No-cloning theorem, Kochen-Specker theorem, and quantum measurement theories

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The usual no-cloning theorem implies that two quantum states are identical or orthogonal if we allow a cloning to be on the two quantum states. Here, we investigate a relation between the no-cloning theorem and the projective measurement theory that the results of measurements are either +1 or −1. We introduce the Kochen-Specker (KS) theorem with the projective measurement theory. We result in the fact that the two quantum states under consideration cannot be orthogonal if we avoid the KS contradiction. Thus the no-cloning theorem implies that the two quantum states under consideration are identical in the case. It turns out that the KS theorem with the projective measurement theory says a new version of the no-cloning theorem. Next, we investigate a relation between the no-cloning theorem and the measurement theory based on the truth values that the results of measurements are either +1 or 0. We return to the usual no-cloning theorem that the two quantum states are identical or orthogonal in the case.

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I. INTRODUCTION

Einstein, Podolsky, and Rosen say that a hidden-variable interpretation of quantum mechanics is essential [1]. The hidden-variable interpretation is a topic of research [2, 3]. One is the Bell theorem [4]. The other is the Kochen-Specker theorem (the KS theorem) [5]. Greenberger, Horne, and Zeilinger discover [6, 7] the GHZ theorem. The Bell theorem and the KS theorem become a simple form (see also Refs. [8–12]).

A new type of the KS theorem (or the KS contradiction) is reported [13]. The results of measurements are either +1 or −1. The KS theorem is a precondition for various quantum information theories. The theorem as a precondition for secure quantum key distribution is discussed [14]. The theorem as a precondition for quantum computing is studied [15]. Therefore it is interesting to study the relation between the KS theorem and various quantum information theories. Here we derive a new version of the no-cloning theorem based on the KS theorem that the results of measurements are either +1 or −1. On the other hand, the measurement theory based on the truth values is studied [16]. The results of measurements are either +1 or 0. We discuss the KS theorem with the measurement theory does not change the no-cloning theorem.

The no-cloning theorem was stated by Wootters and Zurek [17] and Dieks [18] in 1982. It has profound implications in quantum computing and related fields. According to Asher Peres and David Kaiser, the publication of the no-cloning theorem is prompted by a proposal of Nick Herbert [19] for a superluminal communication device using quantum entanglement. The literature concerning quantum cloning topic can be seen in Ref. [20].

Our discussion may provide a new insight for the good security of quantum cryptography when we use the projective measurement theory. The no-cloning theorem in the case implies that the two quantum states under consideration are identical even though an eavesdropper allows a cloning to be on the two quantum states. A probability that the eavesdropper selects an unknown and identical quantum state is very small.

In this paper, we investigate a relation between the no-cloning theorem and the projective measurement theory that the results of measurements are either +1 or −1. We introduce the Kochen-Specker (KS) theorem with the projective measurement theory. We result in the fact that the two quantum states under consideration cannot be orthogonal if we avoid the KS contradiction. Thus the no-cloning theorem implies that the two quantum states under consideration are identical in the case. It turns out that the KS theorem with the projective measurement theory says a new version of the no-cloning theorem. Next, we investigate a relation between the no-cloning theorem and the measurement theory based on the truth values that the results of measurements are either +1 or 0. We return to the usual no-cloning theorem that the two quantum states are identical or orthogonal in the case.

The paper is organized as follows:

In Section II, we review the no-cloning theorem.

In Section III, we discuss the projective measurement theory improves the no-cloning theorem.

In Section IV, we discuss the measurement theory based on the truth values does not change the no-cloning theorem.

Section V concludes the paper.

II. REVIEW OF THE NO-CLONING THEOREM

We review the no-cloning theorem as follows:

$$U|\phi\rangle_A|e\rangle_B = |\phi\rangle_A|\phi\rangle_B. \quad (1)$$

U is the time evolution operator. Alice has a quantum state $|\phi\rangle_A$. Bob has a quantum state $|e\rangle_B$. Bob's state changes into $|\phi\rangle_B$ by using the time evolution operator. Thereby Alice's state is cloned into Bob's state. Let us consider the inner product of them. The inner product is explained as follows: A generalization of the scalar product. Any product $\langle u, v\rangle$ of vectors which satisfies the following conditions. It must be distributive over addition, i.e. $\langle u, v_1 + v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle$, be reflexive, i.e. $\langle v, u\rangle = \langle u, v\rangle^*$, $1\frac{1}{2}$-linear, i.e. $\langle au, bv\rangle = a^*b\langle u, v\rangle$ and strictly positive, i.e. if $\langle v, v\rangle = 0$ then $v = 0$, where the star means the complex conjugation [21]. Then we have

$$\begin{aligned}\langle e|_B\langle\phi|_A|\psi\rangle_A|e\rangle_B &= \langle e|_B\langle\phi|_A U^\dagger U|\psi\rangle_A|e\rangle_B \\ &= \langle\phi|_B\langle\phi|_A|\psi\rangle_A|\psi\rangle_B. \end{aligned} \quad (2)$$

