On the Ricci Scalar, Ricci Tensor and the Riemannian Curvature Tensor

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Abstract

The article in the first two sections proves decisively that the Ricci scalar and the norm of the Ricci tensor are constants on the manifold. In the subsequent sections Ricci tensor and Riemannian curvature tensor turn out to be null tensors. The Ricci scalar works out to zero.

Introduction

The Ricci scalar[1] and the norm of the Ricci tensor[2] are not only invariants but they are constants on a given manifold, independent of the space time coordinates. This idea is established in the initial stages of the article. Subsequent calculations show that the Ricci tensor and Riemannian curvature tensor are the null tensors. Consequently the Ricci scalar works out to zero.

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Part I

Ricci Scalar

First we write the Field equations[3][4]:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \ (1)$$

$R_{\alpha\beta}$:Ricci Tensor ;R Ricci Scalar ;

$g_{\alpha\beta}$:metric coefficients; $T_{\alpha\beta}$:stress energy tensor

$$g^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} g^{\alpha\beta}$$

$$R - \frac{1}{2} R \times 4 = \frac{8\pi G}{c^4} T_{\alpha\beta} g^{\alpha\beta}$$

$$-R = \frac{8\pi G}{c^4} T_{\alpha\beta} g^{\alpha\beta} \ (2)$$

Equation (2) expresses a standard result

We differentiate each side of equation (2)with respect to $x^\gamma; \gamma = 0,1,2,3$

$$\frac{\partial R}{\partial x^\gamma} = -\frac{8\pi G}{c^4}\frac{\partial}{\partial x^\gamma}\left(T_{\alpha\beta}g^{\alpha\beta}\right)$$

We use the formula

$$\frac{\partial f}{\partial x^i} \equiv \nabla_i f$$

$$\nabla_\beta R = -\frac{8\pi G}{c^4}\nabla_\beta\left(g_{\alpha\beta}T^{\alpha\beta}\right) = -\frac{8\pi G}{c^4}\mathrm{T}^{\alpha\beta}\nabla_\beta g_{\alpha\beta} + g_{\alpha\beta}\nabla_\beta \mathrm{T}^{\alpha\beta}$$

$$\nabla_\beta R = 0; \beta = 0,1,2,3$$

$$-\frac{8\pi G}{c^4}\nabla_\beta\left(g_{\alpha\beta}T^{\alpha\beta}\right) = \frac{\partial R}{\partial x^i} = 0 \quad (3)$$

R is independent of space time coordinates.

<u>Detailed Explanation</u>

Covariant derivative of a scalar is equivalent to its partial derivative

We prove

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = A_{\alpha\beta}\nabla_\gamma B^{\alpha\beta} + B_{\alpha\beta}\nabla_\gamma A_{\alpha\beta} \quad (4)$$

Proof:

We consider the following relations

$$\nabla_\gamma A^{\alpha\beta} = A^{\alpha\beta};_\gamma = \frac{\partial A^{\alpha\beta}}{\partial x^\gamma} + \Gamma_{\gamma s}{}^{\alpha}A^{s\beta} + \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}$$

$$\nabla_\gamma B_{\alpha\beta} = B_{\alpha\beta};_\gamma = \frac{\partial B_{\alpha\beta}}{\partial x^\gamma} + \Gamma^s{}_{\gamma\alpha}B_{s\beta} + \Gamma^s{}_{\gamma\beta}B_{\alpha s}$$

[The above relations do not assume $A^{\alpha\beta}$ and $B_{\alpha\beta}$ as symmetric tensors]

We obtain,

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha}A^{s\beta} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^s{}_{\gamma\alpha}B_{s\beta} + \Gamma^s{}_{\gamma\beta}B_{\alpha s}\right)$$

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\alpha\beta}A^{\alpha\beta}\right)=B_{\alpha\beta}\left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}-\Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right)+A^{\alpha\beta}\left(\Gamma^{s}{}_{\gamma\alpha}B_{s\beta}+\Gamma^{s}{}_{\gamma\beta}B_{\alpha s}\right)+A^{\alpha\beta}\nabla_{\gamma}B_{\alpha\beta}+B^{\alpha\beta}\nabla_{\gamma}A_{\alpha\beta}$$

$$=-\Gamma_{\gamma s}{}^{\alpha}g^{s\beta}B_{\alpha\beta}-\Gamma_{\gamma s}{}^{\beta}g^{\alpha s}B_{\alpha\beta}+\Gamma^{s}{}_{\gamma\alpha}A^{\alpha\beta}T_{s\beta}+\Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{s\alpha}+A^{\alpha\beta}\nabla_{\gamma}B_{\alpha\beta}+B^{\alpha\beta}\nabla_{\gamma}A_{\alpha\beta}$$

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\alpha\beta}A^{\alpha\beta}\right)=\left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}B_{\alpha\beta}+\Gamma^{s}{}_{\gamma\alpha}A^{\alpha\beta}B_{s\beta}\right)+\left(\Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{\alpha s}-\Gamma_{\gamma s}{}^{\beta}A^{\alpha s}B_{\alpha\beta}\right)+A^{\alpha\beta}\nabla_{\gamma}B_{\alpha\beta}+B^{\alpha\beta}\nabla_{\gamma}A_{\alpha\beta}\quad(5)$$

[In the above α, s, β are dummy indices]

We work out the two parentheses separately.

