Lagrangian of general relativity over three dimensional subspace

Malik Al Matwi

Department of Mathematics Science, Ritsumeikan University,
1–1 Matsugaoka 6-Chome, Otsu-city, Shiga-ken , Japan, 2019.
malik.matwi@hotmail.com

Abstract

We restrict the Einstein-Hamiltonian Lagrangian to $3D$ surface $S^3(\sigma_1, \sigma_2, \sigma_3)$, we let this surface be immersed in arbitrary $4D$ spacetime manifold M at constant time x^0. The gauge theory of general relativity asserts that the Einstein-Hamiltonian Lagrangian is invariant under infinitesimal variation of that surface, this determines the surface. By that we get continuity equation in arbitrary $4D$ spacetime, then we search for Lagrangian and equation of motion that give same continuity equation according to canonical field theory.

Key words: gravitational Lagrangian on three dimensions surface, continuity equation, canonical gravitational Lagrangian with dependence only on F.

Introduction

In general relativity, we define local Lorentz frame $dx^I = e^I_\mu dx^\mu$, where dx^μ is local charts on arbitrary curved spacetime M, and e^I_μ is the gravitational field. The Lagrangian of general relativity is invariant under continuous transformations of local Lorentz frame, $SO(3,1)$ Lie algebra, and invariant under continuous transformations of arbitrary curved spacetime M.

Because the group $SO(3)$ is subgroup of $SO(3,1)$, then we can restrict the Lagrangian of general relativity to $3D$ surface S^3. Therefore the Lagrangian has internal gauge symmetry of group $SO(3)$. That allows us to determine the surface $S^3(\sigma_i)$ using the action principle for the Lagrangian part that is restricted to the surface $S^3(\sigma_i)$, we consider the remaining terms of GR Lagrangian as responsible for time revolution, that changes the surface $S^3(\sigma_i)$ during the time. It is like to write $S^3_t(\sigma_i)$, which means we have different surfaces at different moments.

1 Continuous equation of Einstein-Hamiltonian action over three dimensional surface

We start with gravitational field Lagrangian of form

$$L(e,\omega) = (16\pi G)^{-1} e^\mu_I e^\nu_J (R_{\mu\nu})^{IJ} e, \tag{1.1}$$

where

$$(R_{\mu\nu})^{IJ} = d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$$

is Riemannian tensor and e^μ_I satisfies $e^I_\mu e^\mu_J = \delta^I_J$. We need to restrict this Lagrangian to $3D$ surface $S^3(\sigma_i)$, where $\sigma_1, \sigma_2, \sigma_3$ are parameters for this surface. Let us choose the surface $S^3(\sigma_i)$ and the gauge $e^\mu_0 = \delta^\mu_0$ to satisfy $(R_{0\mu})^{IJ} = 0$, so this Lagrangian becomes

$$(16\pi G)^{-1} e^a_i e^b_j (R_{ab})^{ij} e,$$

where i and j are Lorentz indices for $I = i = 1,2,3$ and let σ^a be local charts on the surface S^3. Because $(R_{ab})^{ij}$ is anti-symmetric tensor, so $e^a_i e^b_j$ is written as

$$\frac{1}{2}\left(e^a_i e^b_j - e^a_j e^b_i\right),$$

so we can write

$$\frac{1}{2}\left(e^a_i e^b_j - e^a_j e^b_i\right) = \varepsilon^{abc}\varepsilon_{ijk}\Sigma^k_c. \tag{1.2}$$

The Lagrangian becomes

$$L(\Sigma, e) = (16\pi G)^{-1}\Sigma_c^k \varepsilon^{abc}\varepsilon_{ijk}(R_{ab})^{ij} e. \tag{1.3}$$

If we do not want to consider this gauge, we can use another method to get same Lagrangian like to separate the Lagrangian eq.(1.1) to two parts like

$$L(e, \omega) = (16\pi G)^{-1} e_I^a e_J^b (R_{ab})^{IJ} e + (16\pi G)^{-1} e_I^0 e_J^a (R_{0a})^{IJ} e,$$

then we consider the part

$$(16\pi G)^{-1} e_I^a e_J^b (R_{ab})^{IJ} e$$

as a main part which has the gauge symmetry of group $SO(3)$, and determines the surface $S^3(\sigma_i)$. We consider the remaining terms

$$(16\pi G)^{-1} e_I^0 e_J^a (R_{0a})^{IJ} e,$$

which must be small comparing with the first term, as responsible for time revolution, that changes the surface $S^3(\sigma_i)$ during the time. But we use self-dual projection, in which we replace $(R_{ab})^{IJ}$ with $P_i^{IJ} DA^i$.

