viXra manuscript No.
(will be inserted by the editor)

Solving Incompletely Predictable problem Riemann hypothesis with Dirichlet Sigma-Power Law as equation and inequation

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Received: March 28, 2019

Abstract Riemann hypothesis proposed all nontrivial zeros to be located on critical line of Riemann zeta function. Treated as Incompletely Predictable problem, we obtain the novel Dirichlet Sigma-Power Law as final proof of solving this problem. This Law is derived as equation and inequation from original Dirichlet eta function (proxy function for Riemann zeta function). Performing a parallel procedure help explain closely related Gram points.

Keywords Dimensional analysis · Dirichlet Sigma-Power Law · Gram points · Incompletely Predictable problems · Inequation · Nontrivial zeros · Riemann hypothesis

Mathematics Subject Classification (2000) 11A41 · 11M26

1 Introduction

Gram and virtual Gram points are dependently calculated using *complex equation* Riemann zeta function, $\zeta(s)$, or its *proxy* Dirichlet eta function, $\eta(s)$, at the critical strip (denoted by $0 < \sigma < 1$). Gram[y=0], Gram[x=0] and Gram[x=0,y=0] points respectively refer to x-axis, y-axis and Origin intercepts at the critical line (denoted by $\sigma = \frac{1}{2}$). Gram[y=0] and Gram[x=0,y=0] points are respectively synonymous with traditional *'Gram points'* and *nontrivial zeros* with the former further discussed in Segment A2, Appendix A. Virtual Gram[y=0] and virtual Gram[x=0] points respectively refer to x-axis and y-axis intercepts at the non-critical lines (denoted by $\sigma \neq \frac{1}{2}$). Virtual Gram[x=0,y=0] points do not exist.

Gram and virtual Gram points are Incompletely Predictable entities. Activities to prove associated open problem in number theory Riemann hypothesis and explain Gram[y=0] and Gram[x=0] points equate to solving Incompletely Predictable problems. Claims from these activities are only meaningful when provided with definitions for relevant terms in Segment A1, Appendix A. Dependently calculated using *complex algorithm* Sieve of Eratosthenes, prime and composite numbers as Incompletely Predictable numbers are also depicted there.

In order of increasing size, arbitrary Set $\mathbf{X}$ can be countable finite set (CFS), countable infinite set (CIS) or uncountable infinite set (UIS). Cardinality of Set $\mathbf{X}$, $|\mathbf{X}|$, measures the

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"number of elements" in Set **X**. E.g. Set **negative Gram[y=0] point** has CFS of negative Gram[y=0] point with |**negative Gram[y=0] point**| = 1, Set **N** has CIS of natural numbers with $|\mathbf{N}| = \aleph_0$, and Set **R** has UIS of real numbers with $|\mathbf{R}| = \mathfrak{c}$ (cardinality of the continuum).

$$\zeta(s) = \frac{e^{(\ln(2\pi)-1-\frac{\gamma}{2})s)}}{2(s-1)\Gamma(1+\frac{s}{2})}\Pi_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} = \pi^{\frac{s}{2}}\frac{\Pi_\rho\left(1-\frac{s}{\rho}\right)}{2(s-1)\Gamma\left(1+\frac{s}{2}\right)}$$

Proposed in 1859 by German mathematician Bernhard Riemann (September 17, 1826 – July 20, 1866), Riemann hypothesis is mathematical statement on $\zeta(s)$ that critical line denoted by $\sigma = \frac{1}{2}$ contains complete Set **nontrivial zeros** with |**nontrivial zeros**| = $\aleph_0$. Alternatively, this hypothesis is geometrical statement on $\zeta(s)$ that generated curves when $\sigma = \frac{1}{2}$ contain complete Set **Origin intercepts** with |**Origin intercepts**| = $\aleph_0$. Depicted in full and abbreviated version, Hadamard product is infinite product expansion of $\zeta(s)$ based on Weierstrass's factorization theorem displaying a simple pole at s = 1. It contains both trivial and nontrivial zeros indicating their common origin from $\zeta(s)$. Set **trivial zeros** occurs at σ = -2, -4, -6, -8, -10,..., ∞ with |**trivial zeros**| = $\aleph_0$ due to Γ function term in denominator. Nontrivial zeros occur at $s = \rho$ with γ denoting Euler-Mascheroni constant.

Remark 1.1. Computationally checking for first 10,000,000,000,000 nontrivial zeros location on critical line implies but does not prove Riemann hypothesis to be true.

Locations of first 10,000,000,000,000 nontrivial zeros on critical line are previously confirmed to be correct. Hardy in 1914[1] and Hardy and Littlewood in 1921[2] showed infinite nontrivial zeros on critical line by considering moments of certain functions related to $\zeta(s)$. This discovery cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of nontrivial zeros located away from this line.

Remark 1.2. We can apply useful concepts from exact and inexact Dimensional analysis homogeneity to well-defined equations and inequations.

Respectively for 'base quantities' such as *length*, *mass* and *time*; their fundamental SI 'units of measurement' meter (m) is defined as distance travelled by light in vacuum for time interval 1/299 792 458 s with speed of light c = 299,792,458 ms^{-1}, kilogram (kg) is defined by taking fixed numerical value Planck constant h to be 6.626 070 15 X 10^{-34} Joules·second (Js) [whereby Js is equal to $\text{kgm}^2\text{s}^{-1}$] and second (s) is defined in terms of $\Delta\text{vCs} = \Delta(^{133}\text{Cs})_{hfs}$ = 9,192,631,770 s^{-1}. Derived SI units such as J and ms^{-1} respectively represent 'base quantities' *energy* and *velocity*. The word 'dimension' is commonly used to indicate all those mentioned 'units of measurement' in well-defined equations.

Dimensional analysis (DA) is an analytic tool with DA homogeneity and non-homogeneity (respectively) denoting valid and invalid equation occurring when 'units of measurements' for 'base quantities' are "balanced" and "unbalanced" across both sides of the equation. E.g. equation 2 m + 3 m = 5 m is valid and equation 2 m + 3 kg = 5 mkg is invalid (respectively) manifesting DA homogeneity and non-homogeneity. Consider kinetic energy (KE) in MJ with m_0 = rest mass in kg and v = velocity in ms^{-1}. In classical mechanics concerning low velocity with v << c, Newtonian KE = $\frac{1}{2}m_0v^2$. In relativistic mechanics concerning high velocity with v ≥ 0.01c, Relativistic KE = $\frac{m_0c^2}{\sqrt{1-(v^2/c^2)}} - m_0c^2$. We arbitrarily divide DA homogeneity into (1) inexact DA homogeneity for ["<100% accurracy"] Newtonian KE equation and (2) exact DA homogeneity for ["100% accurracy"] Relativistic KE equation.

Let (2n) and (2n-1) be 'base quantities' in Dirichlet Sigma-Power Laws formatted in simplest forms as equations and inequations. E.g. DA on exponent $\frac{1}{2}$ in $(2n)^{\frac{1}{2}}$ in simplest form is correct but DA on exponent $\frac{1}{4}$ in equivalent $(2^2n^2)^{\frac{1}{4}}$ *not* in simplest form is incorrect. Then fractional exponents as 'units of measurement' given by $(1-\sigma)$ for equations and $(\sigma + 1)$ for inequations when $\sigma = \frac{1}{2}$ will coincide with exact DA homogeneity[1]; and $(1 - \sigma)$ for equations and $(\sigma + 1)$ for inequations when $\sigma \neq \frac{1}{2}$ will coincide with inexact DA homogeneity[2]. Respectively for equations and inequations, exact DA homogeneity at $\sigma = \frac{1}{2}$ denotes $\sum$(all fractional exponents) as $2(1 - \sigma)$ and $2(\sigma + 1)$ equates to ["exact"] whole number '1' and '3'; and inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ denotes $\sum$(all fractional exponents) as $2(1 - \sigma)$ and $2(\sigma + 1)$ equates to ["inexact"] fractional number '$\neq$1' and '$\neq$3'.

Footnote 1,2: Exact and inexact DA homogeneity occur in Dirichlet Sigma-Power Laws as equations or inequations for Gram[y=0] points, Gram[x=0] points and nontrivial zeros. *Law of Continuity* is a heuristic principle *whatever succeed for the finite, also succeed for the infinite*. Then these Dirichlet Sigma-Power Laws which inherently manifest themselves on finite and infinite time scale should "succeed for the finite, also succeed for the infinite".

Outline of proof for Riemann hypothesis. To simultaneously satisfy two mutually inclusive conditions: I. *With rigid manifestation of exact DA homogeneity*, Set **nontrivial zeros** with |**nontrivial zeros**| = $\aleph_0$ is located on critical line (viz. $\sigma = \frac{1}{2}$) when $2(1 - \sigma)$ [or $2(\sigma + 1)$] as $\sum$(all fractional exponents) = whole number '1' [or '3'] in Dirichlet Sigma-Power Law[3] as equation [or inequation]. II. *With rigid manifestation of inexact DA homogeneity*, Set **nontrivial zeros** with |**nontrivial zeros**| = $\aleph_0$ is not located on non-critical lines (viz. $\sigma \neq \frac{1}{2}$) when $2(1 - \sigma)$ [or $2(\sigma + 1)$] as $\sum$(all fractional exponents) = fractional number '$\neq$1' [or '$\neq$3'] in Dirichlet Sigma-Power Law[3] as equation [or inequation].

Footnote 3: Derived from original $\eta(s)$ (*proxy* for $\zeta(s)$) as equation or inequation, this Law symbolizes end-result proof on Riemann hypothesis.

Riemann hypothesis mathematical foot-prints. Six identifiable steps to prove Riemann hypothesis: *Step 1* Use $\eta(s)$, *proxy* for $\zeta(s)$, in critical strip. *Step 2* Apply Euler formula to $\eta(s)$. *Step 3* Obtain "simplified" Dirichlet eta function which intrinsically incorporates *actual location [but not actual positions]* of all nontrivial zeros[4]. *Step 4* Apply Riemann integral to "simplified" Dirichlet eta function in discrete (summation) format. *Step 5* Obtain Dirichlet Sigma-Power Law in continuous (integral) format as equation or inequation. *Step 6* Perform exact and inexact DA homogeneity on all of their fractional exponents.

