# Energy in Closed Systems

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**Abstract**

Is there any form of energy that develops surreptitiously/mystically from within a closed system that does not exchange energy with its environment? The article investigates such an issue.

Introduction

What is the possibility of energy[1] developing from within a closed system that does not exchange energy? Our aim here is to explore such an issue.

**Basic calculations**

In a closed system of interacting particles where there is no exchange of energy with the external environment, $dw = 0$ . We have,

$$dw = \sum_i dw_i = \sum_i \vec{F}_i d\vec{r}_i = \sum_i m_i \frac{d\vec{v}_i}{dt} d\vec{r}_i \quad (1)$$

$dw$:total infinitesimal work done on the system by external+internal forces, covering all the particles

$$dw = \sum_i m_i \frac{d\vec{v}_i}{dt}\frac{d\vec{r}_i}{dt} dt = \sum_i \frac{m_i \vec{v}_i d\vec{v}_i}{dt} dt$$

$$= \sum_i m_i \vec{v}_i d\vec{v}_i = \frac{1}{2}\sum_i m_i d(\vec{v}_i \vec{v}_i)$$

$$dw = \frac{1}{2}\sum_i m_i d{v_i}^2$$

$$dw = d\left(\sum_i \frac{1}{2} m_i {v_i}^2\right) \quad (2)$$

For an isolated system[energy is not flowing into the system or flowing out of it]external work is zero; internal work is also expectedly zero. Else the energy of the closed system would change by itself.

We rewrite (1)only for internal forces[external force on each particle being zero for a closed system]

$$dw = \sum_i dw_i = \sum_i \vec{F}_i d\vec{r}_i = \sum_i m_i \frac{d\vec{v}_i}{dt} d\vec{r}_i$$

$$dw = 0 \Rightarrow \sum_i \frac{1}{2} m_i {v_i}^2 = constant \quad (3)$$

The above implies

$$\frac{d\left(\sum_i \frac{1}{2} m_i {v_i}^2\right)}{dt} dt = 0 \Rightarrow \frac{d\left(\sum_i \frac{1}{2} m_i {v_i}^2\right)}{dt} = 0$$

Therefore the constant is time independent of time and similarly of other variables.

Let’s now consider a particle in a conservative field:

$$\vec{E} = -\nabla U$$

$$dW = \vec{E}.d\vec{r} = -\nabla U.d\vec{r} = -dU$$

if the potential function is independent of time

If U depends on time then $dU = \frac{\partial U}{\partial t} dt + \nabla U.d\vec{r} \Rightarrow \nabla U.d\vec{r} = dU - \frac{\partial U}{\partial t} dt; \nabla U.d\vec{r} \neq dU$

We have for time independent potentials

$$dW = -dU$$

Again

$$dW = KE_f - KE_i$$

The last equation is the work energy theorem which is universally valid

$$KE_f - KE_i = -dU$$

$$KE_f - KE_i = U_i - U_f$$

$$KE_i + U_i = KE_f + U_f \quad (4)$$

The above is valid for a <u>time independent</u> conservative field. If the mass[magnitude of source] of one body is much larger than the other bodies involved then the above two conditions are satisfied to a good/high degree of approximation. The potentials of the smaller bodies in motion are ignorable. Thus time independence is achieved.

**Inverse square Law, Gravitation**

Potential energy in a three body gravitational system:

$$dW = \vec{F}_1 d\vec{r}_1 + \vec{F}_2 d\vec{r}_2 + \vec{F}_3 d\vec{r}_3$$

$$= \left(\vec{F}_{12} + \vec{F}_{13}\right) d\vec{r}_1 + \left(\vec{F}_{21} + \vec{F}_{23}\right) d\vec{r}_2 + \left(\vec{F}_{31} + \vec{F}_{32}\right) d\vec{r}_3$$

where $\vec{F}_{ij}$ is the force on the $ith$ particle from the $j\ th$ one due to gravity .