With the second term in the first parenthesis to the right we interchange as follows

$$\alpha \leftrightarrow s$$

$$\left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}B_{\alpha\beta}+\Gamma^{s}{}_{\gamma\alpha}A^{\alpha\beta}B_{s\beta}\right)=\left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}T_{\alpha\beta}+\Gamma^{\alpha}{}_{\gamma s}A^{s\beta}B_{\alpha\beta}\right)=0$$

We do not have to worry about reflections on the left side of (5)because alpha and beta on the left side also disappear on contraction.

Indeed recalling (5) and using the relation:$B_{\alpha\beta}A^{\alpha\beta}=B_{\mu\nu}A^{\mu\nu}$we may rewrite it [equation (5)] in the following form :

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\mu\nu}A^{\mu\nu}\right)=B_{\alpha\beta}\left(\nabla_{\gamma}A^{\alpha\beta}-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}-\Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right)+A^{\alpha\beta}\left(\nabla_{\gamma}B_{\alpha\beta}+\Gamma^{s}{}_{\gamma\alpha}B_{s\beta}+\Gamma^{s}{}_{\gamma\beta}B_{\alpha s}\right)+A^{\alpha\beta}\nabla_{\gamma}B_{\alpha\beta}+B^{\alpha\beta}\nabla_{\gamma}A_{\alpha\beta}$$

There is no α, β on the left side of the above.

With the second term in the second parenthesis

$$\beta \leftrightarrow s$$

$$\left(\Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{\alpha s}-\Gamma_{\gamma s}{}^{\beta}A^{\alpha s}B_{\alpha\beta}\right)=\left(\Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{\alpha s}-\Gamma_{\gamma\beta}{}^{s}A^{\alpha\beta}B_{\alpha s}\right)=0$$

$$\frac{\partial}{\partial x^{\gamma}}\left(B_{\alpha\beta}A^{\alpha\beta}\right)=A^{\alpha\beta}\nabla_{\gamma}B_{\alpha\beta}+B^{\alpha\beta}\nabla_{\gamma}A_{\alpha\beta}\quad(6)$$

From(3) R is independent of space and time coordinates. Incidentally with the Einstein Hilbert Action[5] we do not treat the Ricci scalar as a constant: it depends on the space-time coordinates through the metric coefficients[implicit dependence].

Ricci Tensor

Recalling equation (1)

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\Rightarrow\ R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} R R^{\alpha\beta} g_{\alpha\beta} = \frac{8\pi G}{c^4} R^{\alpha\beta} T_{\alpha\beta}$$

$$\Rightarrow R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} R^2 = \frac{8\pi G}{c^4} R^{\alpha\beta} T_{\alpha\beta}$$

$$\Rightarrow\ R^{\alpha\beta} R_{\alpha\beta} = \frac{8\pi G}{c^4} R^{\alpha\beta} T_{\alpha\beta} + \frac{1}{2} R^2$$

$$\frac{\partial}{\partial x^\beta}(R^{\alpha\beta} R_{\alpha\beta}) = \frac{8\pi G}{c^4} \frac{\partial}{\partial x^\beta}\left(R^{\alpha\beta} T_{\alpha\beta}\right) + \frac{1}{2} \frac{\partial}{\partial x^\beta}(R^2)$$

R being a constant

$$\frac{\partial}{\partial x^\beta}(R^2) = 0$$

Therefore

$$\frac{\partial}{\partial x^\beta}(R^{\alpha\beta} R_{\alpha\beta}) = \frac{8\pi G}{c^4} \frac{\partial}{\partial x^\beta}\left(R^{\alpha\beta} T_{\alpha\beta}\right)$$

$$\Rightarrow \nabla_\beta\left(R^{\alpha\beta} T_{\alpha\beta}\right) = \frac{8\pi G}{c^4}\left(R^{\alpha\beta} \nabla_\beta T_{\alpha\beta} + T^{\alpha\beta} \nabla_\beta R_{\alpha\beta}\right)$$

$$\Rightarrow \frac{\partial}{\partial x^i}\left(R^{\alpha\beta} T_{\alpha\beta}\right) = \frac{8\pi G}{c^4} R^{\alpha\beta} \nabla_\beta T_{\alpha\beta} + \frac{8\pi G}{c^4} T^{\alpha\beta} \nabla_\beta R_{\alpha\beta}$$

$$\frac{\partial}{\partial x^i}\left(R^{\alpha\beta} T_{\alpha\beta}\right) = \frac{8\pi G}{c^4} T^{\alpha\beta} \nabla_\beta R_{\alpha\beta}\ (7)$$

Applying covariant differentiation on (1)and remembering that R is a constant and that $\nabla_\beta g^{\alpha\beta} = 0$