Footnote 4: Respectively Gram[y=0] points, Gram[x=0] points and nontrivial zeros are Incompletely Predictable entities with their actual positions determined by setting [defined below] $\sum Im\{\eta(s)\} = 0$, $\sum Re\{\eta(s)\} = 0$ and $\sum ReIm\{\eta(s)\} = 0$ to *dependently* calculate relevant positions of all preceding entities in the neighborhood. Respectively actual location of Gram[y=0] points, Gram[x=0] points and nontrivial zeros; and virtual Gram[y=0] points, virtual Gram[x=0] points and "absent" nontrivial zeros occur precisely at $\sigma = \frac{1}{2}$; and $\sigma \neq \frac{1}{2}$.

2 Riemann zeta and Dirichlet eta functions

L-functions form an integral part of 'L-functions and Modular Forms Database' with far-reaching implications. In perspective, $\zeta(s)$ is simplest example of an L-function. $\zeta(s)$ is a function of complex variable s (= $\sigma \pm \imath t$) that analytically continues sum of infinite series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$. The common convention is to write s as $\sigma + \imath t$ with $\imath = \sqrt{-1}$, and σ and t real. Valid for $\sigma > 0$, we write $\zeta(s)$ as $\mathrm{Re}\{\zeta(s)\} + \imath{\cdot}\mathrm{Im}\{\zeta(s)\}$ and note that $\zeta(\sigma + \imath t)$ when $0 < t < +\infty$ is the complex conjugate of $\zeta(\sigma - \imath t)$ when $-\infty < t < 0$.

Also known as alternating zeta function, $\eta(s)$ must act as *proxy* for $\zeta(s)$ in critical strip (viz. $0 < \sigma < 1$) containing critical line (viz. $\sigma = \frac{1}{2}$) because $\zeta(s)$ only converges when $\sigma > 1$. This implies $\zeta(s)$ is undefined to left of this region in critical strip which then requires $\eta(s)$ representation instead. They are related to each other as $\zeta(s) = \gamma \cdot \eta(s)$ with proportionality factor $\gamma = \frac{1}{(1-2^{1-s})}$ and $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots$.

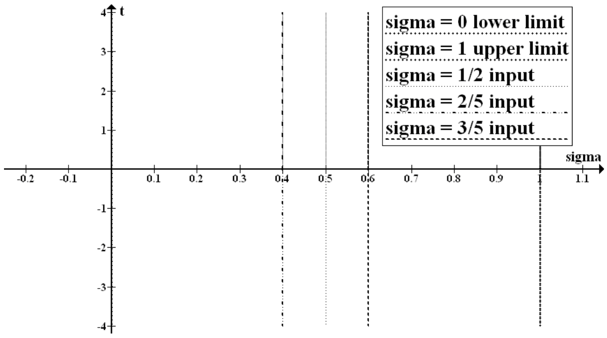

Fig. 1 INPUT for $\sigma = \frac{1}{2}$, $\frac{2}{5}$, and $\frac{3}{5}$. $\zeta(s)$ has CIS of Completely Predictable trivial zeros at σ = all negative even numbers and CIS of Incompletely Predictable nontrivial zeros at $\sigma = \frac{1}{2}$ for various t values.

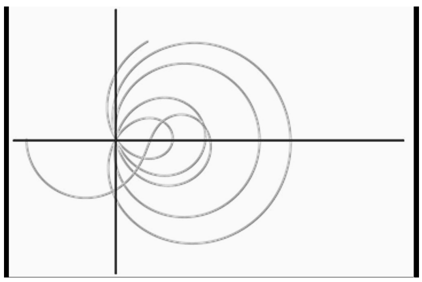

Fig. 2 OUTPUT for $\sigma = \frac{1}{2}$. Schematically depicted polar graph of $\zeta(\frac{1}{2} + \imath t)$ plotted along critical line for real values of t running from 0 to 34, horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Note presence of Origin intercepts which are totally absent in Figures 3 and 4 [with identical axes definitions].

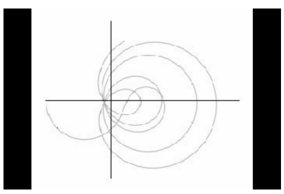

Fig. 3 OUTPUT for $\sigma = \frac{2}{5}$.

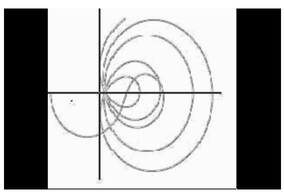

Fig. 4 OUTPUT for $\sigma = \frac{3}{5}$.

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (1)\\ &= \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\\ &= \Pi_{p\ prime}\frac{1}{(1-p^{-s})}\\ &= \frac{1}{(1-2^{-s})}\cdot\frac{1}{(1-3^{-s})}\cdot\frac{1}{(1-5^{-s})}\cdot\frac{1}{(1-7^{-s})}\cdot\frac{1}{(1-11^{-s})}\cdots\frac{1}{(1-p^{-s})}\cdots\end{aligned}$$

Eq. (1) is defined for only $1 < \sigma < \infty$ region where $\zeta(s)$ is absolutely convergent. There are no zeros located here. In Eq. (1), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] can also represent $\zeta(s)$.

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\cdot\Gamma(1-s)\cdot\zeta(1-s) \qquad (2)$$

With $\sigma = \frac{1}{2}$ as symmetry line of reflection, Eq. (2) is Riemann's functional equation valid for $-\infty < \sigma < \infty$. It can be used to find all trivial zeros on horizontal line at $\imath t = 0$ occurring when σ = -2, -4, -6, -8, -10,…, ∞ whereby $\zeta(s) = 0$ because factor $\sin(\frac{\pi s}{2})$ vanishes. Γ is gamma function, an extension of factorial function [a product function denoted by ! notation whereby n! = $n(n-1)(n-2)\ldots(n-(n-1))$] with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer, $\Gamma(n) = (n-1)!$

$$\begin{aligned}\zeta(s) &= \frac{1}{(1-2^{1-s})}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} \qquad (3)\\ &= \frac{1}{(1-2^{1-s})}\left(\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\cdots\right)\end{aligned}$$

Eq. (3) is defined for all $\sigma > 0$ values except for simple pole at σ = 1. As alluded to above, ζ(s) without $\frac{1}{(1-2^{1-s})}$ viz. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ is η(s). It is a holomorphic function of s defined by analytic continuation and is mathematically defined at σ = 1 whereby analogous trivial zeros with presence only for η(s) [but not for ζ(s)] on vertical straight line σ = 1 are found at $s = 1 \pm \imath\cdot\frac{2\pi k}{\ln(2)}$ where k = 1, 2, 3, 4, 5, …, ∞.

Figure 1 depict complex variable s (= $\sigma \pm \imath t$) as INPUT with x-axis denoting real part Re{s} equating to σ; and y-axis denoting imaginary part Im{s} equating to t. Figures 2, 3 and 4 respectively depict $\zeta(s)$ as OUTPUT for real values of t running from 0 to 34 at $\sigma = \frac{1}{2}$ (critical line), $\sigma = \frac{2}{5}$ (non-critical line), and $\sigma = \frac{3}{5}$ (non-critical line) with x-axis denoting real part $\mathrm{Re}\{\zeta(s)\}$ and y-axis denoting imaginary part $\mathrm{Im}\{\zeta(s)\}$. There are infinite types-of-spirals possibilities associated with each and every σ value arising from all the infinite σ values in critical strip. Mathematically proving all nontrivial zeros location on critical line as denoted by solitary $\sigma = \frac{1}{2}$ value equates to geometrically proving all Origin intercepts occurrence at solitary σ = $\frac{1}{2}$ value. Both result in rigorous proof for Riemann hypothesis.

3 Prerequisite lemma, corollary and propositions for Riemann hypothesis

We treat $\eta(s)$, *proxy* function for $\zeta(s)$, as unique mathematical object with key properties and behaviors. Containing all x-axis, y-axis and Origin intercepts, $\eta(s)$ as original equation will intrinsically incorporate *actual location [but not actual positions]* of all Gram[y=0] points, Gram[x=0] points and nontrivial zeros. Proofs on lemma, corollary and propositions below on nontrivial zeros depict exact and inexact DA homogeneity in both derived equation and inequation. Parallel procedure on Gram[y=0] and Gram[x=0] points in Appendix B depict exact and inexact DA homogeneity in similarly derived equations and inequations.

Lemma 3.1. "Simplified" Dirichlet eta function is derived directly from Dirichlet eta function with Euler formula application and it will intrinsically incorporate actual location [but not actual positions] of all nontrivial zeros.

Proof. Denote complex number ($\mathbb{C}$) as z = x + $\imath$·y. Then z = Re(z) + $\imath$·Im(z) with Re(z) = x and Im(z) = y; modulus of z, |z| = $\sqrt{Re(z)^2 + Im(z)^2}$ = $\sqrt{x^2+y^2}$; and $|z|^2$ = x^2+y^2.

Euler formula is commonly stated as $e^{\imath x} = \cos x + \imath \cdot \sin x$. Euler identity (where $x = \pi$) is $e^{\imath\pi} = \cos\pi + \imath \cdot \sin\pi = -1 + 0$ [or stated as $e^{\imath\pi} + 1 = 0$]. The n^s of $\zeta(s)$ is expanded to $n^s = n^{(\sigma+\imath t)} = n^\sigma \mathrm{e}^{t\ln(n)\cdot\imath}$ since $n^t = e^{t\ln(n)}$. Apply Euler formula to n^s result in $n^s = n^\sigma(\cos(t\ln(n)) + \imath \cdot \sin(t\ln(n))$. This is written in trigonometric form [now designated by short-hand notation $n^s(Euler)$] whereby n^σ is modulus and $t\ln(n)$ is polar angle (argument).

Apply $n^s(Euler)$ to Eq. (1). Then $\zeta(s)$ =Re$\{\zeta(s)\} + \imath$·Im$\{\zeta(s)\}$ with $Re\{\zeta(s)\}$ = $\sum_{n=1}^{\infty} n^{-\sigma}\cos(t\ln(n))$ and $Im\{\zeta(s)\}$ = $\sum_{n=1}^{\infty} n^{-\sigma}\sin(t\ln(n))$. As Eq. (1) is defined only for $\sigma>1$ where zeros never occur, we will not carry out further treatment here.

