiv q_\mu(\boldsymbol{j},\boldsymbol{m})+\delta_{q\in\boldsymbol{j}}\times\begin{Bmatrix}+1 & m_q=0\\ -1 & m_q+\beta \bmod \mu_q=0\Rightarrow\ m_q\neq 0\\ 0 & m_q+\beta \bmod \mu_q\neq 0, m_q\neq 0\end{Bmatrix}$$

$$\sum_{\beta=1}^{\mu_q-1}\delta_{q_\mu,R^{\beta}_{q\in\boldsymbol{j}}\,q_\mu(\boldsymbol{j},\boldsymbol{m})}\equiv\begin{Bmatrix}\left(\mu_q-1\right)\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1} & m_q=0\\ \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1}+\left(\mu_q-2\right)\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})} & m_q\neq 0\end{Bmatrix}$$

$$\sum_{\beta=1}^{\mu_q-1}\delta_{q_\mu,R^{\beta}_{q\notin j}\,q_\mu(\boldsymbol{j},\boldsymbol{m})}\equiv\left(\mu_q-1\right)\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}$$

$$\Delta_{q\in\sigma_\mu,\boldsymbol{j}}\left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\,\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)\equiv\delta_{n_\mu,n_\mu(\boldsymbol{j})}\sum_{\beta=1}^{\mu_q-1}\left(\delta_{q_\mu,R^{\beta}_{q}\,q_\mu(\boldsymbol{j},\boldsymbol{m})}-\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)$$

$$\Delta_{\substack{q\in\sigma_\mu\\ q\notin\boldsymbol{j}}}\left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\,\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)\equiv\delta_{n_\mu,n_\mu(\boldsymbol{j})}\sum_{\beta=1}^{\mu_q-1}\left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}-\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)=0$$

$$\Rightarrow\quad\Delta_{q\in\sigma_\mu}\left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\,\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)\equiv\delta_{q\in\boldsymbol{j}}\,\delta_{n_\mu,n_\mu(\boldsymbol{j})}\sum_{\beta=1}^{\mu_q-1}\left(\delta_{q_\mu,R^{\beta}_{q}\,q_\mu(\boldsymbol{j},\boldsymbol{m})}-\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)$$

$$\equiv\delta_{q\in\boldsymbol{j}}\,\delta_{n_\mu,n_\mu(\boldsymbol{j})}\left(\begin{Bmatrix}\left(\mu_q-1\right)\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1} & m_q=0\\ \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1}+\left(\mu_q-2\right)\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})} & m_q\neq 0\end{Bmatrix}-\left(\mu_q-1\right)\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)$$

$$\equiv\delta_{q\in\boldsymbol{j}}\,\delta_{n_\mu,n_\mu(\boldsymbol{j})}\times\begin{Bmatrix}\left(\mu_q-1\right)\left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1}-\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) & m_q=0\\ \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1}-\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})} & m_q\neq 0\end{Bmatrix}$$

$$\sum_{q\in\sigma_\mu}\Delta_q\prod_{\nu}\left(\delta_{n_\nu,n_\nu(\boldsymbol{j})}\,\delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right)\equiv\left(\prod_{\nu\neq\mu}\left(\delta_{n_\nu,n_\nu(\boldsymbol{j})}\,\delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right)\right)\sum_{q\in\sigma_\mu}\Delta_q\left(\delta_{n_\nu,n_\nu(\boldsymbol{j})}\,\delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right)$$

$$\sum_{q\in\sigma_\mu} \Delta_q \left(\delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right) \equiv \sum_{q\in\sigma_\mu} \delta_{q\in\boldsymbol{j}}\, \delta_{n_\mu,n_\mu(\boldsymbol{j})} \times \begin{Bmatrix} (\mu_q - 1)\left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) & m_q = 0 \\ \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})} & m_q \neq 0 \end{Bmatrix}$$

$$\equiv \delta_{n_\mu,n_\mu(\boldsymbol{j})} \sum_{q\in\sigma_\mu,\boldsymbol{j}} \left((\mu_q - 1)\left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) \delta_{m_q,0} + \left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) \overline{\delta_{m_q,0}}\right)$$