$dW = (\vec{F}_{12} + \vec{F}_{13})d\vec{r}_1 + (\vec{F}_{21} + \vec{F}_{23})d\vec{r}_2 + (\vec{F}_{31} + \vec{F}_{32})d\vec{r}_3$

$= (\vec{F}_{12}d\vec{r}_1 + \vec{F}_{21}d\vec{r}_2) + (\vec{F}_{13}d\vec{r}_1 + \vec{F}_{31}d\vec{r}_3) + (\vec{F}_{23}d\vec{r}_2 + \vec{F}_{32}d\vec{r}_3)$

$= (\vec{F}_{12}d\vec{r}_1 - \vec{F}_{12}d\vec{r}_2) + (\vec{F}_{13}d\vec{r}_1 - \vec{F}_{13}d\vec{r}_3) + (\vec{F}_{23}d\vec{r}_2 - \vec{F}_{23}d\vec{r}_3)$

$= \vec{F}_{12}(d\vec{r}_1 - d\vec{r}_2) + \vec{F}_{13}(d\vec{r}_1 - d\vec{r}_3) + \vec{F}_{23}(d\vec{r}_2 - d\vec{r}_3)$

$= \vec{F}_{12}d(\vec{r}_1 - \vec{r}_2) + \vec{F}_{13}d(\vec{r}_1 - \vec{r}_3) + \vec{F}_{23}d(\vec{r}_2 - \vec{r}_3)$

$= \vec{F}_{12}d\vec{r}_{12} + \vec{F}_{13}d\vec{r}_{13} + \vec{F}_{23}d\vec{r}_{23}; \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

$\mathrm{dw} = Gm_1m_2\frac{\vec{r}_{21}}{{r_{21}}^3}d\vec{r}_{12} + Gm_2m_3\frac{\vec{r}_{31}}{{r_{31}}^3}d\vec{r}_{13} + Gm_3m_1\frac{\vec{r}_{32}}{{r_{32}}^3}d\vec{r}_{23}$ (5)

$= -Gm_1m_2\frac{1}{2{r_{12}}^3}d(\vec{r}_{12}.\vec{r}_{12}). -Gm_2m_3\frac{\vec{r}_{13}}{2{r_{13}}^3}d(\vec{r}_{13}.\vec{r}_{13}) - Gm_3m_1\frac{\vec{r}_{23}}{2{r_{23}}^3}d(\vec{r}_{23}.\vec{r}_{23})$

$dw = -Gm_1m_2\frac{1}{2{r_{12}}^3}d{r_{12}}^2. -Gm_2m_3\frac{1}{2{r_{13}}^3}d{r_{13}}^2 - Gm_3m_1\frac{1}{2{r_{23}}^3}d{r_{23}}^2$

$w = Gm_1m_2\frac{1}{r_{12}} + Gm_2m_3\frac{1}{r_{13}} + Gm_3m_1\frac{1}{r_{23}} + constant$ (6)

'w' excludes external work[it is work done by the system]

For a closed system

$$\sum_i \frac{1}{2}m_i{v_i}^2 = constant$$

$$Gm_1m_2\frac{1}{r_{12}} + Gm_2m_3\frac{1}{r_{13}} + Gm_3m_1\frac{1}{r_{23}} = constant$$

Therefore

$$\sum_i \frac{1}{2}m_i{v_i}^2 - \left(Gm_1m_2\frac{1}{2r_{12}} + Gm_2m_3\frac{1}{2r_{13}} + Gm_3m_1\frac{1}{2r_{23}}\right) = constant \ (7)$$

*The two terms on the left side have to be individually constant if the system is closed[considering gravity to be the only internal force in operation]. We should not think of any exchange between the kinetic and the potential parts contrary to what we do in theory and what we observe.*

If w=0

$$Gm_1m_2\frac{1}{r_{12}}+Gm_2m_3\frac{1}{r_{13}}+Gm_3m_1\frac{1}{r_{23}}=constant\ (8)$$

for a 'n' body interaction[closed system] with $dw=0$ we have

$$\sum_{i,j;i\neq j}Gm_im_j\frac{1}{r_{ij}}=constant\ (9)$$

the above constant is time independent.