$\nabla_\beta T_{\alpha\beta} = 0$, we have by differentiating the field equations ,

$$\nabla_\beta R_{\alpha\beta} - \frac{1}{2} R \nabla_\beta g_{\alpha\beta} = \frac{8\pi G}{c^4} \nabla_\beta T_{\alpha\beta}$$

$$\Rightarrow \nabla_\beta R_{\alpha\beta} = 0$$

Therefore from(7) we have,

$$\frac{\partial}{\partial x^\beta}(R^{\alpha\beta} R_{\alpha\beta}) = \frac{8\pi G}{c^4} \frac{\partial}{\partial x^\beta}\left(R^{\alpha\beta} T_{\alpha\beta}\right) = 0$$

$$R^{\alpha\beta}R_{\alpha\beta} = R^{\mu\nu}R_{\mu\nu}$$

$$\frac{\partial}{\partial x^{\beta}}(R^{\mu\nu}R_{\mu\nu}) = 0$$

$$R^{\mu\nu}R_{\mu\nu} = C, \text{constant} \quad (8)$$

Norm of the Ricci tensor is a constant on the manifold, independent of space and time coordinates..

Riemannian Curvature Tensor

We start with the formula for the Riemannian Curvature Tensor

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}} - \frac{\partial^2 g_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}} + \frac{\partial^2 g_{\beta\gamma}}{\partial x^{\alpha}\partial x^{\delta}}\right) \quad (9)$$

In the orthogonal coordinates $R_{\alpha\beta\gamma\delta} = 0$ for four distinct indices α, β, γ and δ;also $R_{\alpha\alpha\gamma\delta} = R_{\alpha\beta\gamma\gamma} = 0$ for all frames of reference, orthogonal or non orthogonal. If any three indices are equal or if all four equal, then $R_{\alpha\beta\gamma\delta} = 0$

Let us now check whether $R_{\alpha\beta\alpha\delta}$, $R_{\alpha\beta\gamma\alpha}$ and $R_{\alpha\beta\beta\alpha}$ are non zero or zero . We transform to some other orthogonal frame of reference. The zeros occur once again [components] every time we transform to some other arbitrary orthogonal reference frame. All the components have to mix in order to produce the zeros. Most important, the transformation rules will also change as we select different frames(orthogonal) of reference.

If a single component—any component---is zero in all frames of reference the tensor will be a null tensor. This happens because the transformation elements $\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}$ become arbitrary covering an infinitude of orthogonal reference frames.

Some zeros occurring in all frames is thus impossible unless

$$R_{\alpha\beta\alpha\delta} = 0; R_{\alpha\beta\gamma\alpha} = 0; R_{\alpha\beta\beta\alpha} = 0 \quad (10)$$

If the Riemannian Tensor is null in one frame of reference it will be null in all other frames of reference.

Alternatively if we analyze in terms of the general coordinate systems[orthogonal or non orthogonal]we shall use $R_{\alpha\alpha\gamma\delta} = R_{\alpha\beta\gamma\gamma} = 0$ for same components in all reference frames [and not $R_{\alpha\beta\gamma\delta} = R_{\alpha\gamma\beta\delta} = 0$]Possibly $R_{\alpha\beta\alpha\delta}$, $R_{\alpha\beta\gamma\alpha}$, $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\gamma\beta\delta}$ and $R_{\alpha\beta\beta\alpha}$ are non zero

The zeros will occur [components] every time we transform to some other arbitrary reference frame. All the components have to mix in order to produce the zeros in the same positions. The transformation elements will also change as we select different frames of reference.

Zeros occurring in all reference is impossible unless $R_{\alpha\beta\alpha\delta}$, $R_{\alpha\beta\gamma\alpha}$, $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\gamma\beta\delta}$ and $R_{\alpha\beta\beta\alpha}$ are each zero.

The Riemannian tensor being zero, the Ricci tensor is also a null tensor and the Ricci scalar stands zero.

That the Ricci tensor is zero has been proved by an alternative method towards the beginning of the section.

Dot Product Preserving Transport

In parallel transport [6] dot product is preserved. We consider here a transport where dot product is preserved but the two vectors individually are not transported parallel to themselves

We have due to the preservation of dot product,

$$t^i \nabla_i \left(g_{\alpha\beta} u^\alpha v^\beta \right) = 0 \; (11)$$

We have

$$t^i \nabla_i u^\alpha \neq 0; t^i \nabla_i v^\beta \neq 0 \;\; (12)$$

since each vector is not transported parallel to itself.

We transform to a frame of reference where t^i has only one non zero component.