Apply $n^s(Euler)$ to Eq. (3). Then $\zeta(s) = \gamma\cdot\eta(s) = \gamma\cdot[Re\{\eta(s)\} + \imath\cdot\mathrm{Im}\{\eta(s)\}]$ with

$Re\{\eta(s)\}$ = $\sum_{n=1}^{\infty}((2n-1)^{-\sigma}\cos(t\ln(2n-1)) - (2n)^{-\sigma}\cos(t\ln(2n)))$;

$Im\{\eta(s)\}$ = $\sum_{n=1}^{\infty}((2n)^{-\sigma}\sin(t\ln(2n)) - (2n-1)^{-\sigma}\sin(t\ln(2n-1)))$;

and proportionality factor γ = $\frac{1}{(1-2^{1-s})}$.

Complex number s in critical strip is designated by s = $\sigma + \imath t$ for $0 < t < +\infty$ and s = $\sigma - \imath t$ for $-\infty < t < 0$. Nontrivial zeros equating to $\zeta(s) = 0$ give rise to our desired $\eta(s)$ = 0. Modulus of $\eta(s)$, $|\eta(s)|$, is defined as $\sqrt{(Re\{\eta(s)\})^2 + (Im\{\eta(s)\})^2}$ with $|\eta(s)|^2$ = $(Re\{\eta(s)\})^2 + (Im\{\eta(s)\})^2$. Mathematically $|\eta(s)|$ = $|\eta(s)|^2$ = 0 is an unique condition giving rise to $\eta(s)$ = 0 occurring only when Re$\{\eta(s)\}$ = Im$\{\eta(s)\}$ = 0 as any non-zero values for Re$\{\eta(s)\}$ and/or Im$\{\eta(s)\}$ will always result in $|\eta(s)|$ and $|\eta(s)|^2$ having non-zero values. Important implication is that sum of $Re\{\eta(s)\}$ and $Im\{\eta(s)\}$ equating to zero [given by Eq. (4)] must always hold when $|\eta(s)|$ = $|\eta(s)|^2$ = 0 and consequently $\eta(s)$ = 0.

$$\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\} = 0 \tag{4}$$

In principle, advocating for existence of theoretical s values leading to non-zero values in Re$\{\eta(s)\}$ and Im$\{\eta(s)\}$ depicted as possibility +Re$\{\eta(s)\}$ = -Im$\{\eta(s)\}$ or -Re$\{\eta(s)\}$ = +Im$\{\eta(s)\}$ could satisfy Eq. (4). Hence reverse implication is not necessarily true as these s values will not result in $|\eta(s)|$ = $|\eta(s)|^2$ = 0. In any event, we need not consider these two possibilities since solving Riemann hypothesis involves nontrivial zeros defined by $\eta(s)$ = 0 with non-zero values in Re$\{\eta(s)\}$ and/or Im$\{\eta(s)\}$ being not compatible with $\eta(s)$ = 0.

Riemann hypothesis proposed all nontrivial zeros to be located on critical line. This location is conjectured to be uniquely associated with presence of exact DA homogeneity in derived equation and inequation of Dirichlet Sigma-Power Law with Eq. (4) intrinsically incorporated into this Law as the $\eta(s) = 0$ definition for nontrivial zeros equates to Eq. (4).

Apply trigonometry identity $\cos(x) - \sin(x) = \sqrt{2}\sin\left(x + \frac{3}{4}\pi\right)$ to $Re\{\eta(s)\} + Im\{\eta(s)\}$ to get Eq. (5) with terms in last line built by mixture of terms from $Re\{\eta(s)\}$ and $Im\{\eta(s)\}$.

$$\sum ReIm\{\eta(s)\} = \sum_{n=1}^{\infty}[(2n-1)^{-\sigma}\cos(t\ln(2n-1)) - (2n-1)^{-\sigma}\sin(t\ln(2n-1))$$
$$-(2n)^{-\sigma}\cos(t\ln(2n)) + (2n)^{-\sigma}\sin(t\ln(2n))]$$
$$= \sum_{n=1}^{\infty}[(2n-1)^{-\sigma}\sqrt{2}\sin(t\ln(2n-1) + \frac{3}{4}\pi) - (2n)^{-\sigma}\sqrt{2}\sin(t\ln(2n) + \frac{3}{4}\pi)] \tag{5}$$

When depicted in terms of Eq. (4), Eq. (5) becomes

$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\sqrt{2}\sin(t\ln(2n) + \frac{3}{4}\pi) = \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\sqrt{2}\sin(t\ln(2n-1) + \frac{3}{4}\pi)$$
$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\sqrt{2}\sin(t\ln(2n) + \frac{3}{4}\pi) - \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\sqrt{2}\sin(t\ln(2n-1) + \frac{3}{4}\pi) = 0 \tag{6}$$

Eq. (6) in discrete (summation) format is a non-Hybrid integer sequence equation – see Appendix C. $\eta(s)$ calculations for all σ values result in infinitely many non-Hybrid integer sequence equations for $0<\sigma<1$ critical strip region of interest with n = 1, 2, 3, 4, 5,..., ∞ as discrete integer number values, or n = 1 to ∞ as continuous real numbers values with Riemann integral application. These equations will geometrically represent entire plane of critical strip, thus (at least) allowing our proposed proof to be of a complete nature.

Eq. (6) being the "simplified" Dirichlet eta function derived directly from $\eta(s)$ will intrinsically incorporate *actual location [but not actual positions]* of all nontrivial zeros. *The proof is now complete for Lemma 3.1*□.

Proposition 3.2. Dirichlet Sigma-Power Law in continuous (integral) format given as equation and inequation can both be derived directly from "simplified" Dirichlet eta function in discrete (summation) format with Riemann integral application.

Proof. In Calculus, integration is reverse process of differentiation viewed geometrically as area enclosed by curve of function and x-axis. Apply definite integral I between points a and b is to compute its value when $\Delta x \longrightarrow 0$, i.e. $I = \lim_{\Delta x \longrightarrow 0}\sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$. This is Riemann integral of function f(x) in interval [a, b] where a<b. Apply Riemann integral to "simplified" Dirichlet eta function (which intrinsically incorporates *actual location [but not actual positions]* of all nontrivial zeros criterion) to obtain Dirichlet Sigma-Power Law in continuous (integral) format. Then Dirichlet Sigma-Power Law will also fullfil this criterion. Due to resemblance to power law functions in σ from s = $\sigma + \imath t$ being exponent of a power function n^{σ}, logarithm scale use, and harmonic ζ(s) series connection in Zipf's law; we elect to call this Law by its given name. A characteristic and crucial step of this Law is its exact formula expression in usual mathematical language [$y = f(x_1, x_2)$ format description for a 2-variable function with $(2n)$ and $(2n-1)$ parameters] consists of $y = f(t, \sigma)$ with n = 1, 2, 3, 4, 5,..., ∞ or n = 1 to ∞ with Riemann integral application; -∞<t<+∞; and $0<\sigma<1$.

With steps of manual integration shown using indefinite integrals [for simplicity], solve the definite integral below based on numerator portion of R1 with $(2n)$ parameter in Eq. (6):

$\int_1^{\infty} \frac{2^{\frac{1}{2}-\sigma} \sin\left(t\ln(2n)+\frac{3\pi}{4}\right)}{n^{\sigma}} dn = \int_1^{\infty} -\frac{\sin(t\ln(2n)) - \cos(t\ln(2n))}{2^{\sigma} n^{\sigma}} dn$. We deduce most other important integrals to be "variations" of this particular integral containing (i) deletion of $(2n)^{-\sigma}$, $\sqrt{2}$ or $\frac{3}{4}\pi$ terms, and/or (ii) interchange of sine and cosine function. We check all derived antiderivatives to be correct using computer algebra system Maxima.

Simplifying and applying linearity, we obtain $2^{\frac{1}{2}-\sigma} \int \frac{\sin\left(t\ln(2n)+\frac{3\pi}{4}\right)}{n^{\sigma}} \mathrm{d}n$.

Now solving $\int \frac{\sin\left(t\ln(2n)+\frac{3\pi}{4}\right)}{n^{\sigma}} \mathrm{d}n$. Substitute $u = t\ln(2n) + \frac{3\pi}{4} \longrightarrow \mathrm{d}n = \frac{n}{t}\,\mathrm{d}u$, use $n^{1-\sigma} = \mathrm{e}^{\frac{(1-\sigma)\left(u - t\ln(2) - \frac{3\pi}{4}\right)}{t}} = \frac{\mathrm{e}^{\frac{(\sigma-1)(4t\ln(2)+3\pi)}{4t}}}{t} \int \mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)\,\mathrm{d}u$.

Now solving $\int \mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)\,\mathrm{d}u$. We integrate by parts twice in a row: $\int \mathtt{f}\mathtt{g}' = \mathtt{f}\mathtt{g} - \int \mathtt{f}'\mathtt{g}$.

