Systems of particles and field in a unified field theory

Jean Claude Dutailly

15 February 2019

Abstract

In previous contributions I have presented a unified theory of particles and field, in the Geometry of General Relativity, which accounts for all the known force fields, as well as the properties of elementary particles, without the need to invoke additional dimension or special physical phenomenon. In this paper the theory is fully detailed, and its focus is on models of systems of elementary particles interacting with the field. The equations are established for continuous systems and solutions, as well as methods to solve the usual cases are exposed in the model of 2 particles. It is then possible to build a clear model of stable systems, such as nuclei and atoms. Discontinuous processes involve discontinuities in the field and I show that they can be represented by particles-like objects, the bosons. Their interaction with particles is formalized in a rigorous but simple way.

Contents

In a previous paper I have proposed a solution to the "Great Unification Theory" problem, that is a representation of particles and fields encompassing the gravitational field, the EM field, the strong and weak interactions, in the framework of the Geometry of General Relativity. It involves a new vision of the usual "axioms" of Quantum Mechanics, which are no more than mathematical consequences of the properties of models commonly used in Physics, and so the proposed theory, meanwhile it benefits from concepts such as the "gauge field" introduced in Modern Physics, is more in line with the usual, and well proven Physics as it has been learnt and taught. There is no extra dimension, no special particle, there is only one Physics, and the scale has the meaning well understood by scientists : it matters only when some aspects can be neglected for a good reason. It is also determinist and embraces clearly a realist vision of the physical world : there are only 3 objects : the Universe (the container) with its geometry, material body composed of particles, and a force field. All have clear properties, as they are usually understood and accepted in the various theories.

The unified field theory has been built, not from abstract Mathematics, but starting from the properties of the objects and the facts as they result from experiments. The known force fields appear as successive layers, in experiments which dig more deeply in physical reality. At first comes the most obvious : the motion of material bodies, governed by the laws of Mechanics, and the Newton's gravitation theory. It has been revolutionize by Relativity, which, starting from the necessity to adjust the Galilean Geometry of the Universe to account for the properties of the EM field, lead to General Relativity and the Einstein's theory of gravitation. It was one of the greatest steps in Theoretical Physics, but it stayed more or less enshrined as it has been expressed at the beginning of the XX° century, with an old and inadequate mathematical formalism, and many simplistic adjustments with Newtonian Mechanics (such as the relativist momentum and the recurring "inertial frames"). Actually what stands firmly is the new representation of the Geometry of the Universe, as it comes out of General Relativity, and the need to review the couple inertia / gravitational charge. It is possible to give a consistent and more manageable representation of Mechanics in the framework of General Relativity by using the tools offered by Mathematics in the last decennium, essentially fiber bundles and Clifford algebras. This leads to the representation by spinors, introduced in a totally abstract way by Dirac with an equally totally different perspective. In a unified theory of field we must make a choice between inertia and the gravitational field, and this is naturally the first which has to go, as it is actually useless. With it the Einstein's Theory of gravitation must be revised. This is all the more necessary that it is not checked by the astronomical observations, which lead to the invention of "dark matter". Actually a consistent theory of gravitation is easily incorporated with spinors and gauge field theories, which is the natural framework when using fiber bundles, and in this framework Einstein's theory appears only as a special case, based on assumptions which are not physically justified (and on which Einstein himself was dubious). The EM field can be easily represented with spinors and gauge field, accounting for its known properties.

The next, and ultimate, step was to account for the nuclear interactions. Of course the starting point is the Standard Model. It implements Relativity, but only in its Special version, and so misses gravitation but also fails to account fully for the motion of material bodies. It incorporates spinors and gauge theories, albeit in an inconsistent manner. Its great innovation is the idea of representations using groups, which is the logical support of gauge theories. The challenge is then to incorporate the groups $SU(2)$, $SU(3)$ associated to the strong and weak forces, in a unified gauge representation. The known elementary particles constitute a bizarre zoo, with numerous symmetries reflected in their behavior under the action of the different fields. Rather than trying to explain these symmetries from a mathematical model, they can be used to set up the constraints which must be met by the unified group. Moreover one of these symmetries is chirality, which is very specific to Clifford algebras. From there it is logical to look for a representation based on Clifford algebras. It requires some new mathematical results (about real and complex algebras, and the exponential notably), but eventually it provides the unified representation.