Let us choose the surface at constant time, so the integral over the time is not efficient, so we integral the Lagrangian only over $3D$ surface of constant time. Using the Lagrangian eq.(1.3), we get the action

$$S(\Sigma, e) = (16\pi G)^{-1} \int \Sigma_c^k \varepsilon^{abc}\varepsilon_{ijk}(R_{ab})^{ij} e d^3\sigma.$$

For $R^{ij} = (R_{ab})^{ij} d\sigma^a \wedge d\sigma^b$ and redefine Σ^i as $\Sigma^i = \Sigma_a^i d\sigma^a$, this action written as

$$S(\Sigma, e) = (16\pi G)^{-1} \int \varepsilon_{ijk}\Sigma^i \wedge R^{ij}.$$

So we rewrite the Lagrangian as 3-form:

$$(16\pi G)^{-1}\varepsilon_{ijk}\Sigma^i \wedge R^{ij}. \tag{1.4}$$

In selfdual representation of Lorentz group, we have the replacement:

$$\frac{1}{2}\varepsilon^i_{jk} R^{jk} \to DA^i.$$

So we rewrite it as

$$\theta = (16\pi G)^{-1}\Sigma_i \wedge DA^i, \tag{1.5}$$

where A^i is complex $SO(3)$ selfdual connection, Σ_i is one-form as defined in eq.(1.2), and D is covariant derivative defined as

$$(DA^i)_{ab} = \partial_{[a}A^i_{b]} + \varepsilon^i_{jk}A^j_{[a}A^k_{b]}.$$

The Lagrangians eq(1.3) and eq(1.5) are the restriction of GR Lagrangian eq(1.1) to the surface $S^3(\sigma^a)$, and they have the gauge symmetry of group $SO(3)$ which is the subgroup of Lorentz group $SO(3,1)$. The action of these Lagrangians must be independent on choosing the surface $S^3(\sigma^a)$, this allow us to determine it. The key idea behind this is that the gravitational field carries all information about the geometry of spacetime, so we let the Lagrangian be independent on choosing that geometry, so on $S^3(\sigma^a)$. We can determine this surface by projection the four form $D\theta$ into the tangent of the surface $S^3(\sigma^a)$, the remaining is one form vector in direction of the normal to this surface. Because the surface is at constant time, the normal is in direction of that time.

$$(16\pi G)D\theta = D\left(\Sigma_i \wedge DA^i\right) = (D\Sigma_i) \wedge DA^i - \Sigma_i \wedge DDA^i. \tag{1.6}$$

The tangent basic on $S^3(\sigma^a)$ is $\partial_a \wedge \partial_b \wedge \partial_c$, we rewrite it as $\varepsilon^{abc}\partial_a\partial_b\partial_c$, the projection of $D\theta$ to this basic is

$$(16\pi G)\left\langle D\theta \mid \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle = \left\langle (D\Sigma_i) \wedge DA^i \mid \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle - \left\langle \Sigma_i \wedge DDA^i \mid \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle,$$

where $\langle dx^\sigma \mid \partial_a\rangle = \delta^\sigma_a$ is inner product.

Starting with the first term

$$\left\langle (D\Sigma_i) \wedge DA^i \mid \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle = \varepsilon^{abc}D_\mu\Sigma_{\nu i}D_\rho A^i_\sigma \left\langle dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \mid \partial_a\partial_b\partial_c\right\rangle.$$

Using $\langle dx^\sigma \mid \partial_a\rangle = \delta^\sigma_a$, with regarding the different permutations, like

$$\left\langle dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \mid \partial_a\partial_b\partial_c\right\rangle = dx^\mu\delta^\nu_c\delta^\rho_b\delta^\sigma_a$$

and

$$\left\langle dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \mid \partial_a\partial_b\partial_c\right\rangle = -dx^\mu\delta^\nu_c\delta^\sigma_b\delta^\rho_a.$$

