Refutation of Ackermann's approach for modal logic and second-order quantification reduction

Abstract: From one source we evaluate the Ackermann rule and from another source three examples in 15 equations of second-order reduction. None of the equations is tautologous. This implies these approaches to map modal clauses of first-order logic to and from second-order logic are *non* tautologous fragments of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓ ,·; \ Not And;
> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ;
< Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇐⇒, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1;
(%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B); (B>A) (A⊢B); (B>A) (A⊨B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Conradie, W.; Goranko, V.; Vakarelov, D. (2006). arxiv.org/pdf/cs/0602024.pdf
Algorithmic correspondence and completeness in modal logic: I. The core algorithm SQEMA.
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Lemma 0.1 (Ackermann's Lemma). Let P be a predicate variable and A(x,z) and B(P) be first-order formulae such that there are no occurrences of P in A(x,z). If P occurs only negatively in B(P) then

∃P (∀x[A(x,z) → P(x)] ∧ B(P)) ≡ B(A(t,z)/P(t)) (0.1.1.1)

LET p, q, r, t, x, y, z: p, A, B, t, x, y, z
(p<(q&(x&z)))>((#(r&p)<(p@p))>((((q&(#x&z))>(%p&#x))&(r&p))=(r&((q&(t&z))\(p&t))))) ;

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TTTT TTTT TTTT TTTT(64),
TTTT TTTT TTTT TTTT,TTTT TTTC TTTT TTTC,(16)
TTTT TTTT TTTT TTTT(16),
TTTT TTTT TTTT TTTT,TTTT TTTC TTTT TTTC,(16)
TTTT TTTT TTTT TTTT(16)
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(0.1.1.2)

and, respectively, if P occurs only positively in B(P), then

∃P (∀x[P(x) → A(x,z)] ∧ B(P)) ≡ B(A(t,z)/P(t)) (0.1.2.1)

(p<(q&(x&z)))>((#(r&p)>(p@p))>(((%p&#x)>(q&(r&p)))=(r&((q&(t&z))\(p&t))))) ;

TFTF TTTT TFTF TTTT(16),TNTT TTTT TNTT TTTT(16)
(0.1.2.2)

where z are parameters, and each occurrence of P(t) in B on the right hand side of the equivalences, for terms t, is substituted by A(t,z).

> **Remark 0.1:** Combining the antecedents of Eqs. 0.1.1.2 and 0.1.2.2 produces the Ackermann rule.
>
> (p<(q&(x&z)))>(((#(r&p)<(p@p))>((((q&(#x&z))>(%p&#x))&(r&p))=(r&((q&(t&z))(p&t)))))
> &((#(r&p)>(p@p))>(((%p&#x)>(q&(r&p)))=(r&((q&(t&z))\(p&t)))))) ;
>
> TFTF TTTT TFTF TTTT(16),TNTT TTTT TNTT TTTT(16),
> TFTF TTTT TFTF TTTT(16),TNTT TTTT TNTT TTTT(16),
> TFTF TTTT TFTF TTTT,TFTF TTTF TFTF TTTF(16),
> TNTT TTTT TNTT TTTT(16),
> TFTF TTTT TFTF TTTT,TFTF TTTF TFTF TTTF(16),
> TNTT TTTT TNTT TTTT(16) (0.2)
>
> The Ackermann rule in Eq. 0.2 is *not* tautologous, hence refuting it.

From: Schmidt, R.A. (2012). The Ackermann approach for modal logic, correspondence theory and second-order reduction. Journal of Applied Logic 10 (2012) 52–74. renate.schmidt@manchester.ac.uk

Example 1. Let us see if we can derive the seriality property, $\forall x \exists y[R(x, y)]$, (1.0.1)

LET p, q, r: x, y, R
(r&(#p&%q)) = (p=p) ; **FFFF FFF**N **FFFF FFF**N (1.0.2)

for the modal axiom **D** = $\forall p[\Box p \rightarrow \Diamond p]$. (1.1.1)
##p>%p ; TTTT TTTT TTTT TTTT (1.1.2)

Its negation is: ¬**D** = $\exists p[\Box p \wedge \Box\neg p]$. (1.2.1.)
~(##p>%#p)=(#%p&#~%p) ; TTTT TTTT TTTT TTTT (1.2.2)

The input is the set containing

1. $\neg a \vee (\Box p \wedge \Box\neg p)$ (1.1)
~q+(#p+#~p) ; TNTN TNTN TNTN TNTN (1.2)

and the goal is to eliminate p, that is, $\Sigma = \{p\}$. Rewriting with respect to the $\wedge$ elimination replacement rule gives us:

2. $\neg a \vee \neg(\neg\Box p \vee \neg\Box\neg p)$ 1, repl. (elim. $\wedge$) (2.1)
~q+~(~#p+~#~p) ; T**F**T**F** T**F**T**F** T**F**T**F** T**F**T**F** (2.2)

and we cross out clause 1. Using the distributivity replacement rule we replace clause 2 by clause 3.

