Energy in Closed Systems

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Abstract

The writing indicates a breakdown of the classical laws . We consider conservation of energy with a many body system in relation to the inverse square law and point out the breakdown. The breakdown is first demonstrated with a two body system and finally with a many body system.

Introduction

. We consider conservation of energy[1] with a many body system in relation to the inverse square law and point out the breakdown. The breakdown is first demonstrated with a two body system and finally with a many body system.

Basic calculations

Infinitesimal work dw is given by,

$$dw = \sum_i dw_i = \sum_i \vec{F}_i d\vec{r}_i = \sum_i m_i \frac{d\vec{v}_i}{dt} d\vec{r}_i \quad (1)$$

dw:total infinitesimal work done on the system by external+internal forces, covering all the particles

$$dw = \sum_i m_i \frac{d\vec{v}_i}{dt}\frac{d\vec{r}_i}{dt} dt = \sum_i \frac{m_i \vec{v}_i d\vec{v}_i}{dt} dt$$

$$= \sum_i m_i \vec{v}_i d\vec{v}_i = \frac{1}{2}\sum_i m_i d(\vec{v}_i \vec{v}_i)$$

$$dw = \frac{1}{2}\sum_i m_i d{v_i}^2$$

$$dw = d\left(\sum_i \frac{1}{2} m_i {v_i}^2\right) \quad (2)$$

Let's now consider a particle in a conservative field:

$$\vec{E} = -\nabla U$$

$$dW = \vec{E}.d\vec{r} = -\nabla U.d\vec{r} = -dU$$

if the potential function is independent of time

If U depends on time then $dU = \frac{\partial U}{\partial t}dt + \nabla U.d\vec{r} \Rightarrow \nabla U.d\vec{r} = dU - \frac{\partial U}{\partial t}dt; \nabla U.d\vec{r} \neq dU$

We have for time independent potentials

$$dW = -dU$$

Again

$$dW = KE_f - KE_i$$

The last equation is the work energy theorem which is universally valid

$$KE_f - KE_i = -dU$$

$$KE_f - KE_i = U_i - U_f$$

$$KE_i + U_i = KE_f + U_f \quad (4)$$

The above is valid for a time independent conservative field. If the mass[magnitude of source] of one body is much larger than the other bodies involved then the above two conditions are satisfied to a good/high degree of approximation. The potentials of the smaller bodies in motion are ignorable. Thus time independence is achieved.

Inverse square Law, Gravitation

Potential energy in a three body gravitational system:

$$dW = \vec{F}_1 d\vec{r}_1 + \vec{F}_2 d\vec{r}_2 + \vec{F}_3 d\vec{r}_3$$

$$= (\vec{F}_{12} + \vec{F}_{13})d\vec{r}_1 + (\vec{F}_{21} + \vec{F}_{23})d\vec{r}_2 + (\vec{F}_{31} + \vec{F}_{32})d\vec{r}_3$$

where $\vec{F}_{ij}$ is the force on the ith particle from the $j\ th$ one due to gravity .

$dW = (\vec{F}_{12} + \vec{F}_{13})d\vec{r}_1 + (\vec{F}_{21} + \vec{F}_{23})d\vec{r}_2 + (\vec{F}_{31} + \vec{F}_{32})d\vec{r}_3$

$= (\vec{F}_{12}d\vec{r}_1 + \vec{F}_{21}d\vec{r}_2) + (\vec{F}_{13}d\vec{r}_1 + \vec{F}_{31}d\vec{r}_3) + (\vec{F}_{23}d\vec{r}_2 + \vec{F}_{32}d\vec{r}_3)$

$= (\vec{F}_{12}d\vec{r}_1 - \vec{F}_{12}d\vec{r}_2) + (\vec{F}_{13}d\vec{r}_1 - \vec{F}_{13}d\vec{r}_3) + (\vec{F}_{23}d\vec{r}_2 - \vec{F}_{23}d\vec{r}_3)$

$= \vec{F}_{12}(d\vec{r}_1 - d\vec{r}_2) + \vec{F}_{13}(d\vec{r}_1 - d\vec{r}_3) + \vec{F}_{23}(d\vec{r}_2 - d\vec{r}_3)$

$= \vec{F}_{12}d(\vec{r}_1 - \vec{r}_2) + \vec{F}_{13}d(\vec{r}_1 - \vec{r}_3) + \vec{F}_{23}d(\vec{r}_2 - \vec{r}_3)$

