The Collatz Conjecture - a proof

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Abstract

Originated by Lothar Collatz in 1937 [1], the conjecture states: given the recursive function, y=3x+1 if x is odd, or y=x/2 if x is even, for any positive integer x, y will equal 1 after a finite number of steps. This analysis examines number form and uses a tree type graph to prove the process.

1. examples

An example for a random selection of 7, using the original method:

(7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1)

An example for a random selection of 12, using the original method:

(12, 6, 3, 10, 5, 16, 8, 4, 2, 1)

2. functions

The recursive function is replaced with function d for odd values (2n-1), with

$$d(x) = 3x+1 = 6n-2 = 2^k y = u \qquad (2.0)$$

and function e for even values, which removes all factors of 2,

$$e(u) = y \qquad (2.1)$$

The function e can be defined as a short program with a loop that repeatedly divides u by 2 until the output is an odd integer. This eliminates the redundancy and clutter of repeated division by 2.
After k divisions by 2, u = y, an odd integer. The value of y becomes the input x, and the cycle is repeated until y=1. The application of e(d(7)) results in (7, 11. 17, 13, 5, 1), the revised format used in this analysis.

3. x-y correspondence

The set of odd integers D is divided into X_1 with the form (4n-1), X_2 with the form (8n-7), and X_3 with the form (8n-3).
Using (2.0),

$$u = d(4n-1) = 12n-2, \qquad (2.2)$$

elements of Y_1 are $\qquad e(u) = 6n-1, \qquad (2.3)$

$$u = d(8n-7) = 24n-20, \qquad (2.4)$$

and elements of Y_2 are $\qquad e(u) = 6n-5. \qquad (2.5)$

There is no corresponding Y_3 since 3x+1 cannot generate (0 mod 3) output, therefore those integers can only begin a sequence of odd integers. X_1 and X_2 are referred to as generation 1 terms.
The elements of X_3 have an interesting property which aids in the solution of the Collatz conjecture. By defining

$$f(x_3) = (x_3-1)/4 = (8n-4)/4 = 2n-1 \qquad (2.6)$$

X_3 is mapped to the set of odd integers D. Using (2.0)

$$d(x_3) = 3(4x+1) +1 = 4(3x+1) = 4u \qquad (2.7)$$

4. reverse sequences

Using the inverse of (2.6), a subset B_x, can be formed for each term in X_1 and X_2, containing a sequence of inflated values which are functionally equivalent to the generation 1 terms. They are referred to as branching or b-terms.
The sequence for B_1 is {1 5 **21** 85 341 **1365** ...}, with bold terms as terminating x, (0 mod 3) values. Beginning with a generation-1 term x as b_1, each successive term b_{r+1} is $4b_r+1$.
The B_x allow forming branches to avoid termination of sequences and tree expansion.
Using
x = (2y-1)/3 for elements from Y1 and
x = (4y-1)/3 for elements from Y2,
and B_x, a reverse sequence of any length can be formed
Beginning with 1, the sequence is (1 1) and terminates since 1 forms a loop.
Using B_1 ={1 5 **21** 85 ...}, the first choice gives (1 5).
Using B_5 ={**3** 13 53 ...}, the first choice would terminate,
but the second choice gives (1 5 13).
Continuing this process with sets B_x corresponding to the current x,
B_{13} = {17 **69** 277 ...},
B_{17} = {11 **45** 181...},
B_{11} = {7 29 **117** ...},
B_7 = {**9** 37 149 ...}, and allowing termination, the modified reverse sequence is
(1 <u>5</u> <u>13</u> 17 11 7 9), or sequence 9 in forward/descending mode.
The underlined terms show two branches or b-terms that avoid terminating the sequence.