$t^{k\prime} \nabla_{k\prime} \left(g_{\alpha\beta}{}' u^{\alpha\prime} v^{\beta\prime} \right) = 0$[no summation on k': prime denotes the new frame of reference and not differentiation]

$$\nabla_k' \left(g_{\alpha\beta}{}' u^{\alpha\prime} v^{\beta\prime} \right) = 0 (13)$$

$$u^{\alpha\prime} v^{\beta\prime} \nabla_{i\prime} \left(g_{\alpha\beta}{}' \right) + g_{\alpha\beta}{}' \nabla_{i\prime} \left(u^{\alpha\prime} v^{\beta\prime} \right) = 0$$

Since $\nabla_i \left(g_{\alpha\beta} \right) = 0$,we have,

$$g_{\alpha\beta}{}' \nabla_i \left(u^{\alpha\prime} v^{\beta\prime} \right) = 0 \;\; (14)$$

The vectors $u^{\alpha\prime}$and $v^{\beta\,\prime}$ and consequently their individual components are arbitrary. Therefore

$$g_{\alpha\beta}' = 0 \Rightarrow g_{\alpha\beta} = 0 \;\; (15)$$

[the null tensor remains null in all frames of reference]That implies that the Riemann tensor , Ricci tensor and the Ricci scalar are all zero valued objects.

Part II

We start with the standard relation given by equation (2)

$$\frac{8\pi G}{c^4} T_{\alpha\beta} g^{\alpha\beta} = -R = \text{Constant}$$

$$T_{\alpha\beta} g^{\alpha\beta} = C \quad (16)$$

[To take note of the fact that $T_{\alpha\beta}$ is a symmetric tensor]

Manifold Independent Properties

$g_{\mu\nu}$ and $g^{\mu\nu}$are not independent:

$$g^{\mu\nu} = \frac{Cofactor\ of\ g_{\mu\nu}}{g}$$

$g^{\mu\nu}$is fully dependent on the set $\{g_{\mu\nu}\}$

$$\sum_{\nu} g_{\mu\nu} g^{\mu\nu} = \frac{\sum_{\nu} g_{\mu\nu} Cofactor\ of\ g_{\mu\nu}}{g} = 1$$

$$\sum_{\mu} \sum_{\nu} g_{\mu\nu} g^{\mu\nu} = 4$$

The above holds for all manifolds

Equation (16)

$$g_{\mu\nu} T^{\mu\nu} = K[\text{constant}]$$

It has been derived consistently from the field equations which hold across all manifolds

Equation(16) holds for all manifolds since it has been derived from the Field Equations

Solution to(16)

$$T^{\mu\nu} = \frac{K}{4} g^{\mu\nu} + \chi^{\mu\nu} \quad (17)$$

such that

$$g_{\mu\nu} \chi^{\mu\nu} = 0 \quad (18)$$

Field Equations

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Using (17) with the field Equations

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \frac{8\pi G}{c^4} \left(\frac{K}{4} g^{\mu\nu} + \chi^{\mu\nu} \right) \quad (19)$$

$$\nabla_\nu\left(R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}\right)=\frac{8\pi G}{c^4}\left(\frac{K}{4}\nabla_\nu g^{\mu\nu}+\nabla_\nu\chi^{\mu\nu}\right)$$

$$\Rightarrow \nabla_\nu\chi^{\mu\nu}=0 \;\; (20)$$

for all manifolds

$$\Rightarrow \frac{\partial\chi^{\mu\nu}}{\partial x^\nu}+\Gamma_{\nu s}{}^{\mu}\chi^{s\nu}+\Gamma_{\nu s}{}^{\nu}\chi^{\mu s}=0$$

The above has to hold across all manifolds that is for different sets of Christoffel symbols and metric tensors produced by different manifolds. The metric coefficients and hence the Christoffel symbols would be represented by arbitrary functions. That is impossible unless

$\chi^{\mu\nu}=0$ or $\chi^{\mu\nu}=g^{\mu\nu}\Rightarrow\chi^{\mu}{}_{\nu}=Cg^{\mu}{}_{\nu}=C\delta^{\mu}{}_{\nu}$ (21)

[For tensors equations the metric coefficients and correspondingly the Christoffel symbols change in a particular manner without any change of the line element, as we pass from one coordinate system to another on the same manifold. The same cannot be asserted for manifold changes]

If $\nabla_\nu\chi^{\mu\nu}=0$ held for all manifolds the field equations would have been modified to

$$R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}+k\chi^{\mu\nu}=\frac{8\pi G}{c^4}T^{\mu\nu}$$

Either side would have become identical on taking covariant derivative

$$\nabla_\nu\left(R^{\mu\nu}-\frac{1}{2}Rg^{\mu\nu}+k\chi^{\mu\nu}\right)=0$$

$$\nabla_i\left(\frac{8\pi G}{c^4}T^{\mu\nu}\right)=0$$

By contrast $\nabla_i g^{\mu\nu}=0$ is a standard result[7] that holds for all manifolds

$$\nabla_i g_{\mu\nu}=\frac{\partial g_{\mu\nu}}{\partial x^i}-\Gamma_{i\mu}{}^{s}g_{s\nu}-\Gamma_{i\nu}{}^{s}g_{\mu s}$$

$$=\frac{\partial g_{\mu\nu}}{\partial x^i}-\frac{1}{2}g^{sp}\left(\frac{\partial g_{pi}}{\partial x^\mu}+\frac{\partial g_{\mu p}}{\partial x^i}-\frac{\partial g_{i\mu}}{\partial x^p}\right)g_{s\nu}-\frac{1}{2}g^{sp}\left(\frac{\partial g_{pi}}{\partial x^\nu}+\frac{\partial g_{\nu p}}{\partial x^i}-\frac{\partial g_{i\nu}}{\partial x^p}\right)g_{\mu s}$$