It is summed up in 2 propositions :

Proposition 1 *The state of elementary particles are represented at each point by a vector ψ, belonging to the complex Clifford algebra $Cl(\mathbb{C}, 4)$, endowed with a real structure. It is measured with respect to a gauge, located at each point, given by an element of the unitary group $U \subset Cl(\mathbb{C}, 4)$, acting on ψ by a left action ϑ.*

Proposition 2 *The field (that is the gravitational field, the EM field, the weak and strong interactions) is represented by a principal connection on the principal bundle P_U.*

These propositions are well in line with the usual concepts of Modern Physics. The newest part if the choice of the complex Clifford algebra. The great advantage of this representation is that the only barrier between the usual forces (EM, gravitation) and the others is the scope of the field : all the theory and the results hold when one drops the components of the field which are not involved. The equations look the same to account for the behavior of elementary particles or star systems. A Clifford algebra can be seen as the demultiplication of an orthogonal frame, and the mathematical representation reflects the successive layers of the force field. The introduction of the complex part is grounded in the specificity of the Universe, with its time vector. In particular it gives a consistent and interesting interpretation of the difference between matter and antimatter : it is linked to the choice of the signature (1,3) or (3,1) of the metric (both are acceptable but are conventional).

The main purpose of this paper is to use the unified theory to formulate models which account for systems of particles and field, in particular for the stability of systems such as nuclei and atoms. I have dedicated a great effort to provide computable equations and explicit methods to solve the usual problems.

To do this it is necessary at first to present, in the most rigorous way, the assumptions and the mathematical framework. This is done in the first part, which encompasses the presentation of the geometry, the particles and the field. It addresses also the lagrangian, which leads to a simple expression with the charges of elementary particles, whose format is given. The content, for the most part, has already be given in previous papers, but it seemed necessary to renew the presentation for a good understanding of the following.

If there were two Physics, it would be the Physics which deal with continuous processes and the Physics which deal with discontinuous processes, because they involve different assumptions and mathematical tools. Particles Physics has accustomed us to a picture where elementary particles are exploding, annihilated, created or combined, exist in a vacuum which is full of virtual particles and can behave as a wave or a field. But the real world exists out of colliders or black holes, and the key fact, as can be checked by everybody, is that matter is extraordinarily resilient. The first law of Chemistry is that elements are conserved along the most violent reactions. Nuclei are insensitive to all usual processes. The electronic shells can combine with each others but, overall, they keep their basic properties and are composed of the same electrons moving around nuclei. If we have quite good models for chemical reactions, the available models for nuclei are poor, and based mainly on phenomenological laws. For all its pretense to be the foundation of a theory of everything, QTF and the Standard Model have little to tell about the existence and properties of the most ordinary of physical objects : matter.

The basic fact is that there is no totally discontinuous process, only transition between different continuous processes. And these transitions are usually of interest for the physicist.

First we study the evolution of a system of particles interacting with the field in the context of continuous processes. The physical assumptions are simple : there is no collision, no creation or annihilation of particles, and are expressed in clear mathematical properties. Then the usual method is the implementation of the Principle of Least Action. It is general, well known, but its implementation with fiber bundles require new mathematical results which I have presented in my book (“Advanced Mathematics for Theoretical Physics") and recalled here. Then by the method of variational derivatives it is possible to sum up the conditions of equilibrium in 4 quite simple equations, encompassing the metric. The challenge is then to provide operational solutions. The key is the additional assumption that the field propagates along Killing curves, that is which preserve the metric. This physical assumption is justified in the first part. Then it is possible to express explicit PDE for the field and the metric. And the way to find solutions is illustrated in the case of 2 particles, which shows that such a system is actually instable.

Using these results it is then possible to study stable systems, systems of particles and field interacting which are stable and behave as a single object. This is obviously the case of nuclei and atoms, but can be extended to stars systems. The conditions for the stability are linked to precise geometric arrangement, and are summed up in an equation. Then the system behave as a single particle, with a precise charge and trajectory. These equations should be useful for nuclei.

The last part deals with discontinuous processes. Their main feature is the apparition of discontinuities in the field, which take the form of bosons. Their representation is given, as well as their interaction with particles.

A comprehensive Annex presents the key mathematical objects and tools which are used. It can be completed by my book on Mathematics (the references are given by Maths.x), as well as my book “Theoretical Physics" (the references are given by Th.Physics x).

Part I
GEOMETRY, PARTICLES AND FIELD

1 Geometry

The first object of Physics is the Universe, seen as the container in which exist everything. Its properties are described and represented in a Theory of Geometry [1], which addresses 3 points : how to locate an event ? how to measure lengths and durations ? how to measure the motion (translation and rotation) of material bodies ?