Thus,

$$\langle\phi|\psi\rangle_A = \langle\phi|\psi\rangle_A\langle\phi|\psi\rangle_B. \quad (3)$$

By omitting subscripts A and B, we have

$$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2. \quad (4)$$

We derive the following proposition

$$\langle\phi|\psi\rangle = 0 \lor \langle\phi|\psi\rangle = 1. \quad (5)$$

Therefore, the no-cloning theorem implies that the two quantum states are identical or orthogonal if we allow a cloning to be on the two quantum states. We derive the following proposition

$$\langle\phi|\psi\rangle^2 = 0 \lor \langle\phi|\psi\rangle^2 = 1. \quad (6)$$

From each a proposition, we have

$$\langle\phi|\psi\rangle^4 = 0 \lor \langle\phi|\psi\rangle^4 = 1. \quad (7)$$

Squaring the both side of Eqn (5) we have (6) and (7).

We cannot assume the two quantum states are orthogonal

$$\langle\phi|\psi\rangle = 0, \quad (8)$$

because we encounter the KS contradiction with the projective measurement theory. This means that we cannot assume $\langle\phi|\psi\rangle = 0$. Hence we have $\langle\phi|\psi\rangle = 1$. The no-cloning theorem implies that the two quantum states under consideration are identical when we consider the KS theorem with the projective measurement theory in the discussion below.

On the other hand, we can assume the two quantum states are orthogonal if we use the measurement theory based on the truth value. We do not encounter the KS contradiction in the case. This means that we can assume $\langle\phi|\psi\rangle = 0, 1$. We return to the usual no-cloning theorem that the two quantum states under consideration are identical or orthogonal.

III. THE PROJECTIVE MEASUREMENT THEORY IMPROVES THE NO-CLONING THEOREM

A. The orthogonal case

We consider a quantum expected value as

$$\langle\phi|\psi\rangle^2 = 0. \quad (9)$$

The above quantum expected value is zero if the two quantum states under consideration $|\phi\rangle$ and $|\psi\rangle$ are orthogonal.

We derive a necessary condition for the quantum expected value given in (9). From the proposition (9) by squaring the both sides, we derive the following proposition concerning quantum mechanics

$$\langle\phi|\psi\rangle^4 = 0. \quad (10)$$

Therefore, we have

$$(\langle\phi|\psi\rangle^4)_{\min} = 0 \land (\langle\phi|\psi\rangle^4)_{\max} = 0. \quad (11)$$

B. The identical case

We consider a quantum expected value as

$$\langle\phi|\psi\rangle^2 = 1. \quad (12)$$

The above quantum expected value is $+1$ if the two quantum states under consideration $|\phi\rangle$ and $|\psi\rangle$ are identical.

We derive a necessary condition for the quantum expected value given in (12). From the proposition (12) by squaring the both sides, we derive the following proposition concerning quantum mechanics

$$\langle\phi|\psi\rangle^4 = 1. \quad (13)$$

Therefore, we have

$$(\langle\phi|\psi\rangle^4)_{\min} = 1 \land (\langle\phi|\psi\rangle^4)_{\max} = 1. \quad (14)$$

C. The orthogonal case is forbidden by the KS theorem with the projective measurement theory

A mean value E satisfies a hidden-variable model if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m}, \quad (15)$$

where l denotes a label and r is the "hidden" result of quantum measurements. The notation $r_l(\langle\phi|\psi\rangle^2)$ means the lth "hidden" outcome of quantum measurements when we would measure the expected value $\langle\phi|\psi\rangle^2 = 0$ in a thoughtful experiment. We can assume the value of r is ± 1.

In what follows, we assume the two quantum states are orthogonal, that is, $\langle\phi|\psi\rangle = 0$. And we derive the KS contradiction when we introduce the hidden-variable model.

Assume the quantum mean value admits the hidden-variable model given in (15). One has the following proposition concerning the hidden-variable model

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m}. \tag{16}$$

We can assume as follows by Strong Law of Large Numbers [22],

$$\langle\phi|\psi\rangle^2(+\infty) = \langle\phi|\psi\rangle^2. \tag{17}$$

Assume the proposition (16) is true. By changing the label l into l', we have the same quantum mean value as follows:

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l'=1}^{m} r_{l'}(\langle\phi|\psi\rangle^2)}{m}. \tag{18}$$

An important note here is that the value of the right-hand-side of (16) is equal to the value of the right-hand-side of (18) because we only change the label l into l'.

