Energy in Closed Systems

Anamitra Palit

palit.anamitra@gmail.com

Abstract

The writing indicates a breakdown of the classical laws . We consider conservation of energy with a many body system in relation to the inverse square law and point out the breakdown. The breakdown is first demonstrated with a two body system and finally with a many body system.

Introduction

. We consider conservation of energy[1] with a many body system in relation to the inverse square law and point out the breakdown. The breakdown is first demonstrated with a two body system and finally with a many body system.

Basic calculations

Infinitesimal work dw is given by,

$$dw = \sum_i dw_i = \sum_i \vec{F}_i d\vec{r}_i = \sum_i m_i \frac{d\vec{v}_i}{dt} d\vec{r}_i \quad (1)$$

dw:total infinitesimal work done on the system by external+internal forces, covering all the particles

$$dw = \sum_i m_i \frac{d\vec{v}_i}{dt}\frac{d\vec{r}_i}{dt} dt = \sum_i \frac{m_i \vec{v}_i d\vec{v}_i}{dt} dt$$

$$= \sum_i m_i \vec{v}_i d\vec{v}_i = \frac{1}{2}\sum_i m_i d(\vec{v}_i \vec{v}_i)$$

$$dw = \frac{1}{2}\sum_i m_i d{v_i}^2$$

$$dw = d\left(\sum_i \frac{1}{2} m_i {v_i}^2\right) \quad (2)$$

Let's now consider a particle in a conservative field:

$$\vec{E} = -\nabla U$$

$$dW = \vec{E}.d\vec{r} = -\nabla U.d\vec{r} = -dU$$

if the potential function is independent of time

If U depends on time then $dU = \frac{\partial U}{\partial t}dt + \nabla U.d\vec{r} \Rightarrow \nabla U.d\vec{r} = dU - \frac{\partial U}{\partial t}dt; \nabla U.d\vec{r} \neq dU$

We have for time independent potentials

$$dW = -dU$$

Again

$$dW = KE_f - KE_i$$

The last equation is the work energy theorem which is universally valid

$$KE_f - KE_i = -dU$$

$$KE_f - KE_i = U_i - U_f$$

$$KE_i + U_i = KE_f + U_f \quad (4)$$

The above is valid for a time independent conservative field. If the mass[magnitude of source] of one body is much larger than the other bodies involved then the above two conditions are satisfied to a good/high degree of approximation. The potentials of the smaller bodies in motion are ignorable. Thus time independence is achieved.

Inverse square Law, Gravitation

Potential energy in a three body gravitational system:

$$dW = \vec{F}_1 d\vec{r}_1 + \vec{F}_2 d\vec{r}_2 + \vec{F}_3 d\vec{r}_3$$

$$= \left(\vec{F}_{12} + \vec{F}_{13}\right)d\vec{r}_1 + \left(\vec{F}_{21} + \vec{F}_{23}\right)d\vec{r}_2 + \left(\vec{F}_{31} + \vec{F}_{32}\right)d\vec{r}_3$$

where $\vec{F}_{ij}$ is the force on the ith particle from the $j\ th$ one due to gravity .

$dW = \left(\vec{F}_{12} + \vec{F}_{13}\right)d\vec{r}_1 + \left(\vec{F}_{21} + \vec{F}_{23}\right)d\vec{r}_2 + \left(\vec{F}_{31} + \vec{F}_{32}\right)d\vec{r}_3$

$= \left(\vec{F}_{12}d\vec{r}_1 + \vec{F}_{21}d\vec{r}_2\right) + \left(\vec{F}_{13}d\vec{r}_1 + \vec{F}_{31}d\vec{r}_3\right) + \left(\vec{F}_{23}d\vec{r}_2 + \vec{F}_{32}d\vec{r}_3\right)$

$= \left(\vec{F}_{12}d\vec{r}_1 - \vec{F}_{12}d\vec{r}_2\right) + \left(\vec{F}_{13}d\vec{r}_1 - \vec{F}_{13}d\vec{r}_3\right) + \left(\vec{F}_{23}d\vec{r}_2 - \vec{F}_{23}d\vec{r}_3\right)$