$= \vec{F}_{12}d\vec{r}_{12} + \vec{F}_{13}d\vec{r}_{13} + \vec{F}_{23}d\vec{r}_{23}; \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

$\mathrm{dw} = Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}$ (5)

$$= -Gm_1m_2\frac{1}{2{r_{12}}^3}d(\vec{r}_{12}.\vec{r}_{12}). -Gm_2m_3\frac{\vec{r}_{13}}{2{r_{13}}^3}d(\vec{r}_{13}.\vec{r}_{13}) - Gm_3m_1\frac{\vec{r}_{23}}{2{r_{23}}^3}d(\vec{r}_{23}.\vec{r}_{23})$$

$$dw = -Gm_1m_2\frac{1}{2{r_{12}}^3}d{r_{12}}^2. -Gm_2m_3\frac{1}{2{r_{13}}^3}d{r_{13}}^2 - Gm_3m_1\frac{1}{2{r_{23}}^3}d{r_{23}}^2$$

$$w = Gm_1m_2\frac{1}{r_{12}} + Gm_2m_3\frac{1}{r_{13}} + Gm_3m_1\frac{1}{r_{23}} + constant \quad (6)$$

'w' excludes external work[it is work done by the system] .It apllies to a closed three body gravitating system.

Comparing (2) and (5) we have for a closed system where external work is zero,

$$d\left(\sum_i \frac{1}{2}m_i{v_i}^2\right) = Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}$$

$$w = \sum_i \frac{1}{2}m_i{v_i}^2 + \mathrm{C} = Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23} + \mathrm{C}' \ (7)$$

If $\sum_i \frac{1}{2}m_i{v_i}^2$ increases $Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}$has to increase

$-\left(Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}\right)$ decreases

For a closed system ,we have

$$\sum_i \frac{1}{2}m_i{v_i}^2 - \left(Gm_1m_2\frac{1}{2r_{12}} + Gm_2m_3\frac{1}{2r_{13}} + Gm_3m_1\frac{1}{2r_{23}}\right) = constant \ (7)(8)$$

We recall

$$dw = -Gm_1m_2\frac{1}{2{r_{12}}^3}d({r_{12}}^2). -Gm_2m_3\frac{\vec{r}_{13}}{2{r_{13}}^3}d({r_{13}}^2) - Gm_3m_1\frac{\vec{r}_{23}}{2{r_{23}}^3}d({r_{23}}^2)$$

$$w = +Gm_1m_2\frac{1}{r_{12}} + C_1(r_{13}, r_{23}) + Gm_2m_3\frac{1}{r_{13}} + C_1(r_{23}, r_{12}) + Gm_3m_1\frac{1}{r_{23}} + C_1(r_{13}, r_{12})$$

$$w = 0 \Rightarrow = Gm_1m_2\frac{1}{r_{12}} + C_1(r_{13}, r_{23}) + Gm_2m_3\frac{1}{r_{13}} + C_2(r_{23}, r_{12}) + Gm_3m_1\frac{1}{r_{23}} + C_3(r_{13}, r_{12}) = 0$$

Differentiating the above with respect to time

$$Gm_1m_2\frac{1}{{r_{12}}^2}\frac{dr_{12}}{dt}+Gm_2m_3\frac{1}{{r_{23}}^2}\frac{dr_{23}}{dt}+Gm_1m_3\frac{1}{{r_{13}}^2}\frac{dr_{13}}{dt} = -\left(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\right)$$

$$Gm_1m_2\frac{r_{12}}{{r_{12}}^3}\frac{dr_{12}}{dt}+Gm_2m_3\frac{r_{23}}{{r_{23}}^3}\frac{dr_{23}}{dt}+Gm_1m_3\frac{r_{13}}{{r_{13}}^3}\frac{dr_{13}}{dt} = -\frac{d\left(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\right)}{dt}\neq 0$$

$$Gm_1m_2\frac{\vec{r}_{12}}{{r_{12}}^3}\frac{d\vec{r}_{12}}{dt}+Gm_2m_3\frac{\vec{r}_{23}}{{r_{23}}^3}\frac{d\vec{r}_{23}}{dt}+Gm_1m_3\frac{\vec{r}_{13}}{{r_{13}}^3}\frac{d\vec{r}_{13}}{dt} = -\frac{d\left(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\right)}{dt}\neq 0\ (11)$$

The functions $C_1, C_2 and\ C_3$ are arbitrary: we can fix them up according to our choice

The above is not true when dw=0. From(5) we expect the right side to be zero when dw=0. Therefore $C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})$ should be independent of time.