We introduce an assumption that Sum rule and Product rule commute with each other. We have

$$\begin{aligned}
&\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m) \\
&= \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\langle\phi|\psi\rangle^2)}{m} \\
&\le \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} |(r_l(\langle\phi|\psi\rangle^2) r_{l'}(\langle\phi|\psi\rangle^2)| \\
&= \frac{\sum_{l=1}^{m}}{m} \frac{\sum_{l'=1}^{m}}{m} (r_l(\langle\phi|\psi\rangle^2))^2 \\
&= 1.
\end{aligned} \tag{19}$$

We use the following fact for the number of elements with property

$$\begin{aligned}
&\|\{l|r_l(\langle\phi|\psi\rangle^2) = 1\}\| = \|\{l'|r_{l'}(\langle\phi|\psi\rangle^2) = 1\}\|, \\
&\|\{l|r_l(\langle\phi|\psi\rangle^2) = -1\}\| = \|\{l'|r_{l'}(\langle\phi|\psi\rangle^2) = -1\}\|,
\end{aligned} \tag{20}$$

and

$$(r_l(\langle\phi|\psi\rangle^2))^2 = 1. \tag{21}$$

We see that the inequality (19) is saturated. We have

$$\begin{aligned}
&\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m) \\
&= \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\langle\phi|\psi\rangle^2)}{m} \\
&\ge \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} (-1) \\
&= (-1) \frac{\sum_{l=1}^{m}}{m} \frac{\sum_{l'=1}^{m}}{m} = -1.
\end{aligned} \tag{22}$$

We use the following fact for the number of elements with property

$$\begin{aligned}
&\|\{l|r_l(\langle\phi|\psi\rangle^2) = 1\}\| = \|\{l'|r_{l'}(\langle\phi|\psi\rangle^2) = -1\}\|, \\
&\|\{l|r_l(\langle\phi|\psi\rangle^2) = -1\}\| = \|\{l'|r_{l'}(\langle\phi|\psi\rangle^2) = 1\}\|,
\end{aligned} \tag{23}$$

and

$$(r_l(\langle\phi|\psi\rangle^2) r_{l'}(\langle\phi|\psi\rangle^2) \ge -1. \tag{24}$$

We see that the inequality (22) is saturated.

Thus we derive propositions concerning the hidden-variable model, that is,

$$\begin{aligned}
&(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\min} = -1, \\
&(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\max} = 1.
\end{aligned} \tag{25}$$

From Strong Law of Large Numbers, we have

$$\begin{aligned}
&(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\min} = -1, \\
&(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\max} = 1.
\end{aligned} \tag{26}$$

Hence we derive the following proposition concerning the hidden-variable model

$$(\langle\phi|\psi\rangle^4)_{\min} = -1 \wedge (\langle\phi|\psi\rangle^4)_{\max} = 1. \tag{27}$$

We cannot assign the truth value "1" for the proposition (11) and the proposition (27), simultaneously. We are thus in the KS contradiction.

Therefore, we cannot assume the two quantum states under consideration are orthogonal

$$\langle\phi|\psi\rangle = 0, \tag{28}$$

when we consider the KS theorem with the projective measurement theory.

D. The identical case avoids the KS contradiction

A mean value E satisfies a hidden-variable model if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m}, \tag{29}$$

where the notation $r_l(\langle\phi|\psi\rangle^2)$ means the lth "hidden" outcome of quantum measurements when we would measure the expected value $\langle\phi|\psi\rangle^2 = 1$ in a thoughtful experiment. We can assume the value of r takes only $+1$ because the expected value is $+1$.

In what follows, we assume the two quantum states are identical, that is, $\langle\phi|\psi\rangle = 1$. And we show that the KS contradiction is avoided when we introduce the hidden-variable model.

Assume the quantum mean value admits the hidden-variable model given in (29). One has the following proposition concerning the hidden-variable model

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m}. \tag{30}$$

Assume the proposition (30) is true. By changing the label l into l', we have the same quantum mean value as follows:

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l'=1}^m r_{l'}(\langle\phi|\psi\rangle^2)}{m}. \quad (31)$$

An important note here is that the value of the right-hand-side of (30) is equal to the value of the right-hand-side of (31) because we only change the label l into l'.

