ABSTRACT Numbers of form $6N - 1$ and $6N + 1$ factor into numbers of the same form. This observation provides elimination sieves for numbers $N$ that lead to primes and prime pairs. The sieves do not explicitly reference primes.

Introduction. All primes except 2 and 3 are of the form $6N - 1$ or $6N + 1$. Also, if any numbers $6N - 1$ or $6N + 1$ factor, their factors are $(6c - 1) * (6d + 1)$ or $(6c + 1) * (6d - 1)$. Sequences of numbers $N$ that give primes or twin or cousin prime pairs appear in the Online Encyclopedia of Integer Sequences\(^1\). In particular, the sequence A067611\(^2\) gives numbers $6cd + c - d$, which are the numbers $N$ for which $6N - 1$ and $6N + 1$ are not both prime. This paper lists the sieves for prime gaps $6k + 2$, $6k - 2$, and $6k$ in matrix form. It includes worksheets that apply these sieves to numbers $N = 1$ to 68.

Twin Primes and Gap 8

Twin primes sieve matrix. Twin primes other than 3 and 5 are of the form $6N - 1$ and $6N + 1$. If the number $6N - 1$ factors, it factors as $(6c - 1) * (6d + 1)$ or $(6c + 1) * (6d - 1)$ which give equations $N = 6cd + c - d$ or $N = 6cd - c + d$. If the number $6N + 1$ factors, it factors as $(6c - 1) * (6d - 1)$ or $(6c + 1) * (6d + 1)$ which give equations $N = 6cd - c - d$ or $6cd + c + d$. Thus, if $6N - 1$ and $6N + 1$ are prime, $N$ cannot be of the form $6cd + c - d$.

The numbers $6cd + c - d$, for $c, d$ positive integers, can be formed into $2 \times 2$ blocks.

$$
\begin{array}{ll}
6cd - c - d & 6cd + c - d \\
6cd - c + d & 6cd + c + d
\end{array}
$$

or

$$
\begin{array}{ll}
(6c - 1)d - c & (6c - 1)d + c \\
(6c + 1)d - c & (6c + 1)d + c
\end{array}
$$

These blocks give the sieve matrix below in which alternate rows are multiples of $6c - 1$ increased or decreased by $c$, and multiples of $6c + 1$ increased or decreased by $c$.

4, 6, 9, 11, 14, 16, 19, 21, 24, 26, ...
6, 8, 13, 15, 20, 22, 27, 29, 34, 36, ...
9, 13, 20, 24, 31, 35, 42, 46, 53, 57, ...
11, 15, 24, 28, 37, 41, 50, 54, 63, 67, ...
14, 20, 31, 37, 48, 54, 65, 71, 82, 88, ...
16, 22, 35, 41, 54, 60, 73, 79, 92, 98, ...
19, 27, 42, 50, 65, 73, 88, 96, 111, 119, ...
21, 29, 46, 54, 71, 79, 96, 104, 121, 129, ...
24, 34, 53, 63, 82, 92, 111, 121, 140, 150, ...
26, 36, 57, 67, 88, 98, 119, 129, 150, 160, ...

\(^1\) OEIS, oeis.org, A046953, A046954, A002822, A056956.
\(^2\) Ibid.
Note that, for example, the third row (or column) contains numbers that differ by 2 from multiples of $11 = 6 \times 2 - 1$, and the eighth row contains numbers that differ by 4 from multiples of $25 = 6 \times 4 + 1$.

A formula for this matrix is
\[ a(m, n) = 6 \times \text{floor}((m+1)/2) \times \text{floor}((n+1)/2) + ((-1)^n) \times \text{floor}((m+1)/2) + ((-1)^m) \times \text{floor}((n+1)/2). \]

Figure 1 shows a worksheet that sieves the numbers 1 to 68. Multiples of $6c - 1$ and $6c + 1$ are marked with a dot, numbers eliminated by the sieve are marked by X. The underlined numbers have no X in their column and give rise to twin primes.

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*Figure 1 - Worksheet for twin primes*

Gap 8 sieve matrix. Primes with gap 8 except for 3 and 11 can be written $6N - 1$ and $6N + 7$. The sieve array for gap 8 consists of 2 X 2 blocks, for $c \geq 1$, $d \geq 1$, which are

- $6cd - c - d - 1$
- $6cd + c - d$
- $6cd - c + d$
- $6cd + c + d - 1$

or

- $(6c - 1)d - c - 1$
- $(6c - 1)d + c$
- $(6c + 1)d - c$
- $(6c + 1)d + c - 1$.

The sieve matrix begins

- $3$, $6$, $8$, $11$, $13$, $16$, $18$, $21$, $23$, $26$, \ldots
A formula is

\[ a(m,n) = 6 \cdot \text{floor}(\frac{m+1}{2}) \cdot \text{floor}(\frac{n+1}{2}) + ((-1)^n \cdot \text{floor}(\frac{m+1}{2}) + ((-1)^m \cdot \text{floor}(\frac{n+1}{2}) - (m+n+1) \mod 2, m,n \geq 1. \]

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Figure 2 - Worksheet for prime pairs with gap 8.

