Refutation of the Gödel class with identity as un-solvable

Abstract: We evaluate the Gödel quantification scheme of $\exists{*}\forall^{n}\exists{*}\varphi$ as tautologous. When it is extended to decidable and undecidable prefix-classes, none is tautologous. This refutes the Gödel class with identity as undecidable, to mean it is in fact solvable as *not* tautologous. Therefore the prefix-classes are *non* tautologous fragments of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **`F`** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪ ; - Not Or; & And, ∧, ∩, · ; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ⤚ ;
< Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↚, ≲;
= Equivalent, ≡, :=, ⟺, ↔, ≜, ≈, ≃; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **`F`** as contradiction, ∅, Null, ⊥, zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1;
(%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆y); (A=B) (A~B); (B>A) (A⊢B); (B>A) (A⊨B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

See: Marcos, J. (2016). Breaking the proof code. youtube.com/watch?v=XykVjsweqpc

The class of sentences of the form

$\exists{*}\forall^{n}\exists{*}\varphi$, where φ is quantifier free, is decidable if (and only if) $n\leq 2$. (1.0)

Remark 1.0: We take $\exists{*}$ to mean zero or more existential quantifiers, and $n\leq 2$ to mean $n=\{0, 1, 2\}$. We rewrite (1.0) by inserting the variables t, u, v, w, x, y, z as needed:

$\exists x\forall^{n}y\exists z\varphi$, where φ is quantifier free, is decidable if (and only if) $n\leq 2$. (1.1)

LET p, q, r, s, t, u, v, w, x, y, z:
φ, q, n,*, t, u, v, w, w, y, z;
* > 0; n ≤ 2; (p@p) 0; (p=p) 3.

(~(s<(q@q))&~(r>(q=q)))>(((%s&x)&((#r&y)&(%s&z)))&p) ;
`TTTT TTTT TTTT TTTT` (1.2)

Remark 1.2: Eq. 1.2 as rendered is tautologous, hence confirming Gödel's asserted proof. However, the antecedent is a contradiction, **`FFFF`**, causing any consequent (here in part **`F`**`N`**`F`**`N`) to imply tautology.

From: Goldfarb, W.D. (1984). The Gödel class with identity is unsolvable.
academia.edu/31504009/The_Godel_Class_with_Identity_is_Unsolvable
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For example, let $G = \forall x \exists u \forall y K$ be any $\forall\exists\forall$-formula of pure quantification theory; we may suppose that the predicate letters of G are distinct from those of F. A straightforward argument shows that G is satisfiable if and only if $F \wedge \forall x \forall y \exists u (Sux \wedge K)$ is satisfiable; and the latter formula has a prenex equivalent in the GCI [Gödel class with identity]. Since the class of $\forall\exists\forall$-formulas is undecidable, we obtain the THEOREM. *The Gödel Class with Identity is undecidable*. (2.1.1) (2.2.1) (2.3.1)

LET p, q, r, s, u, x, y:
F,G,K,S, u, x, y.

q=((#x&(%u&#y))&r) ; **FFFF FFFF FFFF FFF**N(128) (2.1.2)

p&(((#x&#y)&%u)&((s&(u&x))&r)) ;

```
FFFF FFFF FFFF FFFF(52),
FNFN FNFN FNFN FNFN( 2),
FFFF FFFF FFFF FFFF( 2),
FNFN FNFN FNFN FNFN( 2),
FFFF FFFF FFFF FFFF( 2),
FNFN FNFN FNFN FNFN( 2),
FFFF FFFF FFFF FFFF( 2),
FNFN FNFN FNFN FNFN( 2)
```
(2.2.2)

Remark 2.3.1: The conjecture is Eqs. 2.2.1 implies 2.2.1: (2.3.1)

(p&(((#x&#y)&%u)&((s&(u&x))&r))) > (q=((#x&(%u&#y))&r)) ;

```
TTTT TTTT TTTT TTTT(52),
TTTT TTTT TTTT TTCT( 2),
TTTT TTTT TTTT TTTT( 2),
TTTT TTTT TTTT TTCT( 2),
TTTT TTTT TTTT TTTT( 2),
TTTT TTTT TTTT TTCT( 2),
TTTT TTTT TTTT TTTT( 2),
TTTT TTTT TTTT TTCT( 2)
```
(2.3.2)

Remark 2.3.2: Eq. 2.3.2 is *not* tautologous, but *not* for the reason as claimed that 2.2.1 is satisfiable (decidable).

The theorem may be sharpened. Using several additional predicate letters, we may construct an infinity axiom and encode $\forall\exists\forall$-formulas while using only one existential quantifier. Hence the Minimal GCI, i.e., the class of formulas with prefixes $\forall x \forall y \exists z$, is undecidable. This settles the decision problem for all prefix-classes of quantification theory with identity, for we now have the following division: (3.0)

Remark 3.0: We rewrite Eqs. 4.1-7.1 by inserting variables as in Remark 1.0.

Decidable prefix-classes: ∃•••∃∀•••∀ and (4.1.1)

∃•••∃∀∃•••∃. (4.2.1)

%p&((%q&#r)&#s) ; and **FFFF FFFF FFFF FFF**N (4.1.2)

%p&((%q&(#r&%s))&%t) ; **FFFF FFFF FFFF FFFF**(2),
FFFF FFFF FFFF FFFN(2) (4.2.2)

Undecidable prefix-classes: ∀∃∀ and (5.1.1)

∀∀∃. (5.2.1)

#p&(%q&#r) ; **FFFF FFF**N **FFFF FFF**N (5.1.2)

#p&(#q&%r) ; **FFFF FFF**N **FFFF FFF**N (5.2.2)

This dividing line differs from that in pure quantification theory, where the
∃••∃∀∀∃••∃ class is decidable, (6.1)

%u&(((%v&#w)&(#x&%y))&%z) ; **FFFF FFFF FFFF FFFF**(126),
NNNN NNNN NNNN NNNN(2) (6.2)

so that the minimal undecidable prefix-classes are ∀∃∀ (7.1.1)

and ∀∀∀∃... (7.2.1)

#p&(%q&#r) ; **FFFF FFF**N **FFFF FFF**N (7.1.2)

#p&((#q&#r)&%s) ; **FFFF FFFF FFFF FFF**N (7.2.2)

The "sharpening" of the *The Gödel Class with Identity as undecidable* produces two decidable and two undecidable prefix classes, none of which is tautologous (Eqs. 4-5). The difference from quantification theory for class decidability is *not* tautologous (Eq. 6), and the minimal prefix-classes for undecidability are *not* tautologous and *not* equivalent (Eqs. 7). This means the Gödel class with identity is decidable, but solvable as *not* tautologous.