Kinematics with Poisson Brackets

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1 Introduction
Kinematics is quite simple, however, Poisson brackets are not. This shows the most convoluted way of deriving kinematic motion.

2 The Hamiltonian
For an object under gravity, the Hamiltonian is obviously

\[ H = \frac{p^2}{2m} + mgy, \]

with \( p \) being the object’s vertical momentum, \( m \) the mass, \( g \) the gravitational constant, and \( y \) being the object’s height.

3 Poisson Brackets
The poisson brackets are defined as

\[ [A, B] = \frac{\partial A}{\partial y} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial y} \]

for our case, with \( A \) and \( B \) being arbitrary variables.

4 The Derivation
The poisson brackets refer to canonical transformations of a system. Since the Hamiltonian moves the system through time, it can be used to derive the object’s motion through time:

\[ [y, H] = \frac{dy}{dt}. \]

This can be exploited to get the vertical position at any time \( t \), starting with the initial position \( y_0 \), and doing the Taylor expansion:

\[ y(t) = y_0 + t[y, H]/1! + t^2[[y, H], H]/2!, \]
which upon evaluation of the brackets resolves to:

\[ y(t) = y_0 + \frac{tp}{m} + \frac{t^2g}{2}. \]

This is usually written as:

\[ y(t) = y_0 + v_0 t + \frac{gt^2}{2}. \]

where \( v_0 \) is the vertical velocity.