$= \vec{F}_{12}(d\vec{r}_1 - d\vec{r}_2) + \vec{F}_{13}(d\vec{r}_1 - d\vec{r}_3) + \vec{F}_{23}(d\vec{r}_2 - d\vec{r}_3)$

$= \vec{F}_{12}d(\vec{r}_1 - \vec{r}_2) + \vec{F}_{13}d(\vec{r}_1 - \vec{r}_3) + \vec{F}_{23}d(\vec{r}_2 - \vec{r}_3)$

$= \vec{F}_{12}d\vec{r}_{12} + \vec{F}_{13}d\vec{r}_{13} + \vec{F}_{23}d\vec{r}_{23}; \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

$\mathrm{dw} = Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}$ (5)

$$= -Gm_1m_2\frac{1}{2{r_{12}}^3}d(\vec{r}_{12}.\vec{r}_{12}).-Gm_2m_3\frac{\vec{r}_{13}}{2{r_{13}}^3}d(\vec{r}_{13}.\vec{r}_{13}) - Gm_3m_1\frac{\vec{r}_{23}}{2{r_{23}}^3}d(\vec{r}_{23}.\vec{r}_{23})$$

$$dw = -Gm_1m_2\frac{1}{2{r_{12}}^3}d{r_{12}}^2.-Gm_2m_3\frac{1}{2{r_{13}}^3}d{r_{13}}^2 - Gm_3m_1\frac{1}{2{r_{23}}^3}d{r_{23}}^2$$

$$w = Gm_1m_2\frac{1}{r_{12}} + Gm_2m_3\frac{1}{r_{13}} + Gm_3m_1\frac{1}{r_{23}} + constant \quad (6)$$

'w' excludes external work[it is work done by the system] .It apllies to a closed three body gravitating system.

Comparing (2) and (5) we have for a closed system where external work is zero,

$$d\left(\sum_i \frac{1}{2}m_i{v_i}^2\right) = Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}$$

$$w = \sum_i \frac{1}{2}m_i{v_i}^2 + \mathrm{C} = Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23} + \mathrm{C}' \ (7)$$

If $\sum_i \frac{1}{2}m_i{v_i}^2$ increases $Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}$has to increase

$-\left(Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}\right)$ decreases

For a closed system ,we have

$$\sum_i \frac{1}{2}m_i{v_i}^2 - \left(Gm_1m_2\frac{1}{2r_{12}} + Gm_2m_3\frac{1}{2r_{13}} + Gm_3m_1\frac{1}{2r_{23}}\right) = constant \ (7)(8)$$

We recall

$$dw = -Gm_1m_2\frac{1}{2{r_{12}}^3}d({r_{12}}^2).-Gm_2m_3\frac{\vec{r}_{13}}{2{r_{13}}^3}d({r_{13}}^2) - Gm_3m_1\frac{\vec{r}_{23}}{2{r_{23}}^3}d({r_{23}}^2)$$

$$w = +Gm_1m_2\frac{1}{r_{12}} + C_1(r_{13}, r_{23}) + Gm_2m_3\frac{1}{r_{13}} + C_1(r_{23}, r_{12}) + Gm_3m_1\frac{1}{r_{23}} + C_1(r_{13}, r_{12})$$

$$w = 0 \Rightarrow= Gm_1m_2\frac{1}{r_{12}} + C_1(r_{13}, r_{23}) + Gm_2m_3\frac{1}{r_{13}} + C_2(r_{23}, r_{12}) + Gm_3m_1\frac{1}{r_{23}} + C_3(r_{13}, r_{12}) = 0$$