$$=\frac{\partial g_{\mu\nu}}{\partial x^i}-\frac{1}{2}g^{sp}g_{s\nu}\left(\frac{\partial g_{pi}}{\partial x^\mu}+\frac{\partial g_{\mu p}}{\partial x^i}-\frac{\partial g_{i\mu}}{\partial x^p}\right)-\frac{1}{2}g^{sp}g_{\mu s}\left(\frac{\partial g_{pi}}{\partial x^\nu}+\frac{\partial g_{\nu p}}{\partial x^i}-\frac{\partial g_{i\nu}}{\partial x^p}\right)$$

$$=\frac{\partial g_{\mu\nu}}{\partial x^i}-\frac{1}{2}\delta_{\nu}{}^{p}\left(\frac{\partial g_{pi}}{\partial x^\mu}+\frac{\partial g_{\mu p}}{\partial x^i}-\frac{\partial g_{i\mu}}{\partial x^p}\right)-\frac{1}{2}\delta_{\mu}{}^{p}\left(\frac{\partial g_{pi}}{\partial x^\nu}+\frac{\partial g_{\nu p}}{\partial x^i}-\frac{\partial g_{i\nu}}{\partial x^p}\right)$$

$$= \frac{\partial g_{\mu\nu}}{\partial x^i} - \frac{1}{2}\left(\frac{\partial g_{\nu i}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^i} - \frac{\partial g_{i\mu}}{\partial x^\nu}\right) - \frac{1}{2}\left(\frac{\partial g_{\mu i}}{\partial x^\nu} + \frac{\partial g_{\nu\mu}}{\partial x^i} - \frac{\partial g_{i\nu}}{\partial x^\mu}\right) = 0$$

$$\nabla_i g_{\mu\nu} = 0 \ (22)$$

Equation (22) holds does not matter what be the manifold. Same is true with the Bianchi identity in that it applies to all manifolds.

So long as we are on the same manifold, the line element is preserved. This is not true for distinct manifolds

Example: A room with a flat floor and a hemispherical roof is considered. A small arc is drawn on the roof and its projection is taken on the floor. With this transformation

$$ds'^2 \neq ds^2$$

Only if

$$ds'^2 = ds^2$$

then $g_{\mu\nu}$behaves as a rank two tensor. Indeed

$$ds'^2 = ds^2$$

$$\Rightarrow \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = g_{\alpha\beta} dx^\alpha dx^\beta$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} d\bar{x}^\mu \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\nu$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\mu d\bar{x}^\nu$$

$$\Rightarrow \bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

Therefore

$$T^{\mu\nu} = \frac{K}{4} g^{\mu\nu} \ \ (23)$$

But $T_{\alpha\beta} = \frac{K}{4} g_{\alpha\beta}$ makes the stress energy tensor trivial

We now consider the Field Equations given by (1)

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = \frac{8\pi G}{c^4}\frac{K}{4}g^{\alpha\beta}$$

$$R^{\alpha\beta} = k'^{g^{\alpha\beta}}(24)$$

Again from the Field equations

$$\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta}$$

$$R_{\alpha\beta}R^{\alpha\beta} - 2\frac{1}{2}Rg_{\alpha\beta}R^{\alpha\beta} + \frac{1}{4}R^2 g^{\alpha\beta}g_{\alpha\beta} = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta}$$

$$R_{\alpha\beta}R^{\alpha\beta} - R^2 + R^2 = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta}$$

$$R_{\alpha\beta}R^{\alpha\beta} = \left(\frac{8\pi G}{c^4}\right)^2 T^{\alpha\beta}T_{\alpha\beta} \quad (25)$$

From relation (8)

$$R_{\alpha\beta}R^{\alpha\beta} = \text{Constant} \quad (26)$$

$$\Rightarrow T^{\alpha\beta}T_{\alpha\beta} = \text{constant[from (8) and (25)]} \quad (27)$$

Now the metric tensor is a diagonal tensor: off diagonal elements are all zero. They are zero for the in all reference frames. Therefore $g_{\alpha\beta} = 0$ for all components diagonal or off diagonal.

$$\bar{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\mu}g_{\alpha\beta} \quad (28)$$

All components $g_{\alpha\beta}$ mix to produce each $\bar{g}_{\mu\nu}$. If $\bar{g}_{\mu\nu} = 0$ for any component in all frames of reference all $g_{\alpha\beta}$ have to be zero because the transformation elements, $\frac{\partial x^\alpha}{\partial \bar{x}^\mu}$,are arbitrary to the extent the transformation is non singular.