It is related to the properties of material bodies, which are the first and main objects through which phenomenon are described :

- a material body occupies a definite location at each time, so they have a motion of translation, from one point to another

- material bodies show a spatial arrangement, they can be rotated, and have a rotational motion.

They have other properties which reveal themselves in their interaction with the force field.

Geometry requires the introduction of another object of Physics : the observer, who proceeds to the experiments and measures. His main property is that he has "free will" : he can choose the experiment, its location in space and time, as well as the standards used to make his measures.

Geometry by itself is local : its representations are linked to an observer, and how the measures done vary from one observer to another, located at the same point or at a proximity which enables to compare the measures done. Cosmology is another topic, it pretends to give a representation of the whole Universe, and raises fundamental issues, and first the definition of the observer. It is not treated here.

1.1 Manifold structure and metric

1.1.1 Location of an event

A "point" in the Universe M is an event. For a given observer its location requires 4 parameters, 3 for the spatial location and one for the time location. Any procedure which enables to locate an event with 4 parameters is acceptable, it defines a chart, that is a map : $\varphi : \mathbb{R}^4 \to M$ specific to the observer. The only condition is that, if 2 charts φ_1, φ_2 are used, there is a mathematical rule which gives the coordinates (ξ_1^α) as function of (ξ_2^α) for the same event. This is precisely the definition of a structure of manifold :

[1] Cosmology is another topic. Its purpose is to give a representation of the whole Universe, and raises fundamental issues, and first the definition of the observer. It is not addressed here.

Proposition 3 *The Universe can be represented by a 4 dimensional manifold M, and any observer has a chart.*

For a given observer we will consider only a limited area Ω, which is mathematically relatively compact.

As a consequence for each observer there is a function $f_o : \Omega \to \mathbb{R}$ such that $f_o(m) = \xi^0$ is the time at which happens an event located at m. We assume that the chart is differentiable, as well as f_o and that $f_o'(m) \neq 0$. Then f_o defines a foliation of Ω in 3 dimensional hypersurfaces $\Omega_3(t) = \{m \in \Omega : f_o(m) = t\}$ which represent the "present" of the observer and his physical space.

The basic property of a material object is that it occupies a definite location at each time. So their trajectory is represented on M by a path : $q : \mathbb{R} \to M$ which we assume to be differentiable, called a world line. There is a unique (up to the choice of an origin) parameter τ, called the proper time of the body, such that $\frac{dq}{d\tau}$ is the tangent to the trajectory on M. This holds also for the observers, and it is assumed that the observers measure their location on their curve by their proper time t, specific to each observer.

1.1.2 Metric

To any manifold is associated a tangent bundle which, locally, is defined by small displacements. The coordinates ξ^β define vector fields $\partial\xi_\beta$ and the coordinates can themselves be defined through the flow of these vector fields : the point m along the axis β at the "distance" ξ^β is $m = \Phi_{\partial\xi_\beta}\left(\xi^\beta, O\right)$ from the origin O of the coordinates, but ξ^β does not imply any measure of length. For this a metric, acting on the tangent bundle, is necessary.

The Principle of Causality tells that 2 events A, B can be not related, A can be the cause of B $(A < B)$ or B be the cause of A $(B < A)$ which gives a, non total, order relation between events, and the most important fact is that this relation does not depend on the observer. This can be represented by the value of $f_o(c(\tau))$ on any path $c(\tau)$ on M. There is an orientation : one goes towards the future if $f_o(c(\tau))$ is increasing, of equivalently if $f_o(c(\tau))\frac{dc}{d\tau} > 0$. More generally the tangent space at any point m is divided in 3 hypercones of apex m, which distinguishes the vectors oriented towards the future, or the past, or along curves between events which are not related. and this distinction does not depend on the observer. These regions are not path connected.

This leads to :

Proposition 4 *There is on M a scalar 2 form g, defined at each point and which does not depend on the observer.*

The metric is the physical part of the Geometry. It acts on a 4 dimensional manifold and is not definite positive. Because it defines disconnected regions its signature must be $(3,1)$ or $(1,3)$.

There is an euclidean metric on the space $\Omega_3(\theta)$ of each observer, it must be consistent with g, (it is induced by g on the hypersurfaces) and we will denote it

g_3. Because spatial distances are measured by positive numbers we will choose $(3,1)$ in the following, but we will see the implications of a different choice.