Therefore the constant in(7) is time independent.

$\sum_{i,j;i\neq j} Gm_im_j\frac{1}{r_{ij}} = constant$,independent of time. Else the right side (8) will be non zero when we require it to be zero[for w=0].

From the Lagrangian formulation

If the potential function is independent of the generalized velocities then we have the relation[2][3] $T+V=$constant; T: total kinetic energy; Vpotential function of the system of particles; we consider the potential function to be velocity independent.:

$$\frac{\partial V}{\partial x_{ij}} = -F_{ij}$$

i;particle index

j:component index[$j=x,y,z$]

$$\sum_{i,j;i\neq j} Gm_im_j\frac{1}{r_{ij}} = V\ \ (9)$$

satisfies

$$\frac{\partial V}{\partial x_{ij}} = -F_{ij}$$

$$\sum_i \frac{1}{2} m_i {v_i}^2 - \frac{1}{2}\sum_{i,j}{}' \frac{G m_i m_j}{r_{ij}} = constant\ (10)$$

Closed Systems

For a closed system where external work is zero we have

$$\sum_i \frac{1}{2} m_i {v_i}^2 + \mathrm{C} = G m_1 m_2 \frac{\vec{r}_{21}}{{r_{21}}^3} d\vec{r}_{12} + G m_2 m_3 \frac{\vec{r}_{31}}{{r_{31}}^3} d\vec{r}_{13} + G m_3 m_1 \frac{\vec{r}_{32}}{{r_{32}}^3} d\vec{r}_{23} + \mathrm{C}'$$

[The right side involves only internal forces: gravitation. Hence the work is internal work.

For an 'n' body system

$$\sum_i \frac{1}{2} m_i {v_i}^2 - \frac{1}{2}\sum_{i,j}{}' \frac{G m_i m_j}{r_{ij}} = constant\ (10)$$

We apply this idea two body system

$$\frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2 - \frac{\mathrm{G} m_1 m_2}{\mathrm{r}} = C \quad (11)$$

Two Body motion

Two gravitating masses m_A and m_B aqre being considered

$$\frac{d^2\vec{r}_A}{dt^2} = G m_B \frac{\vec{r}_{AB}}{{r_{AB}}^3}$$

$$\frac{d^2\vec{r}_B}{dt^2} = -G m_A \frac{\vec{r}_{AB}}{{r_{AB}}^3}$$

$$\frac{d^2\vec{r}_B}{dt^2} - \frac{d^2\vec{r}_A}{dt^2} = -G(m_A + m_B)\frac{\vec{r}_{AB}}{{r_{AB}}^3}$$

$$\frac{d^2\vec{r}_{AB}}{dt^2} = -G(m_A + m_B)\frac{\vec{r}_{AB}}{{r_{AB}}^3}\ (12)$$

$$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$$

$$\frac{d^2\vec{r}_{AB}}{dt^2} . \frac{d\vec{r}_{AB}}{dt} = -G(m_A + m_B)\frac{\vec{r}_{AB}}{{r_{AB}}^3} . \frac{d\vec{r}_{AB}}{dt}$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -\frac{1}{2}G(m_A+m_B)\frac{1}{{r_{AB}}^3}\frac{d(\vec{r}_{AB}.\vec{r}_{AB})}{dt}$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -\frac{1}{2}G(m_A+m_B)\frac{1}{{r_{AB}}^3}\frac{d({r_{AB}}^2)}{dt}$$

$$\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -G(m_A+m_B)\frac{2r_{AB}}{{r_{AB}}^3}\frac{dr_{AB}}{dt}$$

$$\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -G(m_A+m_B)\frac{2}{{r_{AB}}^2}\frac{dr_{AB}}{dt}$$

$$d\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -G(m_A+m_B)\frac{2}{{r_{AB}}^2}dr_{AB}$$

$$\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = \frac{2G(m_A+m_B)}{r_{AB}} + E \ (13)$$