Differentiating the above with respect to time

$$Gm_1m_2\frac{1}{{r_{12}}^2}\frac{dr_{12}}{dt}+Gm_2m_3\frac{1}{{r_{23}}^2}\frac{dr_{23}}{dt}+Gm_1m_3\frac{1}{{r_{13}}^2}\frac{dr_{13}}{dt}$$
$$=-\left(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\right)$$

$$Gm_1m_2\frac{r_{12}}{{r_{12}}^3}\frac{dr_{12}}{dt}+Gm_2m_3\frac{r_{23}}{{r_{23}}^3}\frac{dr_{23}}{dt}+Gm_1m_3\frac{r_{13}}{{r_{13}}^3}\frac{dr_{13}}{dt}$$
$$=-\frac{d\left(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\right)}{dt}\neq 0$$

$$Gm_1m_2\frac{\vec{r}_{12}}{{r_{12}}^3}\frac{d\vec{r}_{12}}{dt}+Gm_2m_3\frac{\vec{r}_{23}}{{r_{23}}^3}\frac{d\vec{r}_{23}}{dt}+Gm_1m_3\frac{\vec{r}_{13}}{{r_{13}}^3}\frac{d\vec{r}_{13}}{dt}$$
$$=-\frac{d\left(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\right)}{dt}\neq 0\ (11)$$

The functions $C_1, C_2 and\ C_3$ are arbitrary: we can fix them up according to our choice

The above is not true when dw=0. From(5) we expect the right side to be zero when dw=0. Therefore $C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})$ should be independent of time.

Therefore the constant in(7) is time independent.

$\sum_{i,j;i\neq j} Gm_im_j\frac{1}{r_{ij}}=constant$,independent of time. Else the right side (8) will be non zero when we require it to be zero[for w=0].

From the Lagrangian formulation

If the potential function is independent of the generalized velocities then we have the relation[2][3] $T+V=$constant; T: total kinetic energy; Vpotential function of the system of particles; we consider the potential function to be velocity independent.:

$$\frac{\partial V}{\partial x_{ij}}=-F_{ij}$$

i;particle index

j:component index[$j=x,y,z$]

$$\sum_{i,j;i\neq j} Gm_im_j\frac{1}{r_{ij}}=V\quad(9)$$

satisfies

$$\frac{\partial V}{\partial x_{ij}} = -F_{ij}$$

$$\sum_i \frac{1}{2} m_i {v_i}^2 - \frac{1}{2} \sum_{i,j} \frac{'G m_i m_j}{r_{ij}} = constant\ (10)$$

Closed Systems

For a closed system where external work is zero we have

$$\sum_i \frac{1}{2} m_i {v_i}^2 + \mathrm{C} = G m_1 m_2 \frac{\vec{r}_{21}}{{r_{21}}^3} d\vec{r}_{12} + G m_2 m_3 \frac{\vec{r}_{31}}{{r_{31}}^3} d\vec{r}_{13} + G m_3 m_1 \frac{\vec{r}_{32}}{{r_{32}}^3} d\vec{r}_{23} + \mathrm{C}'$$

[The right side involves only internal forces: gravitation. Hence the work is internal work.

For an ‘n’ body system

$$\sum_i \frac{1}{2} m_i {v_i}^2 - \frac{1}{2} \sum_{i,j} \frac{'G m_i m_j}{r_{ij}} = constant\ (10)$$

We apply this idea two body system

$$\frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2 - \frac{\mathrm{G} m_1 m_2}{\mathrm{r}} = C \quad (11)$$

Two Body motion

Two gravitating masses m_A and m_B aqre being considered

$$\frac{d^2 \vec{r}_A}{dt^2} = G m_B \frac{\vec{r}_{AB}}{{r_{AB}}^3}$$

$$\frac{d^2 \vec{r}_B}{dt^2} = -G m_A \frac{\vec{r}_{AB}}{{r_{AB}}^3}$$

$$\frac{d^2 \vec{r}_B}{dt^2} - \frac{d^2 \vec{r}_A}{dt^2} = -G(m_A + m_B) \frac{\vec{r}_{AB}}{{r_{AB}}^3}$$

$$\frac{d^2 \vec{r}_{AB}}{dt^2} = -G(m_A + m_B) \frac{\vec{r}_{AB}}{{r_{AB}}^3}\ (12)$$