In general, we write

$$\langle dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \mid \partial_a \partial_b \partial_c \rangle = dx^{[\mu} \delta_c^\nu \delta_b^\rho \delta_a^{\sigma]},$$

where the bracket [....] is anti-symmetry permutations of indices. We get

$$\begin{aligned} &4 \left\langle (D\Sigma_i) \wedge DA^i \,\middle|\, \varepsilon^{abc} \partial_a \partial_b \partial_c \right\rangle = \\ &-\varepsilon^{abc} D_a \Sigma_{bi} D_c A^i_\mu dx^\mu + \varepsilon^{abc} D_a \Sigma_{bi} D_\mu A^i_c dx^\mu - \varepsilon^{abc} D_a \Sigma_{\mu i} D_b A^i_c dx^\mu + \varepsilon^{abc} D_\mu \Sigma_{ai} D_b A^i_c dx^\mu. \end{aligned}$$

The second term is

$$\left\langle \Sigma_i \wedge DDA^i \,\middle|\, \varepsilon^{abc} \partial_a \partial_b \partial_c \right\rangle = \varepsilon^{abc} \Sigma_{\mu i} D_\nu D_\rho A^i_\sigma \langle dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \mid \partial_a \partial_b \partial_c \rangle .$$

Doing same thing, we get

$$\begin{aligned} &4 \left\langle \Sigma_i \wedge DDA^i \,\middle|\, \varepsilon^{abc} \partial_a \partial_b \partial_c \right\rangle = \\ &-\varepsilon^{abc} \Sigma_{ai} D_b D_c A^i_\mu dx^\mu + \varepsilon^{abc} \Sigma_{ai} D_b D_\mu A^i_c dx^\mu - \varepsilon^{abc} \Sigma_{ai} D_\mu D_b A^i_c dx^\mu + \varepsilon^{abc} \Sigma_{\mu i} D_a D_b A^i_c dx^\mu. \end{aligned}$$

Adding the two terms, we get

$$\begin{aligned} &4(16\pi G) \left\langle D\theta \,\middle|\, \varepsilon^{abc} \partial_a \partial_b \partial_c \right\rangle = \\ &-\varepsilon^{abc} D_a \Sigma_{bi} D_c A^i_\mu dx^\mu + \varepsilon^{abc} D_a \Sigma_{bi} D_\mu A^i_c dx^\mu - \varepsilon^{abc} D_a \Sigma_{\mu i} D_b A^i_c dx^\mu + \varepsilon^{abc} D_\mu \Sigma_{ai} D_b A^i_c dx^\mu \\ &+\varepsilon^{abc} \Sigma_{ai} D_b D_c A^i_\mu dx^\mu - \varepsilon^{abc} \Sigma_{ai} D_b D_\mu A^i_c dx^\mu + \varepsilon^{abc} \Sigma_{ai} D_\mu D_b A^i_c dx^\mu - \varepsilon^{abc} \Sigma_{\mu i} D_a D_b A^i_c dx^\mu. \end{aligned}$$

Define curvature

$$F^i_{\mu c} = \frac{1}{2} \left(D_\mu A^i_c - D_c A^i_\mu \right),$$

and

$$\Sigma_i^{bc} = \varepsilon^{bca} \Sigma_{ai} = \varepsilon^{abc} \Sigma_{ai}.$$

And using them in the last formula, we get

$$\begin{aligned} 4(16\pi G) \left\langle D\theta \,\middle|\, \varepsilon^{abc} \partial_a \partial_b \partial_c \right\rangle &= 2\varepsilon^{abc} (D_a \Sigma_{bi}) F^i_{\mu c} dx^\mu - 2(D_a \Sigma_{\mu i}) F^{ai} dx^\mu \\ &+ 2(D_\mu \Sigma_{ai}) F^{ai} dx^\mu + 2\Sigma_i^{bc} D_b F^i_{c\mu} dx^\mu + 2\Sigma_{ai} D_\mu F^{ai} dx^\mu - 2\Sigma_{\mu i} D_a F^{ai} dx^\mu. \end{aligned}$$

Using

$$\varepsilon^{abc} (D_a \Sigma_{bi}) F^i_{\mu c} dx^\mu = -\varepsilon^{acb} (D_a \Sigma_{bi}) F^i_{\mu c} dx^\mu = -(D_a \Sigma_i^{ac}) F^i_{\mu c} dx^\mu,$$

we get

$$4(16\pi G)\left\langle D\theta \,\middle|\, \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle =$$
$$-2(D_a\Sigma_i^{ac})F^i_{\mu c}dx^\mu - 2(D_a\Sigma_{\mu i})F^{ai}dx^\mu + \quad 2(D_\mu\Sigma_{ai})F^{ai}dx^\mu - 2\Sigma_i^{bc}D_bF^i_{\mu c}dx^\mu$$
$$+2\Sigma_{ai}D_\mu F^{ai}dx^\mu - 2\Sigma_{\mu i}D_aF^{ai}dx^\mu.$$