Gap 6k + 2 matrices. Prime pairs with gap 6k + 2 are of the form 6N – 1 and 6N + 6k + 1. The sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

\begin{align*}
6cd - c - d - k & \quad 6cd + c - d \\
6cd - c + d & \quad 6cd + c + d - k
\end{align*}

or

\begin{align*}
(6c - 1)d - c - k & \quad (6c - 1)d + c \\
(6c + 1)d - c & \quad (6c + 1)d + c - k
\end{align*}
These are sieves for twin primes when \( k = 0 \), and for prime pairs with gap 8 when \( k = 1 \).

**Cousin Primes**

Cousin primes sieve matrix. Cousin primes, prime pairs with gap 4, except for 3 and 7, are of the form \( 6N + 1 \) and \( 6N + 5 \). The gap 4 sieve array consists of 2 X 2 blocks, for \( c \geq 1 \), \( d \geq 1 \), which are

\[
\begin{align*}
6cd - c - d & \quad 6cd + c - d - 1 \\
6cd - c + d - 1 & \quad 6cd + c + d
\end{align*}
\]

or

\[
\begin{align*}
(6c - 1)d - c & \quad (6c - 1)d + c - 1 \\
(6c + 1)d - c - 1 & \quad (6c + 1)d + c
\end{align*}
\]

The sieve array begins

\[
\begin{align*}
4, & \quad 5, \quad 9, \quad 14, \quad 15, \quad 19, \quad 20, \quad 24, \quad 25, \quad ... \\
5, & \quad 8, \quad 12, \quad 15, \quad 19, \quad 22, \quad 26, \quad 29, \quad 33, \quad 36, \quad ... \\
9, & \quad 12, \quad 20, \quad 23, \quad 31, \quad 34, \quad 42, \quad 45, \quad 53, \quad 56, \quad ... \\
10, & \quad 15, \quad 23, \quad 28, \quad 36, \quad 41, \quad 49, \quad 54, \quad 62, \quad 67, \quad ... \\
14, & \quad 19, \quad 31, \quad 36, \quad 48, \quad 53, \quad 65, \quad 70, \quad 82, \quad 87, \quad ... \\
15, & \quad 22, \quad 34, \quad 41, \quad 53, \quad 60, \quad 72, \quad 79, \quad 91, \quad 98, \quad ... \\
19, & \quad 26, \quad 42, \quad 49, \quad 65, \quad 72, \quad 88, \quad 95, \quad 111, \quad 118, \quad ... \\
20, & \quad 29, \quad 45, \quad 54, \quad 70, \quad 79, \quad 95, \quad 104, \quad 120, \quad 129, \quad ... \\
24, & \quad 33, \quad 53, \quad 62, \quad 82, \quad 91, \quad 111, \quad 120, \quad 140, \quad 149, \quad ... \\
25, & \quad 36, \quad 56, \quad 67, \quad 87, \quad 98, \quad 118, \quad 129, \quad 149, \quad 160, \quad ... \\
... & \\
\end{align*}
\]

A formula is

\[a(m,n) = 6 \times \text{floor}((m+1)/2) \times \text{floor}((n+1)/2) + ((-1)^n) \times \text{floor}((m+1)/2) + ((-1)^m) \times \text{floor}((n+1)/2) - (m+n) \mod 2, \ m, n \geq 1.\]
Gap 6k – 2 matrices. Prime pairs with gap 6k – 2 are of the form 6N + 1 and 6N + 6k – 1. The gap 6k – 2 sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

\[
\begin{align*}
6cd - c - d & \quad 6cd + c - d - k \\
6cd - c + d - k & \quad 6cd + c + d \\
\end{align*}
\]

or

\[
\begin{align*}
(6c - 1)d + c & \quad (6c - 1)d + c - k \\
(6c + 1)d - c - k & \quad (6c + 1)d + c.
\end{align*}
\]

For k = 1, this gives the sieve array for cousin primes.

