On the general solution for the quintic Duffing oscillator equation

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Abstract

This paper shows for the first time that exact and general solution for the quintic Duffing oscillator equation may be computed in a straightforward manner within the framework of the generalized Sundman transformation theory introduced recently by authors of this work. A major advantage of the applied method is that it intimately relates such an oscillator equation to the quadratic anharmonic oscillator equation with well-known exact solutions.

1. Introduction

The quintic Duffing oscillator equation has been intensively investigated in the literature by several authors [1-4]. According to [3, 4] such an oscillator equation is not exactly integrable. Thus, until now, only approximate and special solutions are calculated. However the purpose of this paper is to show that exact and general solution to the quintic Duffing oscillator equation may be computed in a simple way, without resorting to the notion of conservative systems. Let the quintic Duffing oscillator equation be in the general form

$$\ddot{x}(t) + \frac{a}{4}x(t)^5 - cx(t) = 0 \tag{1}$$

where a and c are arbitrary parameters. To compute the exact and general solution of equation (1), it appears firstly to show that equation (1) is a nonlocal transformation of the well-known quadratic anharmonic oscillator equation (section 2) so that the desired solution may be obtained from that of the quadratic polynomial differential equation secondly (section 3). Finally a conclusion is addressed for the work.

2. Nonlcal transformation of the quadratic anharmonic oscillator equation

On the basis of the nonlinear differential equation theory developped by Akande et al. [5], consider the quadratic anharmonic oscillator equation

$$y''(\tau) + ay(\tau)^2 = c \tag{2}$$

Then the following theorem may be proved.

Theorem 1. Consider the generalized Sundman transformation

$$y(\tau) = \int g(x)^l \, dx, \quad d\tau = e^{\gamma\varphi(x)} dt \tag{3}$$

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Then the application of (3) to equation (2) may yield

$$\ddot{x}(t)+ag(x)^{l}\left[\int g(x)^{l}dx\right]^{2}=cg(x)^{l} \qquad (4)$$

Proof. From (3) the first derivative of $y(\tau)$ may be written as

$$\frac{dy}{d\tau}=y'(\tau)=\dot{x}g(x)^{l}e^{-\gamma\varphi(x)} \qquad (5)$$

so that one may compute the second derivative of $y(\tau)$ in the form

$$\frac{d^{2}y}{d\tau^{2}}=y''(\tau)=\left[\ddot{x}+\left(l\frac{g'(x)}{g(x)}-\gamma\varphi'(x)\right)\dot{x}^{2}\right]g(x)^{l}e^{-2\gamma\varphi(x)} \qquad (6)$$

Substituting (6) into equation (2) and taking into account (3), yields

$$\ddot{x}+\dot{x}^{2}\left[l\frac{g'(x)}{g(x)}-\gamma\varphi'(x)\right]+a\frac{e^{2\gamma\varphi(x)}}{g(x)^{l}}\left[\int g(x)^{l}dx\right]^{2}-c\frac{e^{2\gamma\varphi(x)}}{g(x)^{l}}=0 \qquad (7)$$

Now making $\gamma\varphi(x)=l\ln g(x)$, where $g(x)\succ 0$, leads to equation (4) of the theorem. Theorem 1 allows the possibility to show that equation (1) is a nonlocal transformation of equation (2). To that end, consider the theorem 2.

Theorem 2. Let $g(x)=x^{2}$, *and* $l=\frac{1}{2}$. *Then equation (4) reduces to equation (1).*

Proof. By application of $g(x)=x^{2}$, equation (4) becomes

$$\ddot{x}+\frac{a}{(2l+1)^{2}}x^{6l+2}-cx^{2l}=0 \qquad (8)$$

Setting $l=\frac{1}{2}$, yields immediately the quintic Duffing oscillator equation (1). Therefore the theorem is proved. To compute then its exact and general solution, it is convenient to consider the solution to equation (2).

3. Exact and general solution to equation (1)

According to [6], the exact and general solution of the quadratic anharmonic oscillator equation (2) may read

$$y(\tau)=a_{3}-(a_{3}-a_{2})sn^{2}(K\tau,p) \qquad (9)$$

where the constants a_2 and a_3 satisfy the system of equations

$$\begin{cases} a_1 + a_2 + a_3 = 0 \\ a_1 a_2 + a_2 a_3 + a_3 a_1 = -\dfrac{3c}{a} \\ a_1 a_2 a_3 = +\dfrac{3K_1}{2a} \end{cases} \tag{10}$$

by defining a_1, a_2 and a_3 as the real roots of the cubic polynomial equation [6]

$$y^3 - \frac{3c}{a} y - \frac{3K_1}{2a} = 0 \tag{11}$$

and K_1 as a constant of integration, so that

$$K^2 = -\frac{2a}{3}\frac{(2a_3 + a_2)}{4} \tag{12}$$

and

$$p^2 = \frac{a_3 - a_2}{2a_3 + a_2} \tag{13}$$

In this regard the nonlocal transformation (3) leads to

$$y(\tau) = a_3 - (a_3 - a_2) sn^2(K\tau, p) = \int x dx$$

which yields the desired exact and general solution $x(t)$ of the quintic Duffing oscillator equation (1) as

$$x^2 = 2a_3 - 2(a_3 - a_2) sn^2(K\tau, p) \tag{14}$$

where the parameter τ is given by

$$\int \frac{d\tau}{\sqrt{a_3 - (a_3 - a_2) sn^2(K\tau, p)}} = \varepsilon\sqrt{2}(t + K_2) \tag{15}$$

where $\varepsilon = \pm 1$, K_2 is a constant of integration and $sn(z,k)$ is a Jacobian elliptic function.

Conclusion

In this work the quintic Duffing oscillator equation is investigated. It is shown for the first time that the exact and general solution of such an equation may be computed in a straightforward manner. The work has shown that the quintic Duffing oscillator equation is nothing but the nonlocal transformation of the quadratic anharmonic oscillator equation with well-known exact analytical properties.

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