With

$$-2(D_a\Sigma_i^{ac})F^i_{\mu c}dx^\mu - 2\Sigma_i^{bc}D_bF^i_{\mu c}dx^\mu = -2D_a\left(\Sigma_i^{ac}F^i_{\mu c}dx^\mu\right),$$

it becomes

$$4(16\pi G)\left\langle D\theta \,\middle|\, \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle = -2D_a(\Sigma_i^{ac}F^i_{\mu c})dx^\mu - 2D_a(\Sigma_{\mu i}F^{ai})dx^\mu + 2D_\mu(\Sigma_{ai}F^{ai})dx^\mu.$$

As we suggested before, we let the normal of surface $S^3(\sigma^a)$ be in direction of the time dx^0, so $\left\langle D\theta \,\middle|\, \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle$ is in direction of that time. Therefore we set $\mu = 0$, thus

$$4(16\pi G)\left\langle D\theta \,\middle|\, \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle = -2D_a(\Sigma_i^{ac}F^i_{0c})dx^0 - 2D_a(\Sigma_{0i}F^{ai})dx^0 + 2D_0(\Sigma_{ai}F^{ai})dx^0.$$

Finally, we set $\left\langle D\theta \,\middle|\, \varepsilon^{abc}\partial_a\partial_b\partial_c\right\rangle = 0$;

$$-2D_a(\Sigma_i^{ac}F^i_{0c})dx^0 - 2D_a(\Sigma_{0i}F^{ai})dx^0 + 2D_0(\Sigma_{ai}F^{ai})dx^0 = 0$$

Because Σ_{0i} is not defined in our case, so we have to omit it or consider it as auxiliary field. We get

$$2D_a(\Sigma_i^{ac}F_c^{0i}) - 2D_a(\Sigma_{0i}F^{ai}) + 2D_0(\Sigma_{ai}F^{ai}) = 0.$$

The term $(\Sigma_{ai}F^{ai})$ is scalar, so $D_0(\Sigma_{ai}F^{ai}) = \partial_0(\Sigma_{ai}F^{ai})$, then

$$D_a(\Sigma_i^{ab}F^{0i}{}_b) - D_a(\Sigma_{0i}F^{ai}) + \partial_0(\Sigma_{ai}F^{ai}) = 0.$$

If we set $\Sigma_{0i} = 0$, we get

$$D_a(\Sigma_i^{ab}F^{0i}{}_b) + \partial_0(\Sigma_{ai}F^{ai}) = 0.$$

The vector $\Sigma_i^{ab}F^{0i}{}_b$ does not carry Lorentz index, so $D_a(\Sigma_i^{ab}F^{0i}{}_b) = \partial_a(\Sigma_i^{ab}F^{0i}{}_b)$, therefore

$$\partial_a(\Sigma_i^{ab}F^{0i}{}_b) + \partial_0(\Sigma_{ai}F^{ai}) = 0.$$

This equation shows that there is a relation between Σ^i and F^i. We can find that relation by considering this equation as continuity equation relates to a Lagrangian like $L(F^{0ai}, DA^i)$ with satisfying the action principle $\delta S(DA^i) = 0$ and the invariance on the surface $S^3(\sigma^a)$, therefore we consider $\Sigma_{ai}F^{ai}$ as energy density T^{00}, and $\Sigma_i^{ab}F^{0i}{}_b = \Sigma^a{}_{bi}F^{0bi}$ as momentum density T^{0a}. Then we search for corresponding canonical Lagrangian and Hamiltonian with canonical relations according to fields theory.

In scalar field ϕ theory, for a Lagrangian

$$L(\phi, \partial\phi) = \pi\partial_0\phi - H(\pi, \phi),$$

with conjugate momentum $\pi = \partial_0\phi = -\partial^0\phi$, we get currents like

$$T^{00} = H(\pi, \phi) = \pi\partial_0\phi - L(\phi, \partial_\mu\phi) \text{ and } T^{0a} = P^a = \partial^0\phi\partial^a\phi = -\pi\partial^a\phi.$$

By comparing our momentum $\Sigma^a{}_{bi}F^{0bi}$ with $\partial^0\phi\partial^a\phi = -\pi\partial^a\phi$, we conclude $\pi^{bi} = -F^{0bi}$, this is because F^{0bi} has a time derivative.