**Sexy Primes**

Sexy primes 6N – 1 and 6N + 5 matrix. One type of pair with gap 6 is of the form 6N – 1 and 6N + 5. The sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

\[
\begin{align*}
6cd + c - d - 1 & \quad 6cd + c - d \\
6cd - c + d - 1 & \quad 6cd - c + d \\
\end{align*}
\]

or

\[
\begin{align*}
(6c - 1)d + c - 1 & \quad (6c - 1)d + c \\
(6c + 1)d - c - 1 & \quad (6c + 1)d - c.
\end{align*}
\]

The sieve matrix begins
5, 6, 10, 11, 15, 16, 20, 21, 25, 26, ...
5, 6, 12, 13, 19, 20, 26, 27, 33, 34, ...
12, 13, 23, 24, 34, 35, 45, 46, 56, 57, ...
10, 11, 23, 24, 36, 37, 49, 50, 62, 63, ...
19, 20, 36, 37, 53, 54, 70, 71, 87, 88, ...
15, 16, 34, 35, 53, 54, 72, 73, 91, 92, ...
26, 27, 49, 50, 72, 73, 95, 96, 118, 119, ...
20, 21, 45, 46, 70, 71, 95, 96, 120, 121, ...
33, 34, 62, 63, 91, 92, 120, 121, 149, 150, ...
25, 26, 57, 58, 87, 88, 118, 119, 149, 150, ...
...

A formula for this array is

\[ a(m,n) = 6 \times \text{floor}((m+1)/2) \times \text{floor}(n+1)/2 + ((-1)^{m+1}) \times \text{floor}(m+1)/2 + ((-1)^m) \times \text{floor}(n+1)/2 - n \mod 2, \ m, n \geq 1. \]

Figure 4 Worksheet for primes $6N - 1$ and $6N + 1$.

Gap 6k matrices of $6N - 1$ type, and $6N - 1$ primes. Some prime pairs with gap 6k are of the form $6N - 1$ and $6N + 6k - 1$. The sieve array consists of 2 X 2 blocks, for $c \geq 1, d \geq 1$, which are

\[ 6cd + c - d - k \quad 6cd + c - d \]
\[ 6cd - c + d - k \quad 6cd - c + d \]

or
(6c - 1)d + c - k  \quad (6c - 1)d + c
(6c + 1)d - c - k  \quad (6c + 1)d - c.

These are sieves for primes of form $6N - 1$ when $k = 0$, and for prime pairs with gap 6 above when $k = 1$.

Figure 5 Worksheet for primes $6N - 1$.

Sexy primes $6N + 1$ and $6N + 7$ matrix. The other type of pair with gap 6 is of the form $6N + 1$ and $6N + 7$. The sieve array for these pairs consists of 2 X 2 blocks, for $c \geq 1$, $d \geq 1$, which are

$$6cd - c - d - 1 \quad 6cd - c - d$$
$$6cd + c + d - 1 \quad 6cd + c + d$$

or

(6c - 1)d - c - 1  \quad (6c - 1)d - c
(6c + 1)d + c - 1  \quad (6c + 1)d + c.

The sieve matrix is

$$3, \ 4, \ 8, \ 9, \ 13, \ 14, \ 18, \ 19, \ 23, \ 24, \ldots$$
$$7, \ 8, \ 14, \ 15, \ 21, \ 22, \ 28, \ 29, \ 35, \ 36, \ldots$$
$$8, \ 9, \ 19, \ 20, \ 30, \ 31, \ 41, \ 42, \ 52, \ 53, \ldots$$
$$14, \ 15, \ 27, \ 28, \ 40, \ 41, \ 53, \ 54, \ 66, \ 67, \ldots$$
$$13, \ 14, \ 30, \ 31, \ 47, \ 48, \ 64, \ 65, \ 81, \ 82, \ldots$$
$$21, \ 22, \ 40, \ 41, \ 59, \ 60, \ 78, \ 79, \ 97, \ 98, \ldots$$
A formula for the matrix is

\[ a(m,n) = 6 \times \text{floor}(\frac{m+1}{2}) \times \text{floor}(\frac{n+1}{2}) + ((-1)^m) \times \text{floor}(\frac{m+1}{2}) + ((-1)^m) \times \text{floor}(\frac{n+1}{2}) - n \mod 2, \quad m, n \geq 1. \]

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**Figure 6 - Worksheet for pairs \(6N + 1\) and \(6N + 7\).**

Gap 6k matrices of \(6N + 1\) type, and \(6N + 1\) primes. Some prime pairs with gap 6k are of the form \(6N + 1\) and \(6N + 6k + 1\). The sieve array for these pairs consists of 2 X 2 blocks, for \(c \geq 1, d \geq 1\), which are

\[
\begin{align*}
6cd - c - d - k & \quad 6cd - c - d \\
6cd + c + d - k & \quad 6cd + c + d
\end{align*}
\]

or

\[
\begin{align*}
(6c - 1)d - c - k & \quad (6c - 1)d - c \\
(6c + 1)d + c - k & \quad (6c + 1)d + c.
\end{align*}
\]

These are sieves for primes of form \(6N + 1\) when \(k = 0\), and for prime pairs with gap 6 above when \(k = 1\).
Note. The matrix for twin primes appears in OEIS\(^3\). My thanks to the editors at OEIS for improvements to the writeups for the other matrices, which ultimately were not accepted by OEIS.

**Bibliography**


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\(^3\) OEIS, oeis.org, A323674.