$$\equiv \delta_{n_\mu,n_\mu(\boldsymbol{j})} \left((\mu - 1)\left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) \sum_{q\in\sigma_\mu,\boldsymbol{j}} \left(1 - \overline{\delta_{m_q,0}}\right) \right.$$

$$\left. + \left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) \sum_{q\in\sigma_\mu,\boldsymbol{j}} \overline{\delta_{m_q,0}}\right)$$

$$\equiv \delta_{n_\mu,n_\mu(\boldsymbol{j})} \left((\mu - 1)\left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)(n_\mu - q_\mu) + \left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) q_\mu\right)$$

$$\boldsymbol{\Delta}\, w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m}) \equiv \sum_{j_1<\cdots<j_n=0}^{D} \sum_\mu \sum_{q\in\sigma_\mu} \Delta_q \prod_\nu \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}$$

$$\equiv \sum_{j_1<\cdots<j_n=0}^{D} \sum_\mu \left(\prod_{\nu\neq\mu} \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right) \sum_{q\in\sigma_\mu} \Delta_q \left(\delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right)$$

$$\equiv \sum_{j_1<\cdots<j_n=0}^{D} \sum_\mu \left(\prod_{\nu\neq\mu} \delta_{n_\nu,n_\nu(\boldsymbol{j})}\, \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right) \delta_{n_\mu,n_\mu(\boldsymbol{j})} \left((\mu - 1)\left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})+1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)(n_\mu - q_\mu) \right.$$

$$\left. + \left(\delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})-1} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) q_\mu\right)$$

$$\boldsymbol{\Delta}\, w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m}) \equiv \sum_{j_1<\cdots<j_n=0}^{D} \left(\prod_\nu \delta_{n_\nu,n_\nu(\boldsymbol{j})}\right) \sum_\mu \left(\prod_{\nu\neq\mu} \delta_{q_\nu,q_\nu(\boldsymbol{j},\boldsymbol{m})}\right) \left((\mu - 1)\left(\delta_{q_\mu-1,q_\mu(\boldsymbol{j},\boldsymbol{m})} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)(n_\mu \right.$$

$$\left. - q_\mu) + \left(\delta_{q_\mu+1,q_\mu(\boldsymbol{j},\boldsymbol{m})} - \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right) q_\mu\right)$$

$$\boldsymbol{\Delta}\, w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m}) \equiv \sum_\mu \left((\mu - 1)\left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu-1} - w_{\boldsymbol{n},\boldsymbol{q}}\right)(n_\mu - q_\mu) + \left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu+1} - w_{\boldsymbol{n},\boldsymbol{q}}\right) q_\mu\right)$$

$$(\mu - 1)\left(\prod_{\nu=2}^{\mu_{\max}} (1-\nu)^{n_\nu - q_\nu}\right) w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu-1} \to -(1-\mu)\left(\prod_{\nu\neq\mu}^{\mu_{\max}} (1-\nu)^{n_\nu - q_\nu}\right)(1-\mu)^{n_\mu-(q_\mu+1)}\, w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\to -\left(\prod_{\nu=2}^{\mu_{\max}} (1-\nu)^{n_\nu - q_\nu}\right) w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)(\mu-1)\ \left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu-1}-w_{\boldsymbol{n},\boldsymbol{q}}\right)$$

$$\to-\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}-(\mu-1)\ \left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\to-\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)(1+\mu-1)\,w_{\boldsymbol{n},\boldsymbol{q}}=-\mu\ \left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu+1}\to\left(\prod_{\nu\neq\mu}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)(1-\mu)^{n_\mu-(q_\mu-1)}\,w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\to\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)(1-\mu)\,w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)\left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu+1}-w_{\boldsymbol{n},\boldsymbol{q}}\right)$$