By considering a Lagrangian like $L(F^{0ai}, DA^i)$ and a Hamiltonian like $H(\pi_{ai}, DA^i)$, the action principle $\delta S(DA^i) = 0$, and the invariance on the surface $S^3(\sigma^a)$ give the currents

$$T^{00} = \pi_{ai}F^{0ai} - L(F^{0ai}, DA^i) \text{ and } T^{0a} = \pi_{bi}F^{abi}.$$

Comparing this with our currents $T^{00} = \Sigma_{ai}F^{ai}$ and $T^{0a} = \Sigma^a{}_{bi}F^{0bi}$, we get

$$T^{0a} = \pi_{bi}F^{abi} = \Sigma^a{}_{bi}F^{0bi}.$$

As usual we set the conjugate momentum as

$$\pi^{bi} = -F^{0bi},$$

so

$$\Sigma^{abi} = -F^{abi}.$$

Inserting $\pi^{ci} = F^{0ci}$ in the Hamiltonian $H = T^{00}$, it becomes

$$H = \pi_{ai}F^{0ai} - L(F^{0ai}, DA^i) = F_{0ai}F^{0ai} - L(F^{0ai}, DA^i),$$

and using energy density $\Sigma_{ai}F^{ai} = \frac{1}{2}\Sigma_{abi}F^{abi}$, we get

$$L(F^{0ai}, DA^i) = \pi_{ai}F^{0ai} - H = F_{0ai}F^{0ai} - \frac{1}{2}\Sigma_{abi}F^{abi},$$

then using $\Sigma^{aci} = -F^{aci}$, it becomes

$$L(F^{0ai}, DA^i) = F_{0ai}F^{0ai} + \frac{1}{2}F_{abi}F^{abi},$$

and setting $F_{0ai}F^{0ai} = \frac{1}{2}(F_{0ai}F^{0ai} + F_{a0i}F^{a0i})$, we get

$$L(F^{0ai}, DA^i) = \frac{1}{2}(F_{0ai}F^{0ai} + F_{a0i}F^{a0i}) + \frac{1}{2}F_{abi}F^{abi} = \frac{1}{2}F_{\mu\nu i}F^{\mu\nu i},$$

or

$$L(F^{0ai}, DA^i)d^4x = \frac{1}{16\pi G}\frac{1}{2}F_{\mu\nu i}F^{\mu\nu i}ed^4x,$$

where $F = DA$ in four dimensions spacetime manifold. This Lagrangian depends only one the curvature DA like electromagnetic Lagrangian. This Lagrangian allows us to consider the spacetime manifold x^μ as flat manifold with fiber dx^i, where $i = 1, 2, 3$ is Lorentz indices in selfdual representation.

References

[1] Carlo Rovelli, Quantum gravity; Cambridge University Pres, Cambridge, United Kingdom, 2004.

[2] Carlo Rovelli and Francesca Vidotto, Covariant Loop Quantum gravity, An elementary introduction to Quantum gravity and Spinfoam Theory; Cambridge University Press, Cambridge, United Kingdom, 2014.

[3] Christopher Pope, Geometry and Group Theory; Texas A&M University Press, Texas, USA, 2008.

[4] Walter Greiner and Joachim Reinhardt, Field Quantization; Springer-Verlag Berlin Heidelberg, Berlin, German, 1996.

[5] Alberto Escalante, Hamiltonian Dynamics for Einstein's Action in $G \to 0$ Limit; International Journal of Theoretical Physics, Springer US, 2009.

[6] Herbert W. Hamber and Ruth M. Williams, Discrete Wheeler-DeWitt equation, Physical Review D, American Physical Society, 2011.

[7] Steffen Gielen and Daniele Oriti, Classical general relativity as BF-Plebanski theory; Classical and Quantum gravity, IOP Publishing, United Kingdom, 2010.

[8] J. E. Rosales-Quintero, Antiself-dual gravity and supergravity from a pure connection formulation; International Journal of Modern Physics A, World Scientific Publishing Co, 2016.

[9] Mark Srednicki, Quantum Field Theory; Cambridge University Press, Cambridge, United Kingdom, 2007.

[10] Michael E. Peskin and Daniel V. Schroeder, An Introduction to Quantum Field Theory; Westview Press, Colorado, USA, 1995.