First time: $\mathrm{f} = \sin(u), \mathrm{g}' = \mathrm{e}^{\frac{(1-\sigma)u}{t}}$

Then $\mathtt{f}' = \boxed{\cos(u)}, \mathtt{g} = \boxed{\frac{(1-\sigma)t\mathrm{e}^{\frac{(1-\sigma)u}{t}}}{\sigma^2 - 2\sigma + 1}}$:

$$= \frac{(1-\sigma)t\mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)}{\sigma^2-2\sigma+1} - \int \frac{(1-\sigma)t\mathrm{e}^{\frac{(1-\sigma)u}{t}} \cos(u)}{\sigma^2-2\sigma+1}\,\mathrm{d}u$$

Second time: $\mathtt{f} = \boxed{\cos(u)}, \mathrm{g}' = \boxed{\frac{(1-\sigma)t\mathrm{e}^{\frac{(1-\sigma)u}{t}}}{\sigma^2-2\sigma+1}}$

Then $\mathtt{f}' = -\sin(u), \mathtt{g} = \frac{t^2\mathrm{e}^{\frac{(1-\sigma)u}{t}}}{\sigma^2-2\sigma+1}$:

$$= \frac{(1-\sigma)t\mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)}{\sigma^2-2\sigma+1} - \left(\frac{t^2\mathrm{e}^{\frac{(1-\sigma)u}{t}} \cos(u)}{\sigma^2-2\sigma+1} - \int -\frac{t^2\mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)}{\sigma^2-2\sigma+1}\,\mathrm{d}u \right)$$

Apply linearity:

$$= \frac{(1-\sigma)t\mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)}{\sigma^2-2\sigma+1} - \left(\frac{t^2\mathrm{e}^{\frac{(1-\sigma)u}{t}} \cos(u)}{\sigma^2-2\sigma+1} + \frac{t^2}{\sigma^2-2\sigma+1} \int \mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)\,\mathrm{d}u \right)$$

As integral $\int \mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)\,\mathrm{d}u$ appears again on Right Hand Side, we can solve for it:

$$= \frac{\frac{(1-\sigma)\mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)}{t} - \mathrm{e}^{\frac{(1-\sigma)u}{t}} \cos(u)}{\frac{\sigma^2-2\sigma+1}{t^2} + 1}$$

Plug in solved integrals: $\frac{\mathrm{e}^{\frac{(\sigma-1)(4t\ln(2)+3\pi)}{4t}}}{t} \int \mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)\,\mathrm{d}u$

$$= \frac{\mathrm{e}^{\frac{(\sigma-1)(4t\ln(2)+3\pi)}{4t}} \left(\frac{(1-\sigma)\mathrm{e}^{\frac{(1-\sigma)u}{t}} \sin(u)}{t} - \mathrm{e}^{\frac{(1-\sigma)u}{t}} \cos(u) \right)}{\left(\frac{\sigma^2-2\sigma+1}{t^2} + 1\right)t}$$

Undo substitution $u = t\ln(2n) + \frac{3\pi}{4}$ and simplifying:

$$= \frac{\mathrm{e}^{\frac{(\sigma-1)(4t\ln(2)+3\pi)}{4t}} \left(\frac{(1-\sigma)\mathrm{e}^{\frac{(1-\sigma)\left(t\ln(2n)+\frac{3\pi}{4}\right)}{t}} \sin\left(t\ln(2n)+\frac{3\pi}{4}\right)}{t} - \mathrm{e}^{\frac{(1-\sigma)\left(t\ln(2n)+\frac{3\pi}{4}\right)}{t}} \cos\left(t\ln(2n)+\frac{3\pi}{4}\right) \right)}{\left(\frac{\sigma^2-2\sigma+1}{t^2} + 1\right)t}$$

Plug in solved integrals: $2^{\frac{1}{2}-\sigma}\int \frac{\sin\left(t\ln(2n)+\frac{3\pi}{4}\right)}{n^{\sigma}}\,\mathrm{d}n$

$$= \frac{2^{\frac{1}{2}-\sigma}\mathrm{e}^{\frac{(\sigma-1)(4t\ln(2)+3\pi)}{4t}}\left(\frac{(1-\sigma)\mathrm{e}^{\frac{(1-\sigma)\left(t\ln(2n)+\frac{3\pi}{4}\right)}{t}}\sin\left(t\ln(2n)+\frac{3\pi}{4}\right)}{t}-\mathrm{e}^{\frac{(1-\sigma)\left(t\ln(2n)+\frac{3\pi}{4}\right)}{t}}\cos\left(t\ln(2n)+\frac{3\pi}{4}\right)\right)}{\left(\frac{\sigma^2-2\sigma+1}{t^2}+1\right)t}$$

By rewriting and simplifying, $\int_1^{\infty}\frac{2^{\frac{1}{2}-\sigma}\sin\left(t\ln(2n)+\frac{3\pi}{4}\right)}{n^{\sigma}}dn$ is finally solved as

$$\left[\frac{(2n)^{1-\sigma}\left((t+\sigma-1)\sin(t\ln(2n))+(t-\sigma+1)\cos(t\ln(2n))\right)}{2\left(t^2+(\sigma-1)^2\right)}+C\right]_1^{\infty} \quad (7)$$

For denominator portion of R1 with $(2n-1)$ parameter in Eq. (6), Eq. (7) equates to

$$\left[\frac{(2n-1)^{1-\sigma}\left((t+\sigma-1)\sin(t\ln(2n-1))+(t-\sigma+1)\cos(t\ln(2n-1))\right)}{2\left(t^2+(\sigma-1)^2\right)}+C\right]_1^{\infty} \quad (8)$$

Dirichlet Sigma-Power Law as equation derived from Eq. (6) is given by:

$$\frac{1}{2\left(t^2+(\sigma-1)^2\right)}\cdot\left[(2n)^{1-\sigma}\left((t+\sigma-1)\sin(t\ln(2n))+(t-\sigma+1)\cos(t\ln(2n))\right)-\right.$$
$$\left.(2n-1)^{1-\sigma}\left((t+\sigma-1)\sin(t\ln(2n-1))+(t-\sigma+1)\cos(t\ln(2n-1))\right)\right]_1^{\infty}=0 \quad (9)$$

Apply Ratio Study to Eq. (6) – see Segment A3, Appendix A. This involves [intentional] incorrect but "balanced" rearrangement of terms in Eq. (6) giving rise to Eq. (10) which is a non-Hybrid integer sequence inequation. Left-hand side contains 'cyclical' sine function in first term (Ratio R1) and 'non-cyclical' power function in second term (Ratio R2).

$$\frac{\sum_{n=1}^{\infty}\sqrt{2}\sin(t\ln(2n)+\frac{3}{4}\pi)}{\sum_{n=1}^{\infty}\sqrt{2}\sin(t\ln(2n-1)+\frac{3}{4}\pi)}-\frac{\sum_{n=1}^{\infty}(2n)^{\sigma}}{\sum_{n=1}^{\infty}(2n-1)^{\sigma}}\neq 0 \quad (10)$$

Apply Riemann integral to selected parts of Eq. (10) without depicting steps of calculation:
$\int_1^{\infty}\sqrt{2}\sin\left(t\ln(2n)+\frac{3\pi}{4}\right)dn=\left[\frac{(2n)\left((t-1)\sin(t\ln(2n))+(t+1)\cos(t\ln(2n))\right)}{2(t^2+1)}+C\right]_1^{\infty}$
and $\int_1^{\infty}\sqrt{2}\sin\left(t\ln(2n-1)+\frac{3\pi}{4}\right)dn=$
$\left[\frac{(2n-1)\left((t-1)\sin(t\ln(2n-1))+(t+1)\cos(t\ln(2n-1))\right)}{2(t^2+1)}+C\right]_1^{\infty}$
$\int_1^{\infty}(2n)^{\sigma}dn=\left[\frac{(2n)^{\sigma+1}}{2(\sigma+1)}+C\right]_1^{\infty}$ and $\int_1^{\infty}(2n-1)^{\sigma}dn=\left[\frac{(2n-1)^{\sigma+1}}{2(\sigma+1)}+C\right]_1^{\infty}$
Dirichlet Sigma-Power Law as inequation derived from Eq. (10) is given by:

$$\left[\frac{(2n)\left((t-1)\sin(t\ln(2n))+(t+1)\cos(t\ln(2n))\right)}{(2n-1)\left((t-1)\sin(t\ln(2n-1))+(t+1)\cos(t\ln(2n-1))\right)}-\frac{(2n)^{\sigma+1}}{(2n-1)^{\sigma+1}}\right]_1^{\infty}\neq 0 \quad (11)$$

Intended derivation of Dirichlet Sigma-Power Law as equation and inequation have been successful. *The proof is now complete for Proposition 3.2*□.

Proposition 3.3. Exact Dimensional analysis homogeneity at $\sigma = \frac{1}{2}$ in Dirichlet Sigma-Power Law as equation and inequation is (respectively) indicated by $\sum$(all fractional exponents) = whole number '1' and '3'.

Proof. Dirichlet Sigma-Power Law as equation for $\sigma = \frac{1}{2}$ value is given by:

$$\frac{1}{2t^2+\frac{1}{2}} \cdot [(2n)^{\frac{1}{2}} \left((t-\frac{1}{2})\sin(t\ln(2n)) + (t+\frac{1}{2})\cos(t\ln(2n)) \right) - (2n-1)^{\frac{1}{2}} \left((t-\frac{1}{2})\sin(t\ln(2n-1)) + (t+\frac{1}{2})\cos(t\ln(2n-1)) \right)]_1^{\infty} = 0 \quad (12)$$

Respectively evaluation of definite integrals Eq. (12), Eq. (24) and Eq. (26) using limit as n $\to +\infty$ for $0 < t < +\infty$ enable countless computations resulting in t values for CIS of nontrivial zeros, Gram[y=0] points and Gram[x=0] points. We illustrate this for Eq. (12) as following expanded antiderivative [depicted as linear combination of sine and cosine waves: $a\sin x + b\cos x = c\sin(x+\varphi)$ with $c = \sqrt{a^2+b^2}$ and $\varphi = \text{atan2}\,(b,a) = \tan^{-1}(\frac{b}{a})$ for a>0].

$$(2\infty)^{\frac{1}{2}} \sin\left((t\ln 2\infty) + \tan^{-1}(\frac{t+\frac{1}{2}}{t-\frac{1}{2}}) \right) - (2\infty-1)^{\frac{1}{2}} \sin\left((t\ln 2\infty - 1) + \tan^{-1}(\frac{t+\frac{1}{2}}{t-\frac{1}{2}}) \right) - 2^{\frac{1}{2}} \sin\left((t\ln 2) + \tan^{-1}(\frac{t+\frac{1}{2}}{t-\frac{1}{2}}) \right) + \frac{t+\frac{1}{2}}{2t^2+\frac{1}{2}} = 0$$

The $(2\infty) > (2\infty\text{-}1)$ relationship involving exponent $\frac{1}{2}$, sin and ln functions shows that as t value solution for nontrivial zeros become larger, initial t values for first and third term tend to progressively increase but that for second and fourth term tend to progressively decrease.