Part III

[Alternative Treatment: Brute Force Calculations]

From (27)we have,

$$T^{\alpha\beta}\nabla_i T_{\alpha\beta} + T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 0$$

$$T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 0; i = 0,1,2,3 \ (29)$$

Indeed

$$T^{\alpha\beta}\nabla_i T_{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} T_{\mu\nu} \nabla_i\left(g_{\alpha k} g_{\beta l} T^{lk}\right)$$

$$= g^{\alpha\mu} g^{\beta\nu} T_{\mu\nu}\left[g_{\alpha k} g_{\beta l} \nabla_i\left(T^{lk}\right) + T^{lk}\nabla_i\left(g_{\alpha k} g_{\beta l}\right)\right]$$

$$= g^{\alpha\mu} g^{\beta\nu} g_{\alpha k} g_{\beta l} T_{\mu\nu} \nabla_i\left(T^{lk}\right)$$

$$= \delta_k{}^{\mu} \delta_l{}^{\nu} \mathrm{T}_{\mu\nu} \nabla_i\left(T^{lk}\right)$$

$$= \mathrm{T}_{\mu\nu} \nabla_i T^{\mu\nu} = \mathrm{T}_{\alpha\beta} \nabla_i T^{\alpha\beta}$$

Therefore

$$T_{\alpha\beta} \nabla_i T^{\alpha\beta} = 0 \ (30)$$

From (8)

$$R_{\alpha\beta} R^{\alpha\beta} = constant$$

We have,

$$R_{\alpha\beta}\nabla_i R^{\alpha\beta} + R^{\alpha\beta}\nabla_i R_{\alpha\beta} = 0$$

Now

$$R^{\alpha\beta}\nabla_i R_{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} R_{\mu\nu} \nabla_i\left(g_{\alpha k} g_{\beta l} R^{lk}\right)$$

$$= g^{\alpha\mu} g^{\beta\nu} R_{\mu\nu}\left[g_{\alpha k} g_{\beta l} \nabla_i\left(R^{lk}\right) + R^{lk}\nabla_i\left(g_{\alpha k} g_{\beta l}\right)\right]$$

$$= g^{\alpha\mu} g^{\beta\nu} g_{\alpha k} g_{\beta l} R_{\mu\nu} \nabla_i\left(R^{lk}\right)$$

$$= \delta_k{}^{\mu} \delta_l{}^{\nu} \mathrm{R}_{\mu\nu} \nabla_i\left(R^{lk}\right)$$

$$= \mathrm{R}_{\mu\nu} \nabla_i R^{\mu\nu} = \mathrm{R}_{\alpha\beta} \nabla_i R^{\alpha\beta}$$

Therefore

$$R_{\alpha\beta} \nabla_i R^{\alpha\beta} = 0 \qquad (31)$$

From (16)

$$T_{\alpha\beta} g^{\alpha\beta} = C$$

$$\Rightarrow g_{\mu\nu} \frac{\partial T^{\mu\nu}}{\partial x^i} + \frac{\partial g_{\mu\nu}}{\partial x^i} T^{\mu\nu} = 0 \qquad (32)$$

From(30)

$$T_{\alpha\beta} \nabla_i T^{\alpha\beta} = 0$$

$$\Rightarrow T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+T_{\alpha\beta}g^{\alpha s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)T^{j\beta}+T_{\alpha\beta}g^{\beta s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)T^{\alpha j}=0$$

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+T_{\alpha\beta}T^{j\beta}g^{\alpha s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)+T_{\alpha\beta}T^{\alpha j}g^{\beta s}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)=0$$

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+T^s{}_\beta T^{j\beta}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)+T^s{}_\beta T^{j\beta}\frac{1}{2}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)=0$$

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)=0\ (33)$$

Now considering the fact that dummy indices can always be interchanged without affecting the value of an expression we have,

$$T^j{}_\beta T^{s\beta}\left(\frac{\partial g_{ji}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^i}-\frac{\partial g_{is}}{\partial x^j}\right)=T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{js}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)\ (34)$$

We have

$$0=2T_{\alpha\beta}\nabla_i T^{\alpha\beta}=2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+2T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{sj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)$$

Applying (34) on the above,

$$2T_{\alpha\beta}\nabla_i T^{\alpha\beta}=2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+T^s{}_\beta T^{j\beta}\left(\frac{\partial g_{si}}{\partial x^j}+\frac{\partial g_{js}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^s}\right)+T^j{}_\beta T^{s\beta}\left(\frac{\partial g_{ji}}{\partial x^s}+\frac{\partial g_{js}}{\partial x^i}-\frac{\partial g_{is}}{\partial x^j}\right)\ (35)$$

$$2T_{\alpha\beta}\nabla_i T^{\alpha\beta}=2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i}+\left(T^j{}_\beta T^{s\beta}-T^s{}_\beta T^{j\beta}\right)\frac{\partial g_{ji}}{\partial x^s}+\left(T^j{}_\beta T^{s\beta}+T^s{}_\beta T^{j\beta}\right)\frac{\partial g_{js}}{\partial x^i}$$
$$+\left(T^s{}_\beta T^{j\beta}-T^j{}_\beta T^{s\beta}\right)\frac{\partial g_{is}}{\partial x^j}\ \ (36)$$