We have

$$\begin{aligned}&\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m)\\ &= \frac{\sum_{l=1}^m r_l(\langle\phi|\psi\rangle^2)}{m} \times \frac{\sum_{l'=1}^m r_{l'}(\langle\phi|\psi\rangle^2)}{m}\\ &= +1. \end{aligned} \quad (32)$$

We use the fact that $r_l(\langle\phi|\psi\rangle^2) = +1$ and $r_{l'}(\langle\phi|\psi\rangle^2) = +1$.

Thus we derive propositions concerning the hidden-variable model, that is,

$$\begin{aligned}&(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\min} = 1,\\ &(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\max} = 1.\end{aligned} \quad (33)$$

From Strong Law of Large Numbers, we have

$$\begin{aligned}&(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\min} = 1,\\ &(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\max} = 1.\end{aligned} \quad (34)$$

Hence we derive the following proposition concerning the hidden-variable model

$$((\langle\phi|\psi\rangle^4)_{\min} = 1 \wedge ((\langle\phi|\psi\rangle^4)_{\max} = 1. \quad (35)$$

We can assign the truth value "1" for the proposition (14) and the proposition (35), simultaneously. We thus avoid the KS contradiction.

Therefore, we assume that the two quantum states under consideration are identical

$$\langle\phi|\psi\rangle = 1. \quad (36)$$

Hence we assume the following case

$$|\phi\rangle = |\psi\rangle. \quad (37)$$

The no-cloning theorem implies that the two quantum states under consideration are identical if we consider the KS theorem in the case.

IV. THE MEASUREMENT THEORY BASED ON THE TRUTH VALUES DOES NOT CHANGE THE NO-CLONING THEOREM

A. The orthogonal case avoids the KS contradiction

A mean value E satisfies a hidden-variable model if it can be written as

$$E = \frac{\sum_{l=1}^m r_l(\langle\phi|\psi\rangle^2)}{m}, \quad (38)$$

where the notation $r_l(\langle\phi|\psi\rangle^2)$ means the lth "hidden" outcome of quantum measurements when we would measure the expected value $\langle\phi|\psi\rangle^2 = 0$ in a thoughtful experiment. We can assume the value of r takes only 0 because the expected value is 0.

In what follows, we assume the two quantum states are orthogonal, that is, $\langle\phi|\psi\rangle = 0$. And we show that the KS contradiction is avoided when we introduce the hidden-variable model.

Assume the quantum mean value admits the hidden-variable model given in (38). One has the following proposition concerning the hidden-variable model

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l=1}^m r_l(\langle\phi|\psi\rangle^2)}{m}. \quad (39)$$

Assume the proposition (39) is true. By changing the label l into l', we have the same quantum mean value as follows:

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l'=1}^m r_{l'}(\langle\phi|\psi\rangle^2)}{m}. \quad (40)$$

An important note here is that the value of the right-hand-side of (39) is equal to the value of the right-hand-side of (40) because we only change the label l into l'.

We have

$$\begin{aligned}&\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m)\\ &= \frac{\sum_{l=1}^m r_l(\langle\phi|\psi\rangle^2)}{m} \times \frac{\sum_{l'=1}^m r_{l'}(\langle\phi|\psi\rangle^2)}{m}\\ &= 0. \end{aligned} \quad (41)$$

We use the fact that $r_l(\langle\phi|\psi\rangle^2) = 0$ and $r_{l'}(\langle\phi|\psi\rangle^2) = 0$.

Thus we derive propositions concerning the hidden-variable model, that is,

$$\begin{aligned}&(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\min} = 0,\\ &(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\max} = 0.\end{aligned} \quad (42)$$

From Strong Law of Large Numbers, we have

$$\begin{aligned}&(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\min} = 0,\\ &(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\max} = 0.\end{aligned} \quad (43)$$

Hence we derive the following proposition concerning the hidden-variable model

$$((\langle\phi|\psi\rangle^4)_{\min} = 0 \wedge ((\langle\phi|\psi\rangle^4)_{\max} = 0. \quad (44)$$

We can assign the truth value "1" for the proposition (11) and the proposition (44), simultaneously. We thus avoid the KS contradiction.

B. The identical case avoids the KS contradiction

A mean value E satisfies a hidden-variable model if it can be written as

$$E = \frac{\sum_{l=1}^m r_l(\langle\phi|\psi\rangle^2)}{m}, \quad (45)$$

where the notation $r_l(\langle\phi|\psi\rangle^2)$ means the lth "hidden" outcome of quantum measurements when we would measure the expected value $\langle\phi|\psi\rangle^2 = 1$ in a thoughtful experiment. We can assume the value of r takes only $+1$ because the expected value is $+1$.