Dirichlet Sigma-Power Law as inequation for $\sigma = \frac{1}{2}$ value is given by:

$$\left[\frac{(2n)\left((t-1)\sin(t\ln(2n)) + (t+1)\cos(t\ln(2n))\right)}{(2n-1)\left((t-1)\sin(t\ln(2n-1)) + (t+1)\cos(t\ln(2n-1))\right)} - \frac{(2n)^{\frac{3}{2}}}{(2n-1)^{\frac{3}{2}}} \right]_1^{\infty} \neq 0 \quad (13)$$

$\sum$(all fractional exponents) as $2(1-\sigma)$ = whole number '1' for Eq. (12) and $2(\sigma+1)$ = whole number '3' for Eq. (13). These findings signify presence of complete set of nontrivial zeros for Eq. (12) and Eq. (13). *The proof is now complete for Proposition 3.3*□.

Corollary 3.4. Inexact Dimensional analysis homogeneity at $\sigma \neq \frac{1}{2}$ [illustrated using $\sigma = \frac{2}{5}$] in Dirichlet Sigma-Power Law as equation and inequation is (respectively) indicated by $\sum$(all fractional exponents) = fractional number '$\neq$1' and '$\neq$3'.

Proof. Dirichlet Sigma-Power Law as equation for $\sigma = \frac{2}{5}$ value is given by:

$$\frac{1}{2t^2+\frac{18}{25}} \cdot [(2n)^{\frac{3}{5}} \left((t-\frac{3}{5})\sin(t\ln(2n)) + (t+\frac{3}{5})\cos(t\ln(2n)) \right) - (2n-1)^{\frac{3}{5}} \left((t-\frac{3}{5})\sin(t\ln(2n-1)) + (t+\frac{3}{5})\cos(t\ln(2n-1)) \right)]_1^{\infty} = 0 \quad (14)$$

Dirichlet Sigma-Power Law as inequation for $\sigma = \frac{2}{5}$ value is given by:

$$\left[\frac{(2n)\left((t-1)\sin(t\ln(2n)) + (t+1)\cos(t\ln(2n))\right)}{(2n-1)\left((t-1)\sin(t\ln(2n-1)) + (t+1)\cos(t\ln(2n-1))\right)} - \frac{(2n)^{\frac{7}{5}}}{(2n-1)^{\frac{7}{5}}} \right]_1^{\infty} \neq 0 \quad (15)$$

$\sum$(all fractional exponents) as $2(1-\sigma)$ = fractional number '$\neq$1' for Eq. (14) and $2(\sigma+1)$ = fractional number '$\neq$3' for Eq. (15). These findings signify absence of complete set of nontrivial zeros for Eq. (14) and Eq. (15). *The proof is now complete for Corollary 3.4*$\square$.

4 Rigorous proof for Riemann hypothesis summarized as Theorem Riemann I – IV

$\zeta(s) = \dfrac{1}{s-1} + \dfrac{1}{2} + 2\displaystyle\int_0^\infty \dfrac{\sin(s\arctan t)}{(1+t^2)^{\frac{s}{2}}\,(e^{2\pi t}-1)}\,\mathrm{d}t$ is integral relation (cf. Abel–Plana summation formula[3],[4]) for all s$\in\mathbb{C}$ and s$\neq$1. This integral is insufficient for our purpose as it involves integration using t [instead of n] for $\zeta(s)$ [instead of $\eta(s)$]. Rigorous proof for Riemann hypothesis is summarized by Theorem Riemann I – IV. One could also obtain this proof solely using Dirichlet Sigma-Power Law as equation. For completeness and further clarification of this proof, we now supply following important mathematical arguments.

For $0<\sigma<1$, then $0<2(1-\sigma)<2$. The only whole number between 0 and 2 is '1' which coincide with $\sigma=\frac{1}{2}$. When $0<\sigma<\frac{1}{2}$ and $\frac{1}{2}<\sigma<1$, then $0<2(1-\sigma)<1$ and $1<2(1-\sigma)<2$.

For $0<\sigma<1$, $2<2(\sigma+1)<4$. The only whole number between 2 and 4 is '3' which coincide with $\sigma=\frac{1}{2}$. When $0<\sigma<\frac{1}{2}$ and $\frac{1}{2}<\sigma<1$, then $2<2(\sigma+1)<3$ and $3<2(\sigma+1)<4$.

Legend: **R** = all real numbers. For $0<\sigma<1$, σ consist of $0<\mathbf{R}<1$. For $0<2(1-\sigma)<2$ and $2<2(\sigma+1)<4$, $2(1-\sigma)$ and $2(\sigma+1)$ must (respectively) consist of $0<\mathbf{R}<2$ and $2<\mathbf{R}<4$. An important caveat is that previously used phrases such as "fractional exponent σ" and "$\sum$(all fractional exponents) = whole number '1' [or '3'] and fractional number '$\neq$1' [or '$\neq$3']", although not incorrect *per se*, should respectively be replaced by "real number exponent σ" and "$\sum$(all real number exponents) = whole number '1' [or '3'] and real number '$\neq$1' [or '$\neq$3']^5" for complete accurracy. We apply this caveat to Theorem Riemann I – IV.

Footnote 5: As whole numbers $\subset$ real numbers, one could also depict this phrase as "$\sum$(all real number exponents) = real number '1' [or '3'] and real number '$\neq$1' [or '$\neq$3']".

Theorem Riemann I. Derived from *proxy* Dirichlet eta function, "simplified" Dirichlet eta function will exclusively contain *de novo* property for actual location [but not actual positions] of all nontrivial zeros.

Proof. The phrase "actual location [but not actual positions] of all nontrivial zeros" can be validly shortened to "actual location of all nontrivial zeros" as used in Theorem Riemann II, III and IV. *The proof for Theorem Riemann I is now complete as it successfully incorporates proof for Lemma 3.1*$\square$.

Theorem Riemann II. Dirichlet Sigma-Power Law [in continuous (integral) format] as equation and inequation which are both derived from "simplified" Dirichlet eta function [in discrete (summation) format] will exclusively manifest exact DA homogeneity in equation and inequation only when real number exponent $\sigma=\frac{1}{2}$.

Proof. *The proof for Theorem Riemann II is now complete as it successfully incorporates proofs from Proposition 3.2 on derivation for equation and inequation of Dirichlet Sigma-Power Law [with both containing de novo property for "actual location of all nontrivial zeros"] and Proposition 3.3 on manifestation of exact DA homogeneity in Dirichlet Sigma-Power Law as equation and inequation when real number exponent* $\sigma=\frac{1}{2}$$\square$.

Theorem Riemann III. Real number exponent $\sigma=\frac{1}{2}$ in Dirichlet Sigma-Power Law as equation and inequation satisfying exact DA homogeneity is identical to σ variable in Riemann hypothesis which propose σ to also have exclusive value of $\frac{1}{2}$ (representing critical line) for "actual location of all nontrivial zeros", thus fully supporting Riemann hypothesis to be true with further clarification by Theorem Riemann IV.

Proof. Since s = $\sigma \pm \imath t$, complete set of nontrivial zeros which is defined by $\eta(s)$ = 0 is exclusively associated with one (and only one) particular $\eta(\sigma \pm \imath t)$ = 0 value solution, and by default one (and only one) particular σ [conjecturally] = $\frac{1}{2}$ value solution. When performing exact DA homogeneity on Dirichlet Sigma-Power Law as equation and inequation [with both containing *de novo* property for "actual location of all nontrivial zeros"], the phrase "If real number exponent σ has exclusively $\frac{1}{2}$ value, only then will exact DA homogeneity be satisfied" implies one (and only one) possible mathematical solution. Theorem Riemann III reflect Theorem Riemann II on presence of exact DA homogeneity for σ = $\frac{1}{2}$ in Dirichlet Sigma-Power Law as equation and inequation. This Law has identical σ variable as that referred to by Riemann hypothesis [whereby σ here uniquely refer to critical line]. *The proof for Theorem Riemann III is now complete as it independently refers to simultaneous association of confirmed (i) solitary σ = $\frac{1}{2}$ value in Dirichlet Sigma-Power Law as equation and inequation satisfying exact DA homogeneity and (ii) critical line defined by solitary σ = $\frac{1}{2}$ value being the "actual location [but with no request to determine actual positions]" of all nontrivial zeros as proposed in original Riemann hypothesis*□.

Theorem Riemann IV. Condition 1. All $\sigma \neq \frac{1}{2}$ values (non-critical lines), viz. $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$ values, exclusively does not contain "actual location of all nontrivial zeros" [manifesting *de novo* inexact DA homogeneity in equation and inequation], together with Condition 2. One (and only one) $\sigma = \frac{1}{2}$ value (critical line) exclusively contains "actual location of all nontrivial zeros" [manifesting *de novo* exact DA homogeneity in equation and inequation], fully support Riemann hypothesis to be true when these two mutually inclusive conditions are met.

Proof. Condition 2 Theorem Riemann IV simply reflect proof from Theorem Riemann III [with Proposition 3.3 incorporated] for "actual location of all nontrivial zeros" exclusively on critical line manifesting *de novo* exact DA homogeneity as $\sum$(all real number exponents) = whole number '1' for equation [or '3' for inequation]. *The proof for Condition 2 Theorem Riemann IV is now complete*□. Corollary 3.4 confirms *de novo* inexact DA homogeneity manifested as $\sum$(all real number exponents) = real number '$\neq$1' for equation [or '$\neq$3' for inequation] by all $\sigma \neq \frac{1}{2}$ values (non-critical lines) that are exclusively not associated with "actual location of all nontrivial zeros". Applying inclusion-exclusion principle: Exclusive presence of nontrivial zeros on critical line for Condition 2 Theorem Riemann IV will now confirm exclusive absence of nontrivial zeros on all non-critical lines for Condition 1 Theorem Riemann IV. *The proof for Condition 1 Theorem Riemann IV is now complete*□.

We logically deduce that explicit mathematical explanation why presence and absence of nontrivial zeros[6] should (respectively) coincide precisely with $\sigma = \frac{1}{2}$ and $\sigma \neq \frac{1}{2}$ [literally the Completely Predictable meta-properties ('overall' *complex properties*)] will require "complex" mathematical arguments. Attempting to provide explicit mathematical explanation with "simple" mathematical arguments would intuitively mean nontrivial zeros have to be (incorrectly and impossibly) treated as Completely Predictable entities.

