Solution of the Lipman-Zariski conjecture.

This is another attempt to fill-in what seems like a very inessential gap in Prof. Hochster’s paper about the graded case, which introduces the relations among graded rings, vector-fields and characteristic cohomology classes.

Let $S$ be a Riemann surface of genus two. Let $\omega$ be a general holomorphic one-form on $S$. There are two distinct points $p, q$ which comprise the zero set of $\omega$. The meromorphic vector-fields with at worst a simple pole at $p, q$ form a one-dimensional vector-space, and it has a basis element $\delta$ such that

$$\langle \omega, \delta \rangle = 1$$

under the pairing between forms and vector-fields. The tangent bundle $T$ of $S$ is an algebraic surface, and $\omega$ induces holomorphic function on $T$. The action of $\omega$ on the local section of $T \rightarrow S$ corresponding to a local vector-field $\sigma$ is $\langle \omega, \sigma \rangle$, so we might use the same symbol $\langle \omega, \cdot \rangle$ to describe the holomorphic function on $T$. That is to say, whenever $s : S \rightarrow T$ is a local section of $T \rightarrow S$ the composite $S \rightarrow T \rightarrow \mathbb{A}$ is $\langle \omega, \sigma \rangle$ where $\sigma$ is the local vector field defined by $s$. When we label the resulting global holomorphic function on $T$ as $\langle \omega, \cdot \rangle$ this is what we mean.

Since the tangent bundle is a functor, the flow on $S \setminus \{p, q\}$ corresponding to $\delta$ extends in a natural way to a flow on the complement of the fibers over $p, q$ on $T$. We continue to call the extended vector field $\delta$. It is locally a cartesian product flow, and being meromorphic is a local property, so just as $\delta$ is a meromorphic vector field on $S$, it extends to a meromorphic vector field on the whole of $T$.

The product

$$\langle \omega, \cdot \rangle \delta$$

corresponds to a holomorphic vector field on $T$, let us call it $\tau$. This holomorphic vector field has a simple zero on the entirety of the zero section $S \subset T$. Recall that the symbol $\langle \omega, \cdot \rangle$ refers to a holomorphic function defined on all of $T$.

A second vector field is the Euler vector field $\epsilon$, which acts on local sections of $T$ by the identity.
The vector fields $\epsilon, \tau$ are clearly transverse away from the poles of $\delta$. And $\epsilon$ restricts to a nonzero vector field along the complement of $S$ in the tangent bundle fibers over $p, q$. They are both zero on the zero-section $S$ and so the only question of transversality which needs consideration is whether $\tau$ is transverse to both fibers.

Working locally, we can translate a local part of the zero section by an element of any one fiber (viewed as a torsor for the additive group of $\mathbb{C}$) to agree with any other local $\delta$ invariant section of the tangent bundle. The flow corresponding to $\tau$ where it is defined is unaffected. Our question of transversality will be answered if we can convince ourselves that the original meromorphic vector-field on $S$ once multiplied by the translated holomorphic function (which has a simple zero at $p$ and $q$) extends to a holomorphic vector-field.

This is certainly true, the meromorphic vector fields with poles at $p, q$ can be converted to holomorphic vector fields which extend across those points by multiplying by rational functions which have a zero at $p, q$. This can be seen as a consequence of the fact the poles $p, q$ are ‘movable’ and a place where the vector field is already deemed to be holomorphic moves in to cover the points $p, q$.

Since the tangent bundle of $S$ is ample, there are global holomorphic functions on $T$ which are constant on the zero section $S$ while embedding the complement of $S$. Specifically, one may use linear combinations of two-tensors, viewed as global sections of $\Omega^2_S$. Thus by choosing a large enough set of global sections of the two-tensors of one-forms on $S$ we may map $T$ to an affine space such that $S$ maps to a point. Since the map is an isomorphism to its image away from the zero-section $S$ the two vector-fields $\delta, \tau$ are still well-defined away from the isolated singular point of the resulting image variety.

In fact, the coordinate ring of the image variety is just the global sections of the coordinate sheaf of $T$ and the global vector fields preserve this. So both extend to vector fields on the singular variety which is the identification space of $T$ with $S$ identified to one point.
The completion of the proof is already contained in J. Lipman’s thesis. The full module of vector fields of the resulting singular affine variety contains the free module of rank two spanned by $\delta$ and $\tau$. The quotient sheaf is supported on one point which has codimension two. If we call the affine coordinate ring $R$ we have

$$0 \to R\tau \oplus R\delta \to \Omega^*_R \to M \to 0$$

where $M$ is supported on the singular point of codimension two.

Dualizing gives

$$0 \to \Omega^{**}_M \to R \oplus R \to Ext^1(M, R) \to 0.$$

Dualizing again gives

$$0 \to R \oplus R \to \Omega^{***}_M \to Ext^1(Ext^1(M, R), R) \to 0.$$

The natural isomorphism between $\Omega^*$ and $\Omega^{**}$ induces

$$M \cong Ext^1(Ext^1(M, R), R).$$

By construction $R$ is normal (it is global sections of the structure sheaf of a normal variety), and since $Ext^1(M, R)$ is supported in codimension two, $0 = Ext^1(Ext^1(M, R), R) \cong M$. Then the derivation sheaf of $R$ is a free module.