5. the tree

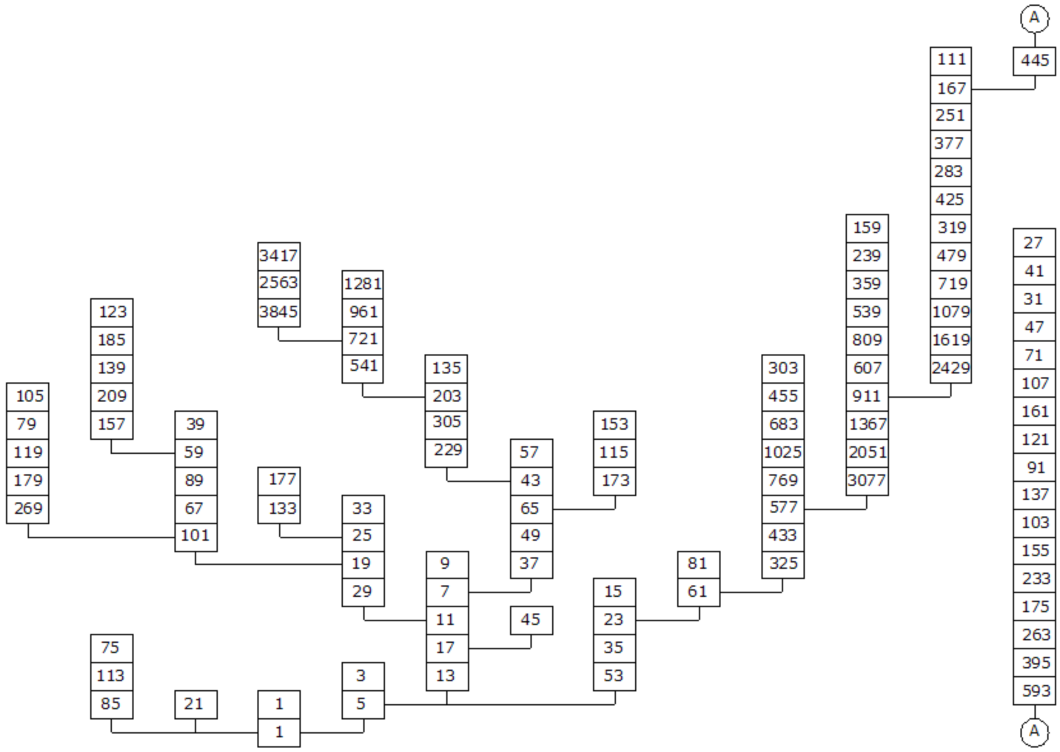

fig.2

Fig.2 shows the initial growth of the tree from a 'trunk' of 1, vertically with each branch terminating in a (0 mod 3) value, and horizontally via the B_x as demonstrated in section 4. The sequence for x=27 is revealed as a composite of 7 branches to the right. The graph only shows some of the branches and has omitted the even values for clarity. The complete tree would have multiple branches at each cell except (o mod 3) values. The complexity of branches would be best visualized in 3 dimensions. The graph is incomplete until even integer selection is considered.

6. even integer selection

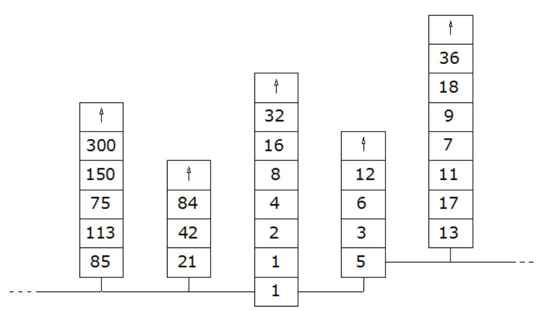

fig.3

In fig.3, each branch is extended without limit with a 2^k progression times the (0 mod 3) value.

7. descending/forward sequences

The simplest method of forming sequences is to begin with a (0 mod 3) integer, and apply d(x) and e(u) from section 2, (with the exception of x = 1). A complete sequence of odd integers ends with a b-term (underlined).

1, 1
3, 5
9, 7, 11, 17, 13
15, 23, 35, 53
21
27, 41, ... 593, 445
33, 25, 19, 29
39, 59, 89, 67, 101
45
51, 77, 29

↓

8. the flow chart

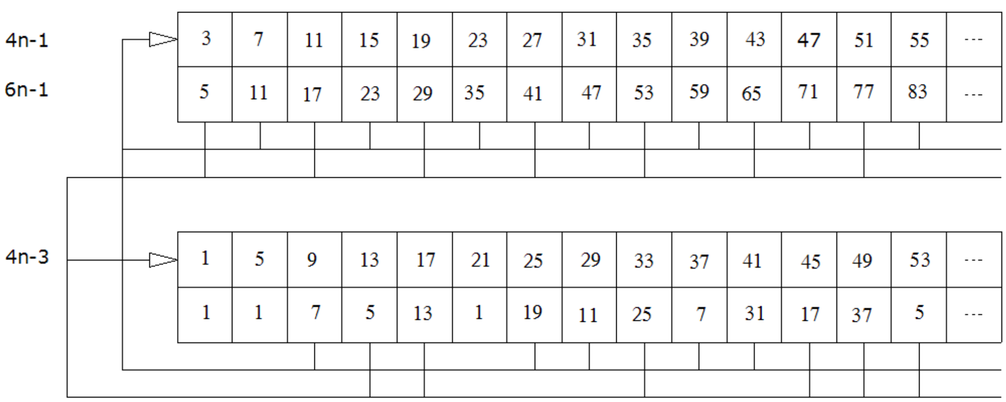

fig.4

Fig.4 shows the flow of elements between the subsets of odd integers {4n-1} and {4n-3}. In an alternating pattern, half of {4n-1} output returns to itself, and half goes to {4n-3}. Less than half of {4n-3} output returns to itself, and less than half goes to {4n-1} due to the terminating values of 1. For each cycle from the upper set to the lower set, each integer advances one position in a sequence until reaching '1'. The {4n-3} section filters out all integers in the set B_1 per cycle.

Conclusion

Using the tree analogy, when ascending from 1, there is a one to many relationship from y to x, via the B_x branching, presenting an unlimited number of possible sequences, as shown in section 4. Descending from a randomly selected term x, there is a one to one relationship from x to y. The current x determines the next term, therefore the entire sequence is pre-determined by the initial selection. Descending results in merging branches, with each ending at the trunk. All odd integers have a corresponding b-term, therefore all sequences merge. The Collatz conjecture applies only to finite length sequences, in the descending mode.

reference

1. Wikipedia.org/Collatz Conjecture, Mar 2018