Footnote 6: Equivalent Completely Predictable meta-properties for Gram and virtual Gram points equate to "Presence of Gram[y=0] and Gram[x=0] points, and virtual Gram[y=0] and virtual Gram[x=0] points (respectively) coincide precisely with $\sigma = \frac{1}{2}$, and $\sigma \neq \frac{1}{2}$".

5 Conclusions

In Hybrid method of Integer Sequence classification, a formula contains non-Hybrid integer sequence or Hybrid integer sequence. Inequation with two 'necessary' Ratio (R) or equation with one 'unnecessary' R contains non-Hybrid integer sequence. Equation with one 'necessary' R contains Hybrid integer sequence. "In the limit" Hybrid integer sequence approach unique Position X, it becomes non-Hybrid integer sequence for all Positions $\geq$ Position X.

Relativistic KE is approximated well by Newtonian KE at low speed. This is obtained from Relativistic KE by binomial approximation or by taking first two terms of Taylor expansion for reciprocal square root. "In the limit" ['$<$100% accuracy'] Newtonian KE at low speed approach ['100% accuracy'] Relativistic KE at high speed, we achieve *perfection*.

Analogy: "In the limit" all three version of Dirichlet Sigma-Power Laws for Gram[y=0] points, Gram[x=0] points and nontrivial zeros as *'$<$100% accuracy'* inequations approach *perfection* as *'100% accuracy'* equations, compliance with inexact DA homogeneity becomes compliance with exact DA homogeneity. We note R1 terms in all inequations contain (2n) and (2n-1) 'base quantities' but these are not endowed with fractional exponent (σ+1) as relevant 'unit of measurement'. In this research paper, we provide relatively elementary proof of Riemann hypothesis and explain closely related Gram points. Harnessed key benefit from successful proof for Riemann hypothesis is often stated as "With this one solution, we have proven five hundred theorems or more at once". This apply to important theorems in number theory that rely on properties of Riemann zeta function or Dirichlet eta function such as location of trivial and nontrivial zeros. E.g., we can delineate prime number theorem by prime counting function $\pi(x)$ [which is defined as number of primes $\leq$ x].

Appendix A: Definitions and Supplementary materials

Exposition on definitions and related commentaries is crucial to help solve Riemann hypothesis and explain closely related Gram points as Incompletely Predictable problems.

Segment A1. *Completely Predictable and Incompletely Predictable numbers*

Completely Unpredictable numbers arising from totally chaotic physical processes give rise to countable infinite set (CIS) of measured true random numbers. In this sense, computational pseudorandom number generators using some deterministic logic are not regarded as sources for true random numbers. Two types of Predictable numbers: CIS of Completely, and CIS of Incompletely Predictable numbers with former "contained" in *simple* equations or algorithms obeying '*Simple* Elementary Fundamental Laws', and later "contained" in *complex* equations or algorithms obeying '*Complex* Elementary Fundamental Laws'.

A Completely, and Incompletely Predictable number is locationally defined as a number whose position is *independently* determined by simple calculations using simple equation or algorithm without, and *dependently* by complex calculations using complex equation or algorithm with needing to know related positions of all preceding numbers in neighborhood. Both types of Predictable number exist as either rational [integers or fractions of two integers] numbers (**Q**) or irrational [algebraic or transcendental] numbers (**R – Q**). A well-defined set of **R – Q** will twice obey relevant location definition in CIS **R – Q** themselves and in CIS numerical digits after decimal point of each **R – Q**.

97 is an Incompletely Predictable number whose precise position is determined by computing positions of all preceding 24 prime numbers (**P**) using complex algorithm Sieve of Eratosthenes to conclude that 97 is 25^{th} **P**. Calculated using simple algorithm, 97 is also [i = (97+1)/2] 49^{th} odd number (**O**) which is a Completely Predictable number. 98 & 99 are respectively [i = 98/2] 49^{th} even number (**E**) & [i = (99+1)/2] 50^{th} **O** which are Completely Predictable numbers calculated using simple algorithm. Determined indirectly using complex algorithm Sieve of Erastosthenes, 98 & 99 are respectively also 72^{nd} & 73^{rd} composite numbers (**C**) which are Incompletely Predictable numbers.

Computing Riemann zeta function (or specifically its *proxy* Dirichlet eta function) and Sieve of Eratosthenes will, respectively, supply Incompletely Predictable nontrivial zeros, Gram[y=0] & Gram[x=0] points and **P** & **C**. CIS of nontrivial zeros (denoted by imaginary part parameter t) = CIS of transcendental numbers = 14.134725, 21.022040, 25.010858, 30.424876, 32.935062, 37.586178,... [rounded off to six decimal places]. CIS of all **P** = Countable Finite Set (CFS) of all even **P** + CIS of all odd **P** = 2, 3, 5, 7, 11, 13,... whereby **P** '2' when treated as **E** is also regarded as a Completely Predictable number.

The three sets of nontrivial zeros, Gram[y=0] points and Gram[x=0] points, respectively, will *dependently* constitute three sets of Origin intercepts (or simultaneous x- & y-axes intercepts), x-axis intercepts and y-axis intercepts. Traditional 'Gram points' [see Segment A2 below] are x-axis intercepts with choice of index 'n' for 'Gram points' historically chosen such that first 'Gram point' corresponds to t value which is larger than (first) nontrivial zero located at t = 14.134725. By convention, first six Gram[y=0] points will occur with following values [rounded off to six decimal places]: at n = -3, t = 0; at n = -2, t = 3.436218; at n = -1, t = 9.666908; at n = 0, t = 17.845599; at n = 1, t = 23.170282; at n = 2, t = 27.670182.

The two sets of **P** 2, 3, 5, 7, 11, 13,... and **C** 4, 6, 8, 9, 10, 12,... will *dependently* constitute set of natural numbers (**N**) 1, 2, 3, 4, 5, 6,... minus first **N** '1'. Whole numbers (**W**) = **N** plus '0'. '0' & '1' are special numbers being neither **P** nor **C** as they represent nothingness (zero) and wholeness (one), and the idea of having factors for '0' & '1' is meaningless. Treating '0' & '1' here as Completely or Incompletely Predictable numbers is also meaningless.

CIS of numbers derived from well-defined simple/complex algorithms or equations are "dual numbers" displayed as Completely & Incompletely Predictable number. Examples of **Q** '2' as **P** (& **E**), '97' as **P** (& **O**), '98' as **C** (& **E**) and '99' as **C** (& **O**) are described above. Examples of **R – Q** are descibed next. First & only negative Gram[y=0] point (by convention at n = -3) with Completely Predictable y = 0 value is obtained by substituting Completely Predictable t = 0 resulting in $\zeta(\frac{1}{2}+\imath t)=\zeta(\frac{1}{2})$ = -1.4603545, an Incompletely Predictable transcendental number [rounded off to seven decimal places] calculated as a limit similar to limit for Euler-Mascheroni constant or Euler gamma – its precise (1^{st}) position can only be determined by computing positions of all preceding (nil) Gram[y=0] points in this case. With exception of this first Gram[y=0] point, all t values from Gram[y=0] points, Gram[x=0] points, and nontrivial zeros (Gram[x=0,y=0] points) are Incompletely Predictable transcendental numbers – these are respectively associated with Completely Predictable x = 0, y = 0, and simultaneous x = 0 & y = 0 values. First 'Gram point' (by convention at n = 0 & is associated with Completely Predictable x = 0 value from Incompletely Predictable t = 17.845599 substitution) is actually the 4^{th} Gram[y=0] point whose precise (4^{th}) position can only be determined by computing positions of all preceding (three) Gram[y=0] points in this case. First nontrivial zero associated with simultaneous x = 0 & y = 0 value [equating to ζ(s) = 0 whereby s = $\sigma+\imath t=\frac{1}{2}+\imath t$] is Completely Predictable occurring with Incompletely Predictable t = 14.134725 value substitution – its precise (1^{st}) position can only be determined by computing positions of all preceding (nil) nontrivial zeros in this case.

Remark A.1. Countable finite set (CFS) of Completely Predictable *simple properties* intrinsically present in simple equations or algorithms help us solve Completely Predictable problems containing countable infinite set (CIS) of Completely Predictable numbers; whereas CFS of Completely Predictable *complex properties* intrinsically present in complex equations or algorithms help us solve Incompletely Predictable problems containing CIS of Incompletely Predictable numbers.

Simple properties are inferred from a phrase like: "...the simple equation or algorithm by itself will intrinsically incorporate actual location [and actual positions] of all Completely Predictable numbers". Solving Completely Predictable problems endowed with simple properties which are amendable to *simple* treatments using *usual* mathematical tools such as Calculus will result in their 'Simple Elementary Fundamental Laws'-based solutions. Complex properties are inferred from a phrase like: "...the complex equation or algorithm by itself will intrinsically incorporate actual location [but not actual positions] of all Incompletely Predictable numbers". Solving Incompletely Predictable problems endowed with complex properties which are amendable to *complex* treatments using *unusual* mathematical tools such as deriving complex equation Dirichlet Sigma-Power Law as well as using *usual* mathematical tools such as Calculus will result in their 'Complex Elementary Fundamental Laws'-based solutions.

Consider x for real number (**R**) values ≥ 1. Let y be Set **R** such that (simple equation) y = 2x or y = 2x - 1. This simple equation will "contain" the complete uncountable infinite set (UIS) of **R** [straight line of infinite length] commencing from Cartesian point (x=1, y=2) or (x=1, y=1). Computing y = 2x or y = 2x - 1 an infinite number of times – a *mathematical impasse* – will not *per se* result in its 'Simple Elementary Fundamental Laws'-based solution for gradient or slope = 2. This gradient (simple property) is obtained by trigonometrically calculating tangent of y = 2x or y = 2x - 1 straight line which = 2 or analyzing y = 2x or y = 2x - 1 equation using Differential Calculus viz. dy/dx = d(2x)/dx or d(2x-1)/dx = 2. Note: applying Integral Calculus from Fundamental Theorem of Calculus to y = 2x or y = 2x - 1 for interval [1, +∞], viz. $\int_1^\infty(2x)dx$ or $\int_1^\infty(2x - 1)dx = [x^2 + C]_1^\infty$ or $[x^2 - x + C]_1^\infty$ = $(\infty^2 + C) - (1^2 + C)$ or $(\infty^2 - \infty + C) - (1^2 - 1 + C) = \infty$ result in 'Simple Elementary Fundamental Laws'-based solution for area (simple property) of infinite size enclosed by the straight line and x-axis.

