Kinematics with Poisson Brackets

Armin Zjajo

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1 Introduction

Kinematics is quite simple, however, Poisson brackets are not. This shows the most convoluted way of deriving kinematic motion.

2 The Hamiltonian

For an object under gravity, the Hamiltonian is obviously

$$H = p^2/2m + mgy,$$

with p being the object's vertical momentum, m the mass, g the gravitational constant, and y being the object's height.

3 Poisson Brackets

The poisson brackets are defined as

$$[A, B] = \frac{\partial A}{\partial y}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial y}$$

for our case, with A and B being arbitrary variables.

4 The Derivation

The poisson brackets refer to canonical transformations of a system. Since the Hamiltonian moves the system through time, it can be used to derive the object's motion through time:

$$[y, H] = \frac{dy}{dt}.$$

This can be exploited to get the vertical position at any time t, starting with the initial position y_0, and doing the Taylor expansion:

$$y(t) = y_0 + t[y, H]/1! + t^2[[y, H], H]/2!,$$

which upon evaluation of the brackets resolves to:

$$y(t) = y_0 + tp/m + t^2 g/2.$$

This is usually written as:

$$y(t) = y_0 + vt + \frac{gt^2}{2}.$$

where v is the vertical velocity.