$$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$$

$$\frac{d^2 \vec{r}_{AB}}{dt^2} . \frac{d\vec{r}_{AB}}{dt} = -G(m_A + m_B) \frac{\vec{r}_{AB}}{{r_{AB}}^3} . \frac{d\vec{r}_{AB}}{dt}$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -\frac{1}{2}G(m_A + m_B)\frac{1}{{r_{AB}}^3}\frac{d(\vec{r}_{AB}.\vec{r}_{AB})}{dt}$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -\frac{1}{2}G(m_A + m_B)\frac{1}{{r_{AB}}^3}\frac{d({r_{AB}}^2)}{dt}$$

$$\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -G(m_A + m_B)\frac{2r_{AB}}{{r_{AB}}^3}\frac{dr_{AB}}{dt}$$

$$\frac{d}{dt}\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -G(m_A + m_B)\frac{2}{{r_{AB}}^2}\frac{dr_{AB}}{dt}$$

$$d\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = -G(m_A + m_B)\frac{2}{{r_{AB}}^2}dr_{AB}$$

$$\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = \frac{2G(m_A + m_B)}{r_{AB}} + E \ (13)$$

E: constant of integration

$$\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 - \frac{2G(m_A + m_B)}{r_{AB}} = E[constant](14)$$

Now,

$$\frac{d\vec{r}_{AB}}{dt} = \frac{d(\vec{r}_B - \vec{r}_A)}{dt} = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} = \vec{v}_B - \vec{v}_A$$

Therefore,

$$\left(\frac{d\vec{r}_{AB}}{dt}\right)^2 = (\vec{v}_B - \vec{v}_A)^2$$

$$\Rightarrow (\vec{v}_B - \vec{v}_B)^2 - \frac{2G(m_A + m_B)}{r_{AB}} = E = E$$

$${v_A}^2 + {v_B}^2 - 2\vec{v}_A.\vec{v}_B - \frac{2G(m_A + m_B)}{r_{AB}} = E \ (15)$$

Momentum of $m_A = \vec{p}_A$;

Momentum of $m_B = \vec{p}_B$

For COM frame

$$\vec{p}_A = -\vec{p}_B = \vec{P}$$

$$|\vec{p}_A| = |\vec{p}_B| = P$$

$${m_A}^2{v_A}^2 = {m_B}^2{v_A}^2 = P^2;$$

Equation (4) stands as

$$\frac{P^2}{{m_A}^2} + \frac{P^2}{{m_B}^2} - 2\frac{\vec{P}.(-\vec{P})}{m_A m_B} + \frac{2K}{r_{AB}} = E$$

$$P^2\left[\frac{1}{{m_A}^2} + \frac{1}{{m_B}^2} + 2\frac{1}{m_A m_B}\right] - \frac{2G(m_A + m_B)}{r_{AB}} = E$$

$$\frac{P^2}{2}\left[\frac{1}{m_A} + \frac{1}{m_B}\right]^2 - \frac{2G(m_A + m_B)}{r_{AB}} = E \ (16)$$

We observe from (5),

$$\frac{1}{2}m_A{v_A}^2 + \frac{1}{2}m_B{v_B}^2 - \frac{Gm_A m_B}{r_{AB}} = E' \ (17)$$

$$\frac{P^2}{2m_A} + \frac{P^2}{2m_B} - \frac{Gm_A m_B}{r_{AB}} = E'$$

$$\frac{P^2}{2}\left[\frac{1}{m_A} + \frac{1}{m_B}\right] - \frac{Gm_A m_B}{r_{AB}} = E' \ (18)$$