Consider x≥1 integer number (**Z**) values for (simple algorithm) y = 2x or y = 2x - 1. We obtain "contained" complete Set **E** or Set **O**. Computing **E** or **O** infinitely often – a *mathematical impasse* – will not *per se* result in 'Simple Elementary Fundamental Laws'-based solution for gap between any two consecutive **E** (**E** gap) or **O** (**O** gap) will both = 2. This gradient-equivalent **E** gaps or **O** gaps (simple property) is obtained by transforming those algorithms from their "discrete" into "continuous" format [viz. "discrete" Δx = 1 $\longrightarrow$ "continuous" Δx = 0] resulting in their gradients using either tangent method or Differential Calculus method. Then **E** or **O** gaps, both = 2, is numerically identical and mathematically equivalent to relevant gradients, both also = 2. Similar method of transforming from "discrete" into "continuous" format to help solve Riemann hypothesis involves applying Riemann integral to discrete-like equation of "simplified" Dirichlet eta function (in summation format) to obtain Dirichlet Sigma-Power Law [which is the continuous-like equation of "simplified" Dirichlet eta function (in integral format)].

Similar to Incompletely Predictable 'varying gaps' [equivalent to 'varying gradients'] between consecutive **P** (**P** gaps) & consecutive **C** (**C** gaps) [relevant to research on Polignac's and Twin prime conjectures], we have Incompletely Predictable 'varying gaps' [equivalent to 'varying gradients'] between consecutive nontrivial zeros (nontrivial zero gaps), consecutive Gram[y=0] points (Gram[y=0] points gaps) & consecutive Gram[x=0] points (Gram[x=0] points gaps). These 'varying gaps' or 'varying gradients' (complex properties) are geometrically related to different shapes/sizes of spirals depicted in Figure 2.

Segment A2. *Gram's Law and traditional 'Gram points'*

Named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 – April 29, 1916), traditional 'Gram points' (Gram[y=0] points) are other conjugate pairs values on critical line defined by $Im\{\zeta(\frac{1}{2} \pm \imath t)\} = 0$. They obey Gram's Rule and Rosser's Rule with interesting characteristic properties as outlined by our brief exposition below.

Z function is used to study Riemann zeta function on critical line. Defined in terms of Riemann-Siegel theta function & Riemann zeta function by $Z(t) = e^{\imath\theta(t)}\zeta(\frac{1}{2} + \imath t)$ whereby

$\theta(t) = \arg(\Gamma(\frac{(2\imath t+1)}{4})) - \frac{\log\pi}{2}t$; it is also called Riemann-Siegel Z function, Riemann-Siegel zeta function, Hardy function, Hardy Z function, & Hardy zeta function.

The algorithm to compute Z(t) is called Riemann-Siegel formula. Riemann zeta function on critical line, $\zeta(\frac{1}{2} + \imath t)$, will be real when $\sin(\theta(t)) = 0$. Positive real values of t where this occurs are called 'Gram points' and can also be described as points where $\frac{\theta(t)}{\pi}$ is an integer. Real part of this function on critical line tends to be positive, while imaginary part alternates more regularly between positive & negative values. That means sign of Z(t) must be opposite to that of sine function most of the time, so one would expect nontrivial zeros of Z(t) to alternate with zeros of sine term, i.e. when θ takes on integer multiples of π. This turns out to hold most of the time and is known as Gram's Rule (Law) – a law which is violated infinitely often though. Thus Gram's Law is statement that nontrivial zeros of Z(t) alternate with 'Gram points'. 'Gram points' which satisfy Gram's Law are called 'good', while those that do not are called 'bad'. A Gram block is an interval such that its very first & last points are good 'Gram points' and all 'Gram points' inside this interval are bad. Counting nontrivial zeros then reduces to counting all 'Gram points' where Gram's Law is satisfied and adding the count of nontrivial zeros inside each Gram block. With this process we do not have to locate nontrivial zeros, and we just have to accurately compute Z(t) to show that it changes sign.

Segment A3. *Ratio Study and Inequations*

A mathematical equation, containing one or more variables, is a statement that values of two ['left-hand side' (LHS) and 'right-hand side' (RHS)] mathematical expressions is related as equality: LHS = RHS; or as inequalities: LHS $<$ RHS, LHS $>$ RHS, LHS $\leq$ RHS, or LHS $\geq$ RHS. A ratio is one mathematical expression divided by another. The term 'unnecessary' Ratio (R) for any given equation is explained by two examples: (1) LHS = RHS and with rearrangement, 'unnecessary' R is given by $\frac{LHS}{RHS} = 1$ or $\frac{RHS}{LHS} = 1$; and (2) LHS $>$ RHS and with rearrangement, 'unnecessary' R is given by $\frac{LHS}{RHS} > 1$ or $\frac{RHS}{LHS} < 1$.

Consider exponent y $\in$ all **R** values & base x $\in$ **R**$\geq$0 values for mathematical expression x^y. Equations such as $x^1 = x$, $x^0 = 1$ & $0^y = 0$ are all valid. Simultaneously letting both x & y = 0 is an incorrect mathematical action because x^y as function of two-variables is not continuous & is thus undefined at Origin. But if we elect to intentionally carry out this "balanced" action [equally] on x & y, we obtain (simple) inequation $0^0 \neq 1$ with associated perpetual obeyance of '=' equality symbol in x^y for all applicable **R** values except when both x & y = 0. The Number '1' value in this inequation is justified by two arguments: I. Limit of x^y value as both x & y tend to zero (from right) is 1 [thus fully satisfying criterion "x^y is right continuous at the Origin"]; and II. Expression x^y is product of x with itself y times [and thus x^0, the "empty product", should be 1 (no matter what value is given to x)].

Mathematical operator 'summation' must obey the law: We can break up a summation across a sum or difference but not across a product or quotient viz, factoring a sum of quotients into a corresponding quotient of sums is an incorrect mathematical action. But if we elect to carry out this action equally on LHS & RHS products or quotients in a suitable equation, we obtain two (unique) 'necessary' R denoted by R1 for LHS and R2 for RHS whereby R1 $\neq$ R2 relationship will always hold. We define 'Ratio Study' as intentionally performing this incorrect [but "balanced"] mathematical action on suitable equation [equivalent to one (non-unique) 'unnecessary' R] to obtain its inequation [equivalent to two (unique) 'necessary' R]. Set $\mathbb{C}$ is a field (but not an ordered field). Thus it is not possible to define a relation between two given (z_1 & z_2) $\mathbb{C}$ as $z_1 < z_2$ since the inequality operation here is not compatible with addition and multiplication. But performing Ratio Study to obtain inequations involving $\mathbb{C}$ does not involve defining a relation between two $\mathbb{C}$.

Appendix B: Prerequisite lemma, corollary and propositions for Gram points

For Gram[y=0] and Gram[x=0] points (and corresponding virtual Gram[y=0] and virtual Gram[x=0] points with totally different values), we apply a parallel procedure carried out on nontrivial zeros but only depict abbreviated treatments and discussions.

Lemma B.1. "Simplified" Gram[y=0] and Gram[x=0] points-Dirichlet eta functions are derived directly from Dirichlet eta function with Euler formula application and (respectively) they will intrinsically incorporate actual location [but not actual positions] of all Gram[y=0] and Gram[x=0] points.

Proof. For Gram[y=0] points, the equivalent of Eq. (4) and Eq. (6) are respectively given by Eq. (16) and Eq. (17) below.

$$\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + 0, \text{ or simply } Im\{\eta(s)\} = 0 \tag{16}$$

$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\sin(t\ln(2n)) = \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\sin(t\ln(2n-1))$$
$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\sin(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\sin(t\ln(2n-1)) = 0 \tag{17}$$

For Gram[x=0] points, the equivalent of Eq. (4) and Eq. (6) are respectively given by Eq. (18) and Eq. (19) below.

$$\sum ReIm\{\eta(s)\} = 0 + Im\{\eta(s)\}, \text{ or simply } Re\{\eta(s)\} = 0 \tag{18}$$

$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\cos(t\ln(2n)) = \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\cos(t\ln(2n-1))$$
$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\cos(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\cos(t\ln(2n-1)) = 0 \tag{19}$$

Eq. (17) and Eq. (19) being the "simplified" Gram[y=0] and Gram[x=0] points-Dirichlet eta functions derived directly from $\eta(s)$ will intrinsically incorporate *actual location [but not actual positions]* of (respectively) all Gram[y=0] and Gram[x=0] points. *The proof is now complete for Lemma B.1*$\square$.

Proposition B.2. Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws in continuous (integral) format given as equations and inequations can both be (respectively) derived directly from "simplified" Gram[y=0] and Gram[x=0] points-Dirichlet eta functions in discrete (summation) format with Riemann integral application.