Now,

$$T^j{}_\beta T^{s\beta}=g_{\beta k}T^{jk}T^{s\beta}$$

$$T^s{}_\beta T^{j\beta}=g_{\beta k}T^{sk}T^{j\beta}=g_{k\beta}T^{s\beta}T^{jk}=T^j{}_\beta T^{s\beta}$$

Therefore

$$T_\beta{}^j T^{s\beta}-T_\beta{}^s T^{j\beta}=0\ \ \ (37)$$

An alternative technique for deriving (37) would be as follows. By the interchange of dummy indices j w and s we may assert that

$$\left(T^j{}_\beta T^{s\beta} - T^s{}_\beta T^{j\beta}\right)\frac{\partial g_{ji}}{\partial x^s} = \left(T^j{}_\beta T^{s\beta} + T^s{}_\beta T^{j\beta}\right)\frac{\partial g_{js}}{\partial x^i}$$

$$\left(T^j{}_\beta T^{s\beta} - T^s{}_\beta T^{j\beta}\right)\left(\frac{\partial g_{ji}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i}\right) = 0$$

$\left(T^j{}_\beta T^{s\beta} - T^s{}_\beta T^{j\beta}\right)$is a tensor while $\left(\frac{\partial g_{ji}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i}\right)$ is not a tensor and their product is not a tensor. The product may not be zero in all frames of reference unless we have (37): $T^j{}_\beta T^{s\beta} - T^s{}_\beta T^{j\beta}$. By quotient law[8], $\left(\frac{\partial g_{ji}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i}\right)$ should be a tensor unless $\left(T^j{}_\beta T^{s\beta} - T^s{}_\beta T^{j\beta}\right) = 0$The only solution is $T_{\alpha\beta} - k g_{\alpha\beta} = 0$..If $\left(\frac{\partial g_{ji}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i}\right) = 0$ in all reference frames it becomes the null tensor. We are considering it to be not so from its transformation.

The alternative technique has been discussed since it will be used in the final stages to bring out important results.

From (36) and (37) we have

$$2T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + \left(T_\beta{}^j T^{s\beta} + T_\beta{}^s T^{j\beta}\right)\frac{\partial g_{js}}{\partial x^i} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + 2T_\beta{}^j T^{s\beta}\frac{\partial g_{js}}{\partial x^i}$$

$$2T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 2T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + 2T_\beta{}^j T^{s\beta}\frac{\partial g_{js}}{\partial x^i} \quad (38)$$

Using $T_{\alpha\beta}\nabla_i T^{\alpha\beta} = 0$ and (38)we have

$$T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + T_\beta{}^j T^{s\beta}\frac{\partial g_{js}}{\partial x^i} = 0$$

$$\Rightarrow T_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + T_k{}^l T^{mk}\frac{\partial g_{lm}}{\partial x^i} = 0 \quad (39)$$

From (32)

$$g_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial g_{lm}}{\partial x^i}T^{lm} = 0$$

$$g_{\alpha\beta}\frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial g_{lm}}{\partial x^i}g_k{}^l T^{mk} = 0 \quad (40)$$

From(38) and(39)

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}\left(T_{\alpha\beta} - k g_{\alpha\beta}\right) + \frac{\partial g_{lm}}{\partial x^i}T^{mk}\left(T_k{}^l - k g_k{}^l\right) = 0$$

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}\left(T_{\alpha\beta} - kg_{\alpha\beta}\right) + \frac{\partial g_{lm}}{\partial x^i} T^{mk} g^{lp}\left(T_{kp} - kg_{kp}\right) = 0$$

$$\frac{\partial T^{\alpha\beta}}{\partial x^i}\left(T_{\alpha\beta} - kg_{\alpha\beta}\right) + \frac{\partial g_{lm}}{\partial x^i} T^{m\alpha} g^{l\beta}\left(T_{\alpha\beta} - kg_{\alpha\beta}\right) = 0$$

$$\left(T_{\alpha\beta} - kg_{\alpha\beta}\right)\left(\frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial g_{lm}}{\partial x^i} T^{m\alpha} g^{l\beta}\right) = 0 \quad (41)$$

.Now the second factor on the left of (36) may be written as:

$$\frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial g_{lm}}{\partial x^i} T^{m\alpha} g^{l\beta} = \frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial (T^{m\alpha} g_{lm})}{\partial x^i} g^{l\beta} - g_{lm} g^{l\beta} \frac{\partial T^{m\alpha}}{\partial x^i}$$

$$= \frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial (T^{m\alpha} g_{lm})}{\partial x^i} g^{l\beta} - \delta_m{}^\beta \frac{\partial T^{m\alpha}}{\partial x^i}$$

$$= \frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial (T^{m\alpha} g_{lm})}{\partial x^i} g^{l\beta} - \frac{\partial T^{\alpha\beta}}{\partial x^i}$$

$$= \frac{\partial T_l{}^a}{\partial x^i} g^{l\beta}$$

Therefore

$$\frac{\partial T^{\alpha\beta}}{\partial x^i} + \frac{\partial g_{lm}}{\partial x^i} T^{m\alpha} g^{l\beta} = \frac{\partial T_l{}^a}{\partial x^i} g^{l\beta}$$