Diving equation by (16) we may solve for r_{AB} which becomes a constant

Many Body Motion

Similar breakdown may be observed for an n body system

$$m_1\frac{d^2\vec{r}_1}{dt^2} = \frac{Gm_1m_2}{{r_{12}}^3}\vec{r}_{12} + \frac{Gm_1m_3}{{r_{13}}^3}\vec{r}_{13} + \frac{Gm_1m_4}{{r_{14}}^3}\vec{r}_{14} \dots\dots\dots \frac{Gm_1m_{n-1}}{{r_{1n-1}}^3}\vec{r}_{1n-1}$$

$$m_2\frac{d^2\vec{r}_2}{dt^2} = \frac{Gm_2m_1}{{r_{21}}^3}\vec{r}_{21} + \frac{Gm_2m_3}{{r_{23}}^3}\vec{r}_{13} + \frac{Gm_2m_4}{{r_{24}}^3}\vec{r}_{14} \dots\dots\dots \frac{Gm_2m_{n-1}}{{r_{2n-1}}^3}\vec{r}_{2n-1}$$

where$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$

By subtrsaction,

$$\frac{d^2\vec{r}_2}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = \frac{Gm_2}{{r_{21}}^3}\vec{r}_{21} - \frac{Gm_1}{{r_{12}}^3}\vec{r}_{12} + \vec{R}$$

$$\frac{d^2\vec{r}_{12}}{dt^2} = -G\frac{m_1+m_2}{{r_{12}}^3}\vec{r}_{12} + \vec{R} \ \ (19)$$

Let

$$\vec{R} = G\frac{m_1+m_2}{{r_{12}}^3}\vec{r}_{12} - G\frac{m_1+m_2}{|\vec{r}_{12}-\vec{r}'|^3}(\vec{r}_{12}-\vec{r}') + \frac{d^2\vec{r}'}{dt^2}(20)$$

where $\vec{r}'$ is a suitable vector satisfying the above equation.

Now we have two body motion

$$\frac{d^2(\vec{r}_{12}-\vec{r}')}{dt^2} = -G\frac{m_1+m_2}{|\vec{r}_{12}-\vec{r}'|^3}(\vec{r}_{12}-\vec{r}') \ \ (21)$$

It is possible to locate an origin from where the pair (m_1, m_2) executes an equivalent two body motion We may write : $\vec{r}_{12}-\vec{r}' = \vec{r}'_{12} = \vec{r}_2{}' - \vec{r}_1{}'$

The equation now stands as

$$\frac{d^2\vec{r}_{12}{}'}{dt^2} = -G\frac{m_1+m_2}{|\vec{r}_{12}|^3}\vec{r}'_{12} \ (22)$$

$$\frac{d^2(\vec{r}_2{}'-\vec{r}_1{}')}{dt^2} = -G\frac{m_1+m_2}{|\vec{r}_2{}'-\vec{r}_1{}''|^3}(\vec{r}_2{}'-\vec{r}_1{}')$$

In a general manner the above equation and hence equation (22) may be obtained from the following two equations[by subtraction]

$$\frac{d^2\vec{r}_2{}'}{dt^2} = -G\frac{m_1}{|\vec{r}_{12}|^3}\vec{r}'_2 + \vec{f}$$

$$\frac{d^2\vec{r}_1{}'}{dt^2} = -G\frac{m_2}{|\vec{r}_{12}|^3}\vec{r}'_1 + \vec{f}$$

By applying an acceleration of $\vec{f}$on the origin we cancel out this acceleration. We nowdo have

$$\frac{d^2\vec{r}_2{}'}{dt^2} = -G\frac{m_1}{|\vec{r}_{12}|^3}\vec{r}'_2$$

$$\frac{d^2\vec{r}_1{}'}{dt^2} = -G\frac{m_2}{|\vec{r}_{12}|^3}\vec{r}'_1$$

We now have our inverse square law and consequently a pair of equations like (16) and (18)!

Conclusion

There is a clear indication of laws getting violated. Classical physics itself is a beacon to new

References

Goldstein H., Poole C, Safko J, Classical Mechanics, Third edition, Addison Wesley,p61-62
Goldstein H., Poole C, Safko J, Classical Mechanics, Third edition, Addison Wesley,p61-62
Goldstein H., Poole C, Safko J, Classical Mechanics, Third edition, Addison Wesley,p 21