Proof. Antiderivatives using (2n) parameter [not previously given] will help obtain all subsequent equations: first two for Gram[y=0] points and second two for Gram[x=0] points.

$$\int_1^{\infty}(2n)^{-\sigma}\sin(t\ln(2n))dn = \left[-\frac{(2n)^{1-\sigma}\left((\sigma-1)\sin(t\ln(2n)) + t\cos(t\ln(2n))\right)}{2\left(t^2+(\sigma-1)^2\right)} + C\right]_1^{\infty}$$
$$\int_1^{\infty}\sin(t\ln(2n))dn = \left[\frac{(2n)(\sin(t\ln(2n)) - t\cos(t\ln(2n)))}{2(t^2+1)} + C\right]_1^{\infty}$$
$$\int_1^{\infty}(2n)^{-\sigma}\cos(t\ln(2n))dn = \left[\frac{(2n)^{1-\sigma}\left(t\sin(t\ln(2n)) - (\sigma-1)\cos(t\ln(2n))\right)}{2\left(t^2+(\sigma-1)^2\right)} + C\right]_1^{\infty}$$

$$\int_1^\infty \cos(t\ln(2n))dn = \left[\frac{(2n)(t\sin(t\ln(2n))+\cos(t\ln(2n)))}{2(t^2+1)}+C\right]_1^\infty$$

For Gram[y=0] points-Dirichlet Sigma-Power Law, the equivalent of Eq. (9) and Eq. (11) are respectively given by Eq. (20) as equation and Eq. (21) as inequation.

$$-\frac{1}{2\left(t^2+(\sigma-1)^2\right)}\cdot[(2n)^{1-\sigma}((\sigma-1)\sin(t\ln(2n))+t\cos(t\ln(2n)))- \\ (2n-1)^{1-\sigma}((\sigma-1)\sin(t\ln(2n-1))+t\cos(t\ln(2n-1)))]_1^\infty=0 \quad (20)$$

$$\left[\frac{(2n)(\sin(t\ln(2n))-t\cos(t\ln(2n)))}{(2n-1)(\sin(t\ln(2n-1))-t\cos(t\ln(2n-1)))}-\frac{(2n)^{\sigma+1}}{(2n-1)^{\sigma+1}}\right]_1^\infty\neq 0 \quad (21)$$

For Gram[x=0] points-Dirichlet Sigma-Power Law, the equivalent of Eq. (9) and Eq. (11) are respectively given by Eq. (22) as equation and Eq. (23) as inequation.

$$\frac{1}{2\left(t^2+(\sigma-1)^2\right)}\cdot[(2n)^{1-\sigma}(t\sin(t\ln(2n))-(\sigma-1)\cos(t\ln(2n)))- \\ (2n-1)^{1-\sigma}(t\sin(t\ln(2n-1))-(\sigma-1)\cos(t\ln(2n-1)))]_1^\infty=0 \quad (22)$$

$$\left[\frac{(2n)(t\sin(t\ln(2n))+\cos(t\ln(2n)))}{(2n-1)(t\sin(t\ln(2n-1))+\cos(t\ln(2n-1)))}-\frac{(2n)^{\sigma+1}}{(2n-1)^{\sigma+1}}\right]_1^\infty\neq 0 \quad (23)$$

Intended derivation of Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws as equations and inequations have been successful. *The proof is now complete for Lemma B.2*□.

Proposition B.3. Exact Dimensional analysis homogeneity at $\sigma=\frac{1}{2}$ in Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws as equations and inequations are (respectively) indicated by ∑(all fractional exponents) = whole number '1' and 3'.

Proof. Gram[y=0] points-Dirichlet Sigma-Power Law as equation for $\sigma=\frac{1}{2}$ value is given by:

$$-\frac{1}{2t^2+\frac{1}{2}}\cdot[(2n)^{\frac{1}{2}}\left(t\cos(t\ln(2n))-\frac{1}{2}\sin(t\ln(2n))\right)- \\ (2n-1)^{\frac{1}{2}}\left(t\cos(t\ln(2n-1))-\frac{1}{2}\sin(t\ln(2n-1))\right)]_1^\infty=0 \quad (24)$$

Gram[y=0] points-Dirichlet Sigma-Power Law as inequation for $\sigma=\frac{1}{2}$ value is given by:

$$\left[\frac{(2n)(\sin(t\ln(2n))-t\cos(t\ln(2n)))}{(2n-1)(\sin(t\ln(2n-1))-t\cos(t\ln(2n-1)))}-\frac{(2n)^{\frac{3}{2}}}{(2n-1)^{\frac{3}{2}}}\right]_1^\infty\neq 0 \quad (25)$$

Gram[x=0] points-Dirichlet Sigma-Power Law as equation for $\sigma = \frac{1}{2}$ value is given by:

$$\frac{1}{2t^2+\frac{1}{2}} \cdot [(2n)^{\frac{1}{2}} \left(t\sin(t\ln(2n)) + \frac{1}{2}\cos(t\ln(2n))\right) - (2n-1)^{\frac{1}{2}} \left(t\sin(t\ln(2n-1)) + \frac{1}{2}\cos(t\ln(2n-1))\right)]_1^{\infty} = 0 \tag{26}$$

Gram[x=0] points-Dirichlet Sigma-Power Law as inequation for $\sigma = \frac{1}{2}$ value is given by:

$$\left[\frac{(2n)(t\sin(t\ln(2n)) + \cos(t\ln(2n)))}{(2n-1)(t\sin(t\ln(2n-1)) + \cos(t\ln(2n-1)))} - \frac{(2n)^{\frac{3}{2}}}{(2n-1)^{\frac{3}{2}}}\right]_1^{\infty} \neq 0 \tag{27}$$

$\sum$(all fractional exponents) as $2(1-\sigma)$ = whole number '1' for Eq. (24) and Eq. (26), and $2(\sigma+1)$ = whole number '3' for Eq. (25) and Eq. (27). These findings signify presence of complete sets of Gram[y=0] points for Eq. (24) and Eq. (25) and Gram[x=0] points for Eq. (26) and Eq. (27). *The proof is now complete for Proposition B.3*$\square$.

Corollary B.4. Inexact Dimensional analysis homogeneity at $\sigma \neq \frac{1}{2}$ [illustrated using $\sigma = \frac{2}{5}$] in Gram[y=0] and Gram[x=0] points-Dirichlet Sigma-Power Laws as equations and inequations are (respectively) indicated by $\sum$(all fractional exponents) = fractional number '$\neq$1' and '$\neq$3'.

Proof. Gram[y=0] points-Dirichlet Sigma-Power Law as equation for $\sigma = \frac{2}{5}$ value is given by:

$$-\frac{1}{2t^2+\frac{18}{25}} \cdot [(2n)^{\frac{3}{5}} \left(t\cos(t\ln(2n)) - \frac{3}{5}\sin(t\ln(2n))\right) - (2n-1)^{\frac{3}{5}} \left(t\cos(t\ln(2n-1)) - \frac{3}{5}\sin(t\ln(2n-1))\right)]_1^{\infty} = 0 \tag{28}$$

Gram[y=0] points-Dirichlet Sigma-Power Law as inequation for $\sigma = \frac{2}{5}$ value is given by:

$$\left[\frac{(2n)(\sin(t\ln(2n)) - t\cos(t\ln(2n)))}{(2n-1)(\sin(t\ln(2n-1)) - t\cos(t\ln(2n-1)))} - \frac{(2n)^{\frac{7}{5}}}{(2n-1)^{\frac{7}{5}}}\right]_1^{\infty} \neq 0 \tag{29}$$

Gram[x=0] points-Dirichlet Sigma-Power Law as equation for $\sigma = \frac{2}{5}$ value is given by:

$$\frac{1}{2t^2+\frac{18}{25}} \cdot [(2n)^{\frac{3}{5}} \left(t\sin(t\ln(2n)) + \frac{3}{5}\cos(t\ln(2n))\right) - (2n-1)^{\frac{3}{5}} \left(t\sin(t\ln(2n-1)) + \frac{3}{5}\cos(t\ln(2n-1))\right)]_1^{\infty} = 0 \tag{30}$$

Gram[x=0] points-Dirichlet Sigma-Power Law as inequation for $\sigma = \frac{2}{5}$ value is given by:

$$\left[\frac{(2n)(t\sin(t\ln(2n)) + \cos(t\ln(2n)))}{(2n-1)(t\sin(t\ln(2n-1)) + \cos(t\ln(2n-1)))} - \frac{(2n)^{\frac{7}{5}}}{(2n-1)^{\frac{7}{5}}}\right]_1^{\infty} \neq 0 \tag{31}$$

$\sum$(all fractional exponents) as $2(1-\sigma)$ = fractional number '$\neq$1' for Eq. (28) and Eq. (30), and $2(\sigma+1)$ = fractional number '$\neq$3' for Eq. (29) and Eq. (31). These findings signify presence of complete sets of virtual Gram[y=0] points for Eq. (28) and Eq. (29) and virtual Gram[x=0] points for Eq. (30) and Eq. (31). *The proof is now complete for Corollary B.4*$\square$.

Appendix C: Hybrid method of Integer Sequence classification

The Hybrid method of Integer Sequence classification enables meaningful division of all integer sequences into either Hybrid or non-Hybrid integer sequences. Our exotic A228186 integer sequence[5] was published on The On-line Encyclopedia of Integer Sequences website in 2013. It is the first ever [infinite length] Hybrid integer sequence synthesized from Combinatorics Ratio. In 'Position i' notation, let i = 0, 1, 2, 3, 4, 5,..., ∞ be complete set of natural numbers. A228186 "Greatest k $>$ n such that ratio R $<$ 2 is a maximum rational number with R = $\frac{CombinationsWithRepetition}{CombinationsWithoutRepetition}$" is equal to [infinite length] non-Hybrid (usual garden-variety) integer sequence A100967[6] except for finite 21 'exceptional' terms at Positions 0, 11, 13, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their values given by relevant A100967 terms plus 1. The first 49 terms [from Position 0 to Position 48] of A100967 "Least k such that binomial(2k+1, k-n) $\geq$ binomial(2k, k)" are listed below: 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091, 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366, 2485, 2606, 2730, 2857, 2987, 3119, 3255, 3394, and 3535. For those 21 'exceptional' terms: at Position 0, A228186 (= 4) is given by A100967 (= 3) + 1; at Position 11, A228186 (= 226) is given by A100967 (= 225) + 1; at Position 13, A228186 (= 304) is given by A100967 (= 303) + 1; at Position 19, A228186 (= 607) is given by A100967 (= 606) + 1; etc. Here is a useful concept: Commencing from Position 0 onwards "in the limit" that this Position approaches 82, A228186 Hybrid integer sequence becomes (and is identical to) A100967 non-Hybrid integer sequence for all Positions $\geq$ 82.

Acknowledgments The author is indebted to Civil Engineer and Mathematician Mr. Rodney Williams, and Software Engineer Mr. Tony O'Hagan in Australia for advices on this research paper (dedicated to his daughter Jelena born 13 weeks early on May 14, 2012).

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