On a manifold endowed with a metric of signature $(3,1)$ the tangent space T_mM at any point is divided in several regions according to the value of $g(m)(V,V)$:

null vectors $g(m)(V,V)=0$

future oriented vectors $g(m)(V,V)<0$

past oriented vectors $g(m)(V,V)>0$

the last two sets are disconnected (Maths.2.3.3).

To define a null, future oriented vector one proceeds by continuity : a null vector $V=V^0+v$ is said to be future oriented if there is a sequence $V_n=V_n^0+v_n$ such that $g(m)(V_n,V_n)<0, V_n^0\to V^0, v_n\to v$. The last continuity being measured in $T_m\Omega_3(t)$ with the euclidean metric g_3.

Spatial distance are measured in units L which are different from the units of time. It implies the existence of a universal constant c such that $\xi^0=ct$. At this step nothing more is assumed on the physical meaning of c.

Practically any chart is built from a spatial chart $x=\varphi_3\left(\xi^1,\xi^2,\xi^3\right)$ of $\Omega_3(0)$ then $\varphi_o(ct,x)=m$.

By taking $\varepsilon_0(m)=\frac{1}{c}gradf_o'(m)$ there is, for each observer, a unique vector field ε_0, future oriented and with length $\langle\varepsilon_0,\varepsilon_0\rangle=-1=g(m)(\varepsilon_0(m),\varepsilon_0(m))$. It is necessarily orthogonal, for g, to the hypersurfaces $\Omega_3(t)$, and conversely these hypersurfaces are spatial : $\forall u,v\in T\Omega_3(t):g(u,v)\geq 0, g(\varepsilon_0,u)=0$.

1.1.3 Velocity of material bodies

We assume that the clocks of all observers "run" at the same rate. Indeed the definition of the second is given in reference to a physical process, without any specification of the observer. For an object or an observer the four dimensional velocity (or velocity in short) with respect to the proper time τ is the derivative $V=\frac{dq}{d\tau}$ which is a vector of TM. It does not depend on the choice of a chart. The proposition that the clocks run at the same rate is equivalent to :

Proposition 5 *For any observer or material object the square $\langle V,V\rangle$ of its velocity $V=\frac{dq}{d\tau}$ for the metric is equal to $-c^2$*

Let us consider an observer who follows a material body in his chart. It will be given by :

$q(t)=\varphi_o(ct,x(t))$ with $x(t)\in\Omega_3(t)$

The derivative with respect to t gives the velocity of the body with respect to the time of the observer :

$V_o=\frac{dq}{dt}=c\varepsilon_0+\overrightarrow{v}$ where $\overrightarrow{v}=\frac{dx}{dt}$ is the spatial speed of the body.

$V_o\in T_{q(t)}M$ is located and measured at the location of the body.

The velocity of the body, with respect to its proper time is : $V=\frac{dq}{d\tau}$.

From $V=\frac{dq}{d\tau}=\frac{dq}{dt}\frac{dt}{d\tau}$ and $\langle V,V\rangle=-c^2$ we get :

$$\frac{dt}{d\tau}=\sqrt{1-\frac{\|v\|^2}{c^2}} \tag{1}$$

where $\|v\|^2 = g_3(q(t))(\overrightarrow{v}, \overrightarrow{v})$

$V_o = V\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$ can be seen as the relative velocity of the body with respect to the observer.

1.1.4 The tetrad

There are orthonormal bases for the metric, called tetrads, at each point. For the observer who defines the chart of M, that we will call the standard observer, the vector $\varepsilon_0(m)$ is imposed (it is in the direction of his velocity), and we assume that he chooses an orthonormal spatial basis $(\varepsilon_i(m))_{i=1}^3$ at each point, the fact that the basis is orthonormal can be checked locally. The choice of the chart and the tetrad is arbitrary, under these conditions.

Another observer, located at the same point m, is represented by another tetrad, whose vector $\widetilde{\varepsilon_0}(m)$ is future oriented but usually distinct from $\varepsilon_0(m)$.

A tetrad is expressed in the basis of the chart : $\varepsilon_i(m) = \sum_{\alpha=0}^3 P_i^\alpha(m)\,\partial\xi_\alpha$.

For the standard observer the matrix $[P(m)]_j^\alpha$ has necessarily the form $\begin{bmatrix} 1 & 0 \\ 0 & [Q]_{3\times3} \end{bmatrix}$ with $\partial\xi_0 = c\partial t$.