$$\to\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)(1-\mu)\,w_{\boldsymbol{n},\boldsymbol{q}}-\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\to\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)(1-\mu-1)\,w_{\boldsymbol{n},\boldsymbol{q}}=-\mu\ \left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)\left((\mu-1)\ \left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu-1}-w_{\boldsymbol{n},\boldsymbol{q}}\right)\ (n_\mu-q_\mu)+\left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu+1}-w_{\boldsymbol{n},\boldsymbol{q}}\right)\ q_\mu\right)$$

$$=-\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)\left(\mu\,w_{\boldsymbol{n},\boldsymbol{q}}\,(n_\mu-q_\mu)+\mu\,w_{\boldsymbol{n},\boldsymbol{q}}\,q_\mu\right)$$

$$=-\mu\ \left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)\left((n_\mu-q_\mu)+q_\mu\right)\ w_{\boldsymbol{n},\boldsymbol{q}}=-\mu\,n_\mu\ \left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}$$

$$\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)\sum_\mu\left((\mu-1)\ \left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu-1}-w_{\boldsymbol{n},\boldsymbol{q}}\right)\ (n_\mu-q_\mu)+\left(w_{\boldsymbol{n},\boldsymbol{q}:q_\mu\to q_\mu+1}-w_{\boldsymbol{n},\boldsymbol{q}}\right)\ q_\mu\right)$$

$$=-\sum_\mu\mu\,D_\mu\ \left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}=-\left(\prod_{\nu=2}^{\mu_{\max}}(1-\nu)^{n_\nu-q_\nu}\right)w_{\boldsymbol{n},\boldsymbol{q}}\ \sum_\mu\mu\,n_\mu$$

$$\boldsymbol{\Delta}\, w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m}) \equiv -w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m}) \sum_{\mu} \mu\, n_\mu$$

So $w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m})$ is an eigenfunction of the gradient operator as required for the solution to be the truncated Green's function.

Calculating the coefficient

The coefficient $w_{\boldsymbol{n},\boldsymbol{q}}(\boldsymbol{m},\boldsymbol{j}) \equiv \sum_{j_1<\cdots<j_n=0}^{D} \prod_\mu \left(\delta_{n_\mu,n_\mu(\boldsymbol{j})}\, \delta_{q_\mu,q_\mu(\boldsymbol{j},\boldsymbol{m})}\right)$ counts the number of ways of choosing n_μ elements from σ_μ such that q_μ of them have $\alpha_q \neq 0$ and so on for all the subspaces $\{\sigma_\mu\}$. This can be split into a product over the subspace labels,

$$w_{\boldsymbol{n},\boldsymbol{q}} = \prod_\mu w_{n_\mu,q_\mu}$$

Each subspace has a total of $D_\mu \equiv \sum_{j\in\sigma_\mu} 1$ elements, of which $d_\mu \equiv \sum_{j\in\sigma_\mu} \overline{\delta_{m_j,0}} = d_\mu(\boldsymbol{m})$ have $\alpha_j \neq 0$. So w_{n_μ,q_μ} counts the number of ways of choosing q_μ objects from d_μ AND choosing $(n_\mu - q_\mu)$ from $D_\mu - d_\mu$.

$$w_{n_\mu,q_\mu}(d_\mu) = \begin{bmatrix} d_\mu \\ q_\mu \end{bmatrix} \begin{bmatrix} D_\mu - d_\mu \\ n_\mu - q_\mu \end{bmatrix} = \frac{d_\mu!\, \left(D_\mu - d_\mu\right)!}{\left(d_\mu - q_\mu\right)!\, q_\mu!\, \left(D_\mu - d_\mu - n_\mu + q_\mu\right)!\, \left(n_\mu - q_\mu\right)!}$$