. The above is not a tensor [derivative of a tenor is not a tensor in the curved space time context]

$$\left(T_{\alpha\beta} - kg_{\alpha\beta}\right)\frac{\partial T_l{}^a}{\partial x^i} g^{l\beta} = 0$$

$\left(T_{\alpha\beta} - kg_{\alpha\beta}\right)$ is a rank two covariant tensor.

$$\left(T_{\alpha\beta} - kg_{\alpha\beta}\right)\frac{\partial T_l{}^a}{\partial x^i} g^{l\beta} = 0 \; (42)$$

$$\left(T_\alpha{}^l - k\delta_\alpha{}^l\right)\frac{\partial T_l{}^a}{\partial x^i} = 0 \quad (43)$$

$\left(T_\alpha{}^l - k\delta_\alpha{}^l\right)$is a rank two mixed tensor.By quotient law $\frac{\partial T_l{}^a}{\partial x^i}$ should be a tensor unless$T_\alpha{}^l - k\delta_\alpha{}^l = 0$.

Therefore the left side of (17) is not a tensor. It may not transform to zero in all frames of reference unless

$$T_\alpha{}^l - k\delta_\alpha{}^l = 0$$

$$g_{l\beta}\left(T_{\alpha}{}^{l} - kg_{\alpha}{}^{l}\right) = 0$$

$$T_{\alpha\beta} = kg_{\alpha\beta}\ (44)$$

It is important to take note of the fact that equation (43) is derived from tensor equations. Hence it preserves form in all reference frames though it is not expected to do so considering the fact that it is not a tensor equation. By division rule $\frac{\partial T_{l}{}^{\alpha}}{\partial x^{i}}$ should be a tensor unless $T_{\alpha\beta} - kg_{\alpha\beta} = 0$The only solution is $T_{\alpha\beta} - kg_{\alpha\beta} = 0$..

Similarly by using (31`) and (32) we may prove

$$R_{\alpha\beta} = k' g_{\alpha\beta}\ \ (45)$$

but we have seen in part I that $g_{\alpha\beta} = 0$. The same result was derived n the manuscript [Dot Product Preserving Transport]

We recall(43)

$$\left(T_{\alpha}{}^{l} - k\delta_{\alpha}{}^{l}\right)\frac{\partial T_{l}{}^{\alpha}}{\partial x^{i}} = 0$$

If $l = \alpha$

$$(T_{\alpha}{}^{\alpha} - k)\frac{\partial T_{\alpha}{}^{\alpha}}{\partial x^{i}} = 0$$

$$T_{\alpha}{}^{\alpha}\frac{\partial T_{\alpha}{}^{\alpha}}{\partial x^{i}} - k\frac{\partial T_{\alpha}{}^{\alpha}}{\partial x^{i}} = 0$$

$$T_{\alpha}{}^{\alpha}dT_{\alpha}{}^{\alpha} - kdT_{\alpha}{}^{\alpha} = 0 \Rightarrow \frac{1}{2}dT_{\alpha}{}^{\alpha 2} - kdT_{\alpha}{}^{\alpha} = 0$$

$$\Rightarrow \frac{1}{2}T_{\alpha}{}^{\alpha 2} - kT_{\alpha}{}^{\alpha} - c = 0 \Rightarrow T_{\alpha}{}^{\alpha 2} - 2kT_{\alpha}{}^{\alpha} - C = 0$$

$$T_{\alpha}{}^{\alpha} = \frac{2k \pm \sqrt{4k^2 + 4C}}{2}\ \ (46)$$

The field is constant .But we have seen$\left(T_{\alpha}{}^{l} - k\delta_{\alpha}{}^{l}\right) = 0$

Therefore

$$T_{\alpha}{}^{\alpha} = \frac{2k \pm \sqrt{4k^2 + 4C}}{2} = k \Rightarrow k^2 + C = 0\ \ (47)$$

The point is the right hand side of equation (43) is not expected to have zero on its right hand side in all frames of reference since it is not a tensor equation. Nevertheless the form of (43)is preserved since

$T_{\alpha\beta} - kg_{\alpha\beta} = 0$ which is a tensor equation.

For $l \neq a$

$$T_{\alpha}{}^{l} \frac{\partial T_{l}{}^{a}}{\partial x^{i}} = 0$$

$\frac{\partial T_{l}{}^{a}}{\partial x^{i}}$ is not a tensor. Therefore

$$T_{\alpha}{}^{l} = 0 \quad (48)$$

The last equation may be compared with $T_{\alpha}{}^{l} = kg^{l}{}_{a} = k\delta^{l}{}_{a} = 0$for $l \neq a$

Conclusion

We have unexpected constants on the manifold like the Ricci Scalar and the norm of the Ricci Tensor. They are independent of the space time coordinates. They are not only invariants but they are also constants. The article renders the fact that the Ricci tensor and the Riemannian curvature tensor are the null tensors.The metric tensor also happens to be a null tensor. There is a requirement for restructuring the subject.

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