Division by Zero Calculus and Singular Integrals

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Abstract: What are the singular integrals? Singular integral equations are presently encountered in a wide range of mathematical models, for instance in acoustics, fluid dynamics, elasticity and fracture mechanics. Together with these models, a variety of methods and applications for these integral equations has been developed. In this paper, we will give the interpretation for the Hadamard finite part of singular integrals by means of the division by zero calculus.

Key Words: Zero, division by zero, division by zero calculus, $0/0 = 1/0 = z/0 = 0$, Laurent expansion, singular integral, Hadamard finite part, Cauchy's principal value.

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1 Division by zero and division by zero calculus

For the long history of division by zero, see [1, 23]. The division by zero with mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $a/0 = 0$

in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [6] for example.

Division by zero is trivial and clear also from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [6, 32].

The simple field structure containing division by zero was established by M. Yamada ([9]). For a simple introduction, see H. Okumura [21].

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [15, 16, 17, 18, 19, 20] for example.

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.5}$$

we **define** the identity, by the division by zero

$$f(a) = C_0. \tag{1.6}$$

Note that here, there is no problem on any convergence of the expansion (1.5) at the point $z = a$, because all the terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$.

Apart from the motivation, we define the division by zero calculus by (1.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

In addition, we will refer to the naturality of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz.$$

Even when the function $f(z)$ has any singularity at the point a, we assume that this formula is valid as the division by zero calculus.

The basic idea of the above may be considered that we can consider the value of a function by some mean value of the function.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [3] and also the references for recent related results.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x = 0) = 1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17: 9: 45-10: 00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [3],

he kindly sent the message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Meanwhile, on ZERO, the authors S. K. Sen and R. P. Agarwal [29] published its long history and many important properties of zero. See also R. Kaplan [5] and E. Sondheimer and A. Rogerson [31] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [28] that

The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea
and that is represented geometrically with the horn torus model [3].

S. K. Sen and R. P. Agarwal [29] referred to the paper [6] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

" Thou shalt not divide by zero " remains valid eternally.

However, in [27] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

They stated in the book many meanings of zero over mathematics, deeply.

In this paper, we will show a very interesting interpretation of singular integrals by the division by zero calculus. This may be considered as the basic relation of ZERO and INFINITY through integrals. Furthermore, we will see a similar nature of singular integrals and the division by zero calculus. It is **discontinuity**.

2 Finite parts of Hadamard in singular integrals

Singular integral equations are presently encountered in a wide range of mathematical models, for instance in acoustics, fluid dynamics, elasticity and fracture mechanics. Together with these models, a variety of methods and applications for these integral equations has been developed. See, for example, [2, 4, 11, 12].

For the numerical calculation of this finite part, see [14], and there, they gave an effective numerical formulas by using the DE (double exponential) formula. See also its references for various methods.

For singular integrals, we will consider their integrals as divergence, however, the Hadamard finite part or Cauhy's principal values give finite values; that is, from divergence values we will consider finite values; for this interesting property, we will be able to give a natural interpretation by the division by zero calculus.

What are singular integrals? For the interrelationship between divergence integrals and finite values in singular integrals, we can obtain an essential answer by means of the division by zero calculus.

Let $F(x)$ be an integrable function on an interval (c,d). The functions $F(x)/(x-a)^n (n = 1, 2, 3..., c < a < d)$ are, in general, not integrable on (c,d). However, for any $\epsilon > 0$, of course, the functions

$$\left(\int_c^{a-\epsilon} + \int_{a+\epsilon}^{d}\right) \frac{F(x)}{(x-a)^n} dx$$

are integrable. For an integrable function $\varphi(x)$ on (a,d), we assume the Taylor expansion

$$F(x) = \sum_{k=0}^{n-1} \frac{F^{(k)}(a)}{k!}(x-a)^k + \varphi(x)(x-a)^n.$$

Then, we have

$$\int_{a+\epsilon}^{d} \frac{F(x)}{(x-a)^n} dx$$

$$= \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{\epsilon^{n-k-1}} - \frac{F^{(n-1)}(a)}{(n-1)!} \log \epsilon$$

$$+\{-\sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{(d-a)^{n-k-1}}$$

$$+\frac{F^{(n-1)}(a)}{(n-1)!} \log(d-a) + \int_{a+\epsilon}^{d} \varphi(x) dx\}.$$

Then, the last term $\{....\}$ is the finite part of Hadamard of the integral

$$\int_a^d \frac{F(x)}{(x-a)^n} dx$$

and is written by

$$\text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx;$$

that is, precisely

$$\text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx$$

$$:= \lim_{\epsilon \to +0} \{ \int_{a+\epsilon}^{d} \frac{F(x)}{(x-a)^n} dx$$

$$- \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{\epsilon^{n-k-1}} + \frac{F^{(n-1)}(a)}{(n-1)!} \log \epsilon \}. \tag{2.1}$$

We do not take the limiting $\epsilon \to +0$, but we put $\epsilon = 0$, in (2.1), then we obtain, by the division by zero calculus, the formula

$$\text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx = \int_a^d \frac{F(x)}{(x-a)^n} dx.$$

The division by zero will give the natural meaning (**definition**) for the above two integrals.

Of course,

$$\text{f. p.} \int_c^d \frac{F(x)}{(x-a)^n} dx := \text{f. p.} \int_c^a \frac{F(x)}{(x-a)^n} dx$$

$$+ \quad \text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx.$$

When $n = 1$, the integral is the Cauchy principal value.

In particular, for the expression (2.1), we have, missing $\log \epsilon$ term, for $n \geq 2$

$$\text{f. p.} \int_c^d \frac{F(x)}{(x-a)^n} dx$$

$$= \lim_{\epsilon \to +0} \{ (\int_c^{a-\epsilon} + \int_{a+\epsilon}^{d}) \frac{F(x)}{(x-a)^n} dx$$

$$- \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1 + (-1)^{n-k}}{\epsilon^{n-k-1}} \}.$$

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