3. $\neg\neg(\neg a \vee \Box p) \vee \neg(\neg a \vee \Box\neg p)$ 2, repl. (distr.) (3.1)
~(~(~q+#p)+~(~q+#~p)) = (p=p) ; TT**FF** TT**FF** TT**FF** TT**FF** (3.2)

Applying the clausify rule we obtain

4. $\neg a \vee \Box p$ 3, repl., cl. (4.1)
~q+#p ; TT**F**N TT**F**N TT**F**N TT**F**N (4.2)

5. $\neg a \vee \Box\neg p$ 3, repl., cl. (5.1)
~q+#~p ; TTN**F** TTN**F** TTN**F** TTN**F** (5.2)

and delete clause 3. p occurs only positively in clause 4 but is shielded by a box operator. Applying the surfacing rule to 4 we obtain

6. $\Box^{u}\neg a \vee p$ 4, surf. (6.1)
#~q+p ; NT**F**T NT**F**T NT**F**T NT**F**T (6.2)

The positive occurrence of p is now unshielded and we can resolve 6 into 5 by applying the Ackermann rule. This replaces clauses 5 and 6 by 7.

7. $\neg a \vee \Box\Box^{u}\neg a$ 6 into 5, Acker. (7.1)
~q+##~q ; TT**FF** TT**FF** TT**FF** TT**FF** (7.2)

Since it does not contain the non-base symbol p, we could stop at this point. However clause 7 can be simplified by using the rewrite rule $\alpha \vee \Box^{\sigma}\Box^{\sigma,u}\alpha \Rightarrow \alpha \vee \Box^{\sigma}\neg\top$ from Table 3.

8. $\neg a \vee \Box\top$ 7, repl. (8.1)
~q+#(#(p@p)) ; TT**FF** TT**FF** TT**FF** TT**FF** (8.2)

The procedure returns {8}. Translating 8 into first-order logic we get:

$\forall x\pi(\neg a \vee \Box\neg\top, x) \equiv \pi(\Box\neg\top, a) = \forall x\neg R(a, x)$ (9.1)
LET p, q, R, s: pi, alpha, R, x
((p&((~q+#(p@p))&#s))=(p&(#(p@p)&q)))=~(r&(p&#s)) ;
TTTT TTTT TCTT TTTC (9.2)

Unskolemization returns $\exists y\forall x[\neg R(y, x)]$. (10.1)
LET p, q, r, s: x, y, R, z
~(r&(%q&#p)) = (p=p) ; TTTT TTTC TTTT TTTC (10.2)

Finally negating gives the expected result: $\forall y\exists x[R(y, x)]$. (11.1)
r&(#q&%p) ; **FFFF FFF**N **FFFF FFF**N (11.2)

Example 3. The modal axiom $\forall p\forall q[\Box(\Box p \equiv q) \to \Diamond\Box\neg p]$ (3.1.1)
#(##p=#q)>%#~#p ; TTTC TTTC TTTC TTTC (3.1.2)

corresponds to $\forall x\exists y\forall z[R(x, y) \wedge \neg R(y, z)]$, (3.2.1)
(r&(#p&#q))&~(r&(#q&#s)) ; **FFFF FFF**N **FFFF FFFF** (3.2.2)

in words, every world has a successor that is a dead-end.

Example 9. The following is the rule version of the axiom from Example 7.

$\forall p \forall q[\Box(p \lor q)/(\Box p \lor \Box q)$. (9.1.1)

#(#p+#q)\(##p+##q) ; TCCC TCCC TCCC TCCC (9.1.2)

We show that it [Eq. 9.1.1] is equivalent to $\forall p[\Diamond p \rightarrow \Box p]$. (9.2.1)

(#(#p+#q)\(##p+##q))=(%#p>##p) ; TCCC TCCC TCCC TCCC (9.2.2)

Excepting the expected modal axiom(s) for D as rendered, the 15 example equations are *not* tautologous. This means the Ackermann approach for modal logic, correspondence theory, and second-order reduction is refuted.