On the integrability of Liénard equations

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Abstract

A simple condition for integrability of Liénard equations is established to compute their general solution explicitly or by quadrature via a nonlocal transformation.

Introduction

It is well known that the interest to investigate analytical properties of Liénard equation

$$\ddot{x} + bf(x)\dot{x} - cg(x) = 0 \qquad (1)$$

where b and c are arbitrary constants, overdot denotes differentiation with respect to t, and $f(x) \neq 0,$ and $g(x) \neq 0,$ are arbitrary functions of $x,$ results from the fact that such an equation arises in many problems of applied mathematics and physics. For instance Liénard equations are encountered in determining traveling solitary wave solution to nonlinear partial differential equations [1-3]. The well known Duffing equation used to model oscillations in mechanical and electrical systems belongs to the class of Liénard equations (1). Although Liénard equation (1) has been the object of an intensive mathematical study in the literature, the problem of integrability in the sense that general solutions may be calculated explicitly or by quadrature has not been completely solved [3-7]. As illustration, it has not been possible for many decades to find the general solution to the well known Bratu differential equation and quintic Duffing oscillator equation [8, 9]. It is also well known that linearizing transformation like point transformation or nonlocal transformation is one of the most mathematical techniques used to secure integrability of nonlinear differential equations [3-7, 10, 11]. Unfortunately it is not known whether there exists a simple condition on $f(x)$ and $g(x)$ to secure the integrability of (1) via nonlocal transformations and of its reduced form where $b = 0$. This is then a drawback in the integrability analysis of nonlinear differential equations since no one is able to answer the question: Does exist a simple condition on $f(x)$ and $g(x)$ to secure the integrability of (1)? This work assumes that such a simple and fairly condition exists through a nonlocal transformation. To demonstrate, a nonlocal transformation [11] is used to establish the integrability criterion (section 2) and integrable equations are considered as illustrative examples (section 3). A conclusion is finally addressed for the developed work.

2. Integrability condition

This part is devoted to show the simple and fairly condition under which the Liénard equation (1) is integrable. To this end, consider the second order linear differential equation

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$$y''(\tau)+by'(\tau)=c \quad (2)$$

where prime means differentiation with respect to argument. The general solution to (2) may read

$$y(\tau)=c_1e^{-b\tau}+\frac{c}{b}\tau+c_2 \quad (3)$$

where c_1 and c_2 are constants of integration. Consider also the nonlocal transformation [10,11]

$$y(\tau)=\int g(x)dx, \qquad d\tau=f(x)dt \quad (4)$$

Now the following theorem may be formulated.

Theorem 1. Let

$$f(x)=g(x) \quad (5)$$

Then by application of nonlocal transformation (4), equation (1) may be reduced to equation (2).

Proof. The first derivative of x, by application of (4) may be written

$$\frac{dx}{dt}=\dot{x}=\frac{dx}{dy}\frac{dy}{d\tau}\frac{d\tau}{dt}=\frac{dy}{d\tau} \quad (6)$$

In this way, the second derivative of x may take the form

$$\frac{d^2x}{dt^2}=\ddot{x}=f(x)\frac{d^2y}{d\tau^2} \quad (7)$$

Substituting (6) and (7) into (1) yields

$$\frac{d^2y}{d\tau^2}f(x)+bf(x)\frac{dy}{d\tau}-cg(x)=0 \quad (8)$$

which may reduce to (2), for $f(x)=g(x)$.

Therefore the theorem is proved, and (5) becomes an integrability condition. Conversely applying (4) to (2) may lead to (1). In such a situation the general solution $x(t)$ to (1) may be computed exactly from (4) using (3) under the integrabilty condition (5).

Consider now the following theorem.

Theorem 2. *If $b=0$, then by application of (4), equation (1) may reduce to*

$$y''(\tau)=c \quad (9)$$

where the solution becomes

$$y(\tau) = \frac{1}{2}c\tau^2 + k_1\tau + k_2 \qquad (10)$$

where k_1 *and* k_2 *are constants of integration.*

Proof. Theorem 2 is a special case of the theorem 1. It suffices to set

$b = 0,$ into (8) to obtain (9) under $f(x) = g(x)$. To abtain (10), it suffices to integrate twice (9). Then the theorem is proved. In such a case the integrability condition (5) disappears and the simple and natural criterion $g(x) \neq 0,$ suffices to ensure the integrability of (1) when $b = 0,$ that is of

$$\ddot{x} - cg(x) = 0 \qquad (11)$$

Equation (11) is of high importance for mathematical physics since it denotes a class of conservative systems when t is the time.

3. Illustrative examples

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