In what follows, we assume the two quantum states are identical, that is, $\langle\phi|\psi\rangle = 1$. And we show that the KS contradiction is avoided when we introduce the hidden-variable model.

Assume the quantum mean value admits the hidden-variable model given in (45). One has the following proposition concerning the hidden-variable model

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m}. \tag{46}$$

Assume the proposition (46) is true. By changing the label l into l', we have the same quantum mean value as follows:

$$\langle\phi|\psi\rangle^2(m) = \frac{\sum_{l'=1}^{m} r_{l'}(\langle\phi|\psi\rangle^2)}{m}. \tag{47}$$

An important note here is that the value of the right-hand-side of (46) is equal to the value of the right-hand-side of (47) because we only change the label l into l'.

We have

$$\begin{aligned}&\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m)\\&= \frac{\sum_{l=1}^{m} r_l(\langle\phi|\psi\rangle^2)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\langle\phi|\psi\rangle^2)}{m}\\&= +1.\end{aligned} \tag{48}$$

We use the fact that $r_l(\langle\phi|\psi\rangle^2) = +1$ and $r_{l'}(\langle\phi|\psi\rangle^2) = +1$.

Thus we derive propositions concerning the hidden-variable model, that is,

$$\begin{aligned}&(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\min} = 1,\\&(\langle\phi|\psi\rangle^2(m) \times \langle\phi|\psi\rangle^2(m))_{\max} = 1.\end{aligned} \tag{49}$$

From Strong Law of Large Numbers, we have

$$\begin{aligned}&(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\min} = 1,\\&(\langle\phi|\psi\rangle^2 \times \langle\phi|\psi\rangle^2)_{\max} = 1.\end{aligned} \tag{50}$$

Hence we derive the following proposition concerning the hidden-variable model

$$(\langle\phi|\psi\rangle^4)_{\min} = 1 \wedge (\langle\phi|\psi\rangle^4)_{\max} = 1. \tag{51}$$

We can assign the truth value "1" for the proposition (14) and the proposition (51), simultaneously. We thus avoid the KS contradiction.

V. CONCLUSIONS

In conclusion, we have investigated a relation between the no-cloning theorem and the projective measurement theory that the results of measurements are either $+1$ or -1. We have introduced the Kochen-Specker (KS) theorem with the projective measurement theory. We have resulted in the fact that the two quantum states under consideration cannot be orthogonal if we avoid the KS contradiction. Thus the no-cloning theorem has implied that the two quantum states under consideration are identical in the case. It has turned out that the KS theorem with the projective measurement theory says a new version of the no-cloning theorem. Next, we have investigated a relation between the no-cloning theorem and the measurement theory based on the truth values that the results of measurements are either $+1$ or 0. We have returned to the usual no-cloning theorem that the two quantum states are identical or orthogonal in the case.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] M. Redhead, *Incompleteness, Nonlocality, and Realism* (Clarendon Press, Oxford, 1989), 2nd ed.
[3] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, The Netherlands, 1993).
[4] J. S. Bell, Physics **1**, 195 (1964).
[5] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).
[6] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989), pp. 69-72.
[7] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).
[8] C. Pagonis, M. L. G. Redhead, and R. K. Clifton, Phys. Lett. A **155**, 441 (1991).
[9] N. D. Mermin, Phys. Today **43**(6), 9 (1990).
[10] N. D. Mermin, Am. J. Phys. **58**, 731 (1990).
[11] A. Peres, Phys. Lett. A **151**, 107 (1990).
[12] N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
[13] K. Nagata and T. Nakamura, Int. J. Theor. Phys. **48**, 3287 (2009).
[14] K. Nagata, Phys. Rev. A **72**, 012325 (2005).
[15] K. Nagata and T. Nakamura, Int. J. Theor. Phys. **55**, 5193 (2016).
[16] K. Nagata and T. Nakamura, Int. J. Theor. Phys. **55**, 3616 (2016).
[17] W. Wootters and W. Zurek, Nature **299**, 802 (1982).
[18] D. Dieks, Phys. Lett. A **92**, 271 (1982).
[19] N. Herbert, Found. Phys. **12**, 1171 (1982).
[20] V. Scarani, S. Iblisdir, N. Gisin, and A. Acin, Rev. Mod. Phys. **77**, 1225 (2005).

[21] C. Clapham and J. Nicholson, *The Concise Oxford Dictionary of Mathematics* (Oxford University Press, 2014), Fifth ed.

[22] In probability theory, the law of large numbers is a theorem that describes the result of performing the same experiment a large number of times. According to the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed. The strong law of large numbers states that the sample average converges almost surely to the expected value.