We recall

$$dw=-Gm_1m_2\frac{1}{2{r_{12}}^3}d({r_{12}}^2).-Gm_2m_3\frac{\vec{r}_{13}}{2{r_{13}}^3}d({r_{13}}^2)-Gm_3m_1\frac{\vec{r}_{23}}{2{r_{23}}^3}d({r_{23}}^2)$$

$$w=+Gm_1m_2\frac{1}{r_{12}}+C_1(r_{13},r_{23})+Gm_2m_3\frac{1}{r_{13}}+C_1(r_{23},r_{12})+Gm_3m_1\frac{1}{r_{23}}+C_1(r_{13},r_{12})$$

$$w=0\Rightarrow=Gm_1m_2\frac{1}{r_{12}}+C_1(r_{13},r_{23})+Gm_2m_3\frac{1}{r_{13}}+C_2(r_{23},r_{12})+Gm_3m_1\frac{1}{r_{23}}+C_3(r_{13},r_{12})=0$$

Differentiating the above with respect to time

$$Gm_1m_2\frac{1}{{r_{12}}^2}\frac{dr_{12}}{dt}+Gm_2m_3\frac{1}{{r_{23}}^2}\frac{dr_{23}}{dt}+Gm_1m_3\frac{1}{{r_{13}}^2}\frac{dr_{13}}{dt}=-\big(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\big)$$

$$Gm_1m_2\frac{r_{12}}{{r_{12}}^3}\frac{dr_{12}}{dt}+Gm_2m_3\frac{r_{23}}{{r_{23}}^3}\frac{dr_{23}}{dt}+Gm_1m_3\frac{r_{13}}{{r_{13}}^3}\frac{dr_{13}}{dt}=-\frac{d\big(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\big)}{dt}\neq 0$$

$$Gm_1m_2\frac{\vec{r}_{12}}{{r_{12}}^3}\frac{d\vec{r}_{12}}{dt}+Gm_2m_3\frac{\vec{r}_{23}}{{r_{23}}^3}\frac{d\vec{r}_{23}}{dt}+Gm_1m_3\frac{\vec{r}_{13}}{{r_{13}}^3}\frac{d\vec{r}_{13}}{dt}=-\frac{d\big(C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})\big)}{dt}\neq 0\ (10)$$

The functions $C_1, C_2 and\ C_3$ are arbitrary: we can fix them up according to our choice

The above is not true when dw=0. From(5) we expect the right side to be zero when dw=0. Therefore $C_1(r_{13},r_{23})+C_2(r_{23},r_{12})+C_3(r_{13},r_{12})$ should be independent of time.

Therefore the constant in(7) is time independent.

$\sum_{i,j;i\neq j} Gm_i m_j \frac{1}{r_{ij}} = constant$,independent of time. Else the right side (8) will be non zero when we require it to be zero[for w=0].

## From the Lagrangian formulation

If the potential function is independent of the generalized velocities then we have the relation[2][3] $T + V$ =constant; $T$: total kinetic energy; $V$potential function of the system of particles; we consider the potential function to be velocity independent.:

$$\frac{\partial V}{\partial x_{ij}} = -F_{ij}$$

$i$;particle index

$j$:component index[$j = x, y, z$]

$$\sum_{i,j;i\neq j} Gm_i m_j \frac{1}{r_{ij}} = V \quad (9)$$

satisfies

$$\frac{\partial V}{\partial x_{ij}} = -F_{ij}$$

$$\sum_i \frac{1}{2} m_i {v_i}^2 - \frac{1}{2}\sum_{i,j} \frac{{}' G m_i m_j}{r_{ij}} = constant \; (10)$$

$\sum_i \frac{1}{2} m_i {v_i}^2$ and $\sum_{i,j} Gm_i m_j \frac{1}{r_{ij}}$ are not constant independently. Energy flows in and out of the system in so far as the Lagrangian formulation is concerned even if we envisage it as a closed one .

## Conclusion

If dw=0 breaks down internally in a closed system, we have to believe in a surreptitious/mystic source of energy. Else we have to do away with the notion of a closed system.

References

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