General solution

Putting it all together,

$$g(\boldsymbol{m}) = -\sum_{\{n_\mu\}} \sum_{\{q_\mu\}} \sum_{n=1}^{D} \frac{(-1)^n\, \delta_{n,\sum_\mu n_\mu}}{\sum_\mu \mu\, n_\mu} \prod_\mu (1-\mu)^{n_\mu - q_\mu} \begin{bmatrix} d_\mu \\ q_\mu \end{bmatrix} \begin{bmatrix} D_\mu - d_\mu \\ n_\mu - q_\mu \end{bmatrix}$$

$$= -\sum_{\{n_\mu\}} \sum_{n=1}^{D} \frac{(-1)^n\, \delta_{n,\sum_\mu n_\mu}}{\sum_\mu \mu\, n_\mu} \sum_{\{q_\mu\}} \prod_\mu (1-\mu)^{n_\mu - q_\mu} \begin{bmatrix} d_\mu \\ q_\mu \end{bmatrix} \begin{bmatrix} D_\mu - d_\mu \\ n_\mu - q_\mu \end{bmatrix}$$

$$= -\sum_{\{n_\mu\}} \sum_{n=1}^{D} \frac{(-1)^n\, \delta_{n,\sum_\mu n_\mu}}{\sum_\mu \mu\, n_\mu} \prod_\mu \sum_q (1-\mu)^{n_\mu - q} \begin{bmatrix} d_\mu \\ q \end{bmatrix} \begin{bmatrix} D_\mu - d_\mu \\ n_\mu - q \end{bmatrix}$$

$$= -\sum_{\substack{\{n_\mu\} \\ \boldsymbol{n}\neq\boldsymbol{0}}} \frac{(-1)^{\sum_\mu n_\mu}}{\sum_\mu \mu\, n_\mu} \prod_\mu \sum_q (1-\mu)^{n_\mu - q} \begin{bmatrix} d_\mu \\ q \end{bmatrix} \begin{bmatrix} D_\mu - d_\mu \\ n_\mu - q \end{bmatrix}$$

$$g(\boldsymbol{m}) = -\sum_{\substack{\{n_\mu\} \\ \boldsymbol{n}\neq\boldsymbol{0}}} \frac{(-1)^{\sum_\mu n_\mu}}{\sum_\mu \mu\, n_\mu} \prod_\mu \sum_q (1-\mu)^{n_\mu - q} \begin{bmatrix} d_\mu \\ q \end{bmatrix} \begin{bmatrix} D_\mu - d_\mu \\ n_\mu - q \end{bmatrix}$$

$$= -\sum_{\substack{\{n_\mu\} \\ \boldsymbol{n}\neq\boldsymbol{0}}} \frac{1}{\sum_\mu \mu\, n_\mu} \prod_\mu \sum_q -(1-\mu)^{n_\mu - q} \begin{bmatrix} d_\mu \\ q \end{bmatrix} \begin{bmatrix} D_\mu - d_\mu \\ n_\mu - q \end{bmatrix}$$

All the dependence of g on $\boldsymbol{m}$ comes from the subspace metrics via $d_\mu = d_\mu(\boldsymbol{m})$, so one can write $g = g(\boldsymbol{m}) = g(\{d_\mu\})$,

$$g(\boldsymbol{m}) = g(\{d_\mu\}) = -\sum_{\substack{\{n_\mu\} \\ \boldsymbol{n}\neq\boldsymbol{0}}} \frac{1}{\sum_\mu \mu\, n_\mu} \prod_\mu \varphi_{n_\mu}^{\mu;D_\mu}(d_\mu)$$

$$\varphi_n^{\mu;D_\mu}(r) \equiv -\sum_q (1-\mu)^{n-q} \begin{bmatrix} r \\ q \end{bmatrix} \begin{bmatrix} D_\mu - r \\ n - q \end{bmatrix}$$

General solution to state equation

A general solution can be constructed from a linear superposition of the truncated Green's functions centred on the microstates $\{\boldsymbol{m}\}$,

$$\rho(\boldsymbol{m}) = \lambda + \sum_{\boldsymbol{m}'} Q(\boldsymbol{m}')\, g(\{d_\mu(\boldsymbol{m}, \boldsymbol{m}')\})$$

$$d_\mu(\boldsymbol{m}, \boldsymbol{m}') \equiv \sum_{j\in\sigma_\mu} \overline{\delta_{m_j, m_j'}}$$

where λ is some constant.