E: constant of integration

$$\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 - \frac{2G(m_A+m_B)}{r_{AB}} = E[constant](14)$$

Now,

$$\frac{d\vec{r}_{AB}}{dt} = \frac{d(\vec{r}_B-\vec{r}_A)}{dt} = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} = \vec{v}_B - \vec{v}_A$$

Therefore,

$$\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = (\vec{v}_B - \vec{v}_A)^2$$

$$\Rightarrow (\vec{v}_B - \vec{v}_B)^2 - \frac{2G(m_A+m_B)}{r_{AB}} = E = E$$

$${v_A}^2 + {v_B}^2 - 2\vec{v}_A.\vec{v}_B - \frac{2G(m_A+m_B)}{r_{AB}} = E \ (15)$$

Momentum of $m_A = \vec{p}_A$;

Momentum of $m_B = \vec{p}_B$

For COM frame

$$\vec{p}_A = -\vec{p}_B = \vec{P}$$

$$|\vec{p}_A| = |\vec{p}_B| = P$$

$$m_A{}^2 v_A{}^2 = m_B{}^2 v_A{}^2 = P^2;$$

Equation (4) stands as

$$\frac{P^2}{m_A{}^2} + \frac{P^2}{m_B{}^2} - 2\frac{\vec{P}.\left(-\vec{P}\right)}{m_A m_B} + \frac{2K}{r_{AB}} = E$$

$$P^2\left[\frac{1}{m_A{}^2} + \frac{1}{m_B{}^2} + 2\frac{1}{m_A m_B}\right] - \frac{2G(m_A + m_B)}{r_{AB}} = E$$

$$\frac{P^2}{2}\left[\frac{1}{m_A} + \frac{1}{m_B}\right]^2 - \frac{2G(m_A + m_B)}{r_{AB}} = E\ (16)$$

We observe from (5),

$$\frac{1}{2}m_A v_A{}^2 + \frac{1}{2}m_B v_B{}^2 - \frac{G m_A m_B}{r_{AB}} = E'\ (17)$$

$$\frac{P^2}{2m_A} + \frac{P^2}{2m_B} - \frac{G m_A m_B}{r_{AB}} = E'$$

$$\frac{P^2}{2}\left[\frac{1}{m_A} + \frac{1}{m_B}\right] - \frac{G m_A m_B}{r_{AB}} = E'\ (18)$$

Diving equation by (16) we may solve for r_{AB} which becomes a constant

Many Body Motion

Similar breakdown may be observed for an n body system

$$m_1\frac{d^2\vec{r}_1}{dt^2} = \frac{Gm_1m_2}{r_{12}{}^3}\vec{r}_{12} + \frac{Gm_1m_3}{r_{13}{}^3}\vec{r}_{13} + \frac{Gm_1m_4}{r_{14}{}^3}\vec{r}_{14} \dots\dots\dots \frac{Gm_1m_{n-1}}{r_{1n-1}{}^3}\vec{r}_{1n-1}$$

$$m_2\frac{d^2\vec{r}_2}{dt^2} = \frac{Gm_2m_1}{r_{21}{}^3}\vec{r}_{21} + \frac{Gm_2m_3}{r_{23}{}^3}\vec{r}_{13} + \frac{Gm_2m_4}{r_{24}{}^3}\vec{r}_{14} \dots\dots\dots \frac{Gm_2m_{n-1}}{r_{2n-1}{}^3}\vec{r}_{2n-1}$$

where$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$

By subtrsaction,

$$\frac{d^2\vec{r}_2}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = \frac{Gm_2}{r_{21}{}^3}\vec{r}_{21} - \frac{Gm_1}{r_{12}{}^3}\vec{r}_{12} + \vec{R}$$

$$\frac{d^2\vec{r}_{12}}{dt^2} = -G\frac{m_1 + m_2}{{r_{12}}^3}\vec{r}_{12} + \vec{R} \quad (19)$$

Let

$$\vec{R} = G\frac{m_1 + m_2}{{r_{12}}^3}\vec{r}_{12} - G\frac{m_1 + m_2}{|\vec{r}_{12} - \vec{r}'|^3}(\vec{r}_{12} - \vec{r}') + \frac{d^2\vec{r}'}{dt^2} (20)$$

where $\vec{r}'$ is a suitable vector satisfying the above equation.