Checking this is a solution,

$$\boldsymbol{\Delta}\, \rho(\boldsymbol{m}) = \sum_{\boldsymbol{m}'} Q(\boldsymbol{m}')\, \boldsymbol{\Delta}\, g(\{d_\mu(\boldsymbol{m}, \boldsymbol{m}')\}) = \sum_{\boldsymbol{m}'} Q(\boldsymbol{m}')\, V \left(\delta_{\boldsymbol{m},\boldsymbol{m}'} - \frac{1}{V}\right) = V\, Q(\boldsymbol{m}) - \sum_{\boldsymbol{m}'} Q(\boldsymbol{m}') = V\, Q(\boldsymbol{m})$$

where charge conservation ensures $\sum_{\boldsymbol{m}'} Q(\boldsymbol{m}') = 0$ in the final expression.

Relating the pseudo-charges to the sampled values

For a set of known sampled points, $\rho_{\boldsymbol{m}_n} = \rho(\boldsymbol{m}_n)$, where $1 \le n \le N$, un-sampled points are to have interpolated values (i.e. with vanishing pseudo charges). The pseudo-charge distribution then has the form,

$$Q(\boldsymbol{m}) \to \sum_{n=1}^{N} Q(\boldsymbol{m}_n)\ \delta_{\boldsymbol{m},\boldsymbol{m}_n}$$

Giving a corresponding solution to the state equation,

$$\rho(\boldsymbol{m}) = \lambda + \sum_{\boldsymbol{m}'} Q(\boldsymbol{m}')\ g(\{d_\mu(\boldsymbol{m}, \boldsymbol{m}')\}) = \lambda + \sum_{\boldsymbol{m}'} \left(\sum_{n=1}^{N} Q(\boldsymbol{m}_n)\ \delta_{\boldsymbol{m}',\boldsymbol{m}_n} \right) g(\{d_\mu(\boldsymbol{m}, \boldsymbol{m}')\})$$

$$= \lambda + \sum_{n=1}^{N} Q(\boldsymbol{m}_n) \sum_{\boldsymbol{m}'} \delta_{\boldsymbol{m}',\boldsymbol{m}_n}\ g(\{d_\mu(\boldsymbol{m}, \boldsymbol{m}')\}) = \lambda + \sum_{n=1}^{N} Q(\boldsymbol{m}_n)\ g(\{d_\mu(\boldsymbol{m}, \boldsymbol{m}_n)\})$$

So, the sample point values and associated pseudo charges are related by,

$$\rho(\boldsymbol{m}_{n'}) = \lambda + \sum_{n=1}^{N} Q(\boldsymbol{m}_n)\ g(\{d_\mu(\boldsymbol{m}_{n'}, \boldsymbol{m}_n)\})$$

Calculating the Pseudo-Charge Values from the Sampled Node Values

Written symbolically in a linear equation, now of N dimensions, the above relationship between the sampled values and pseudo-charges is,

$$\boldsymbol{r} = \lambda\,\boldsymbol{1} + \boldsymbol{\gamma}\,\boldsymbol{q}$$

where $r_n \equiv \rho(\boldsymbol{m}_n)$, $\gamma_{n'n} \equiv g(\{d_\mu(\boldsymbol{m}_{n'}, \boldsymbol{m}_n)\})$, $q_n \equiv Q(\boldsymbol{m}_n)$ and $\boldsymbol{1} \equiv (\{1\})$. In addition, there is the constraint that the total pseudo charge for the entire net is vanishing, i.e. $\sum_n q_n \equiv \boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{q} = 0$. Including this in the above constraint gives the linear vector equation,

$$\begin{pmatrix} \boldsymbol{r} \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\gamma} & \boldsymbol{1} \\ \boldsymbol{1}^{\boldsymbol{T}} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{q} \\ \lambda \end{pmatrix}$$

Inverting this linear equation then yields the pseudo-charges $\boldsymbol{q}$ and constant background value.

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