Now we have two body motion

$$\frac{d^2(\vec{r}_{12} - \vec{r}')}{dt^2} = -G\frac{m_1 + m_2}{|\vec{r}_{12} - \vec{r}'|^3}(\vec{r}_{12} - \vec{r}') \quad (21)$$

It is possible to locate an origin from where the pair (m_1, m_2) executes an equivalent two body motion We may write : $\vec{r}_{12} - \vec{r}' = \vec{r}'_{12} = \vec{r}_2{}' - \vec{r}_1{}'$

The equation now stands as

$$\frac{d^2\vec{r}_{12}{}'}{dt^2} = -G\frac{m_1 + m_2}{|\vec{r}_{12}|^3}\vec{r}'_{12} \; (22)$$

$$\frac{d^2(\vec{r}_2{}' - \vec{r}_1{}')}{dt^2} = -G\frac{m_1 + m_2}{|\vec{r}_2{}' - \vec{r}_1{}''|^3}(\vec{r}_2{}' - \vec{r}_1{}')$$

In a general manner the above equation and hence equation (22) may be obtained from the following two equations[by subtraction]

$$\frac{d^2\vec{r}_2{}'}{dt^2} = -G\frac{m_1}{|\vec{r}_{12}|^3}\vec{r}'_2 + \vec{f}$$

$$\frac{d^2\vec{r}_1{}'}{dt^2} = -G\frac{m_2}{|\vec{r}_{12}|^3}\vec{r}'_1 + \vec{f}$$

By applying an acceleration of $\vec{f}$on the origin we cancel out this acceleration. We nowdo have

$$\frac{d^2\vec{r}_2{}'}{dt^2} = -G\frac{m_1}{|\vec{r}_{12}|^3}\vec{r}'_2$$

$$\frac{d^2\vec{r}_1{}'}{dt^2} = -G\frac{m_2}{|\vec{r}_{12}|^3}\vec{r}'_1$$

We now have our inverse square law and consequently a pair of equations like (16) and (18)!

Physical Independence of forces

First we write the following equations [in an n body motion] for their partial interactions:

$$m_1 \frac{d^2\vec{r}_{12}}{dt^2} = Gm_1 m_2 \frac{\vec{r}_{12}}{{r_{12}}^3} \quad (23.1)$$

$$m_1 \frac{d^2\vec{r}_{13}}{dt^2} = Gm_1 m_2 \frac{\vec{r}_{13}}{{r_{13}}^3} \quad (23.2)$$

$$m_1 \frac{d^2\vec{r}_{14}}{dt^2} = Gm_1 m_2 \frac{\vec{r}_{14}}{{r_{14}}^3} \quad (23.3)$$

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$$m_1 \frac{d^2\vec{r}_{1n}}{dt^2} = Gm_1 m_2 \frac{\vec{r}_{1n}}{{r_{1n}}^3} \quad (23.n)$$

Each of the above equations expresses an independent two body motion which leads to equations of the type (16) and (18)!Each holds for every instant of time as if the partial intractions were proceeding independently.

We also have

$$\frac{d^2(\vec{r}_{12} + \vec{r}_{13} + \cdots . + \vec{r}_{1n})}{dt^2} = \frac{d^2\vec{r}_1}{dt^2}$$

[total acceleration= sum of partial accelerations]

$$\vec{r}_1 = \vec{r}_{12} + \vec{r}_{13} + \cdots . + \vec{r}_{1n} + Ct + D$$

We solve $\vec{r}_{12}, \vec{r}_{13}, \vec{r}_{14} \ldots \ldots . \vec{r}_{1n}$ from (23.1),(23.2)…..(23.n).Then we apply

$$\vec{r}_1 = \vec{r}_{12} + \vec{r}_{13} + \cdots . + \vec{r}_{1n} + Ct + D$$

to obtain the many body solution subject to the difficulty mentioned[equations [(16) and (18)]

For validation(cross checking) we may consider the following

$$\vec{r}_j = \vec{r}_{j2} + \vec{r}_{j3} + \cdots . + \vec{r}_{jn} + C_j t + D_j$$

$$\sum_j \vec{r}_j = t \sum_j C_j + \sum_j D_j$$

For Center of Mass frame

$$\sum_j m_j \vec{r}_j = 0$$

Conclusion

There is a clear indication of laws getting violated. Classical physics itself is a beacon to new

References

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