The scalar product on TM can be computed from the components of the tetrad. By definition :

$g_{\alpha\beta}(m) = \langle\partial\xi_\alpha, \partial\xi_\beta\rangle = \sum_{ij=0}^3 \eta_{ij}\,[P']_\alpha^i\,[P']_\beta^j$ with $[P'] = [P]^{-1}, [Q'] = [Q]^{-1}$

The induced metric on the cotangent bundle is denoted with upper indexes :$g^* = \sum_{\alpha\beta} g^{\alpha\beta}\partial\xi_\alpha \otimes \partial\xi_\beta$ and its matrix is $[g]^{-1}$: $g^{\alpha\beta}(m) = \sum_{ij=0}^3 \eta^{ij}\,[P]_i^\alpha\,[P]_j^\beta$

$$[g]^{-1} = [P]\,[\eta]\,[P]^t \Leftrightarrow [g] = [P']^t\,[\eta]\,[P'] \qquad (2)$$

with

$$[\eta] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & [I]_{3\times3} \end{bmatrix}$$

In the standard chart of the observer :

$[g] = [P']^t\,[\eta]\,[P'] = \begin{bmatrix} -1 & 0 \\ 0 & [g_3] \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & [Q']^t\,[Q'] \end{bmatrix}$

$[g]^{-1} = [P]\,[\eta]\,[P]^t = \begin{bmatrix} -1 & 0 \\ 0 & [g_3]^{-1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & [Q]\,[Q]^t \end{bmatrix}$

and $[g_3]$ is definite positive.

g_3 depends on the coordinates in the chart, *including the time t.*

The metric defines a volume form on M. Its expression in any chart is, by definition :

$\varpi_4(m) = \varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 = \sqrt{|\det[g]|}d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

$[g] = [P']^t\,[\eta]\,[P'] \Rightarrow \det[g] = -(\det[P'])^2 \Rightarrow \sqrt{|\det[g]|} = \det[P']$

$$\varpi_4 = \sqrt{|\det[g]|}d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \qquad (3)$$

and it induces a volume form on each hypersurface $\Omega_3(t)$:

$\varpi_3 = \sqrt{|\det[g_3]|}d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

1.1.5 Rotation

All material bodies, up to the molecular level, have a definite spatial organization, such that an observer can measure their "arrangement" with respect to an orthonormal basis in the hypersurfaces $\Omega_3(t)$. We assume that this property is shared by all material bodies, including elementary particles. It is represented as follows :

Proposition 6 *To any material body is attached, at its location, a tetrad* $(e_i(\tau))_{i=0}^3$ *such that* $e_0(\tau)$ *is in the direction of its velocity* $\frac{dq}{d\tau}$.

This assumption acknowledges that the physical universe has "a relief" : this is not just a juxtaposition of points. In particular even if an elementary particle is represented as a point, it can have a rotational motion.

The spatial arrangement of a material body with respect to an observer is measured as a rotation of $(e_i(\tau))_{i=1}^3$ with respect to $(\varepsilon_i(t))_{i=1}^3$. It must be at the same location. If the observer and the body share the same world line then $\tau = t, e_0(\tau) = \varepsilon_0(t)$ but this is not necessarily so. A rotation is measured in the 4 dimensional vector space tangent to the location of the body, it preserves the tetrads, it is represented by an element s of the orthogonal group $SO(3,1)$.

If one can measure a rotation, there is also a rotational motion, and it is measured by the derivative $\frac{ds(t)}{dt}$ or, more efficiently, by $s^{-1} \cdot \frac{ds}{dt}$ which sums up to measure the instantaneous rotational motion ω with respect to the previous arrangement. The quantity $s^{-1} \cdot \frac{ds}{dt}$ belongs to the Lie algebra of the group of rotations, and does not depend on the observer. But $SO(3,1)$ and the spin group $Spin(3,1)$ have the same Lie algebra, and this has a physical meaning : a rotation with axis r and rotational speed $\frac{ds}{dt}$ has the same measure as a rotation with axis $-r$ and rotational speed $-\frac{ds}{dt}$. In Euclidean Geometry it is usually said that the 2 instantaneous rotational motions cannot be distinguished (there is no universal way to distinguish r from $-r$) but this is no longer true in Relativity : the velocities are necessarily future oriented. Moreover it is clear that going from one rotational motion to the other would be physically felt. As a consequence the right group for measuring rotations in Relativity is the spin group $Spin(3,1)$, which is equivalent to the product $SO(3,1) \times \{+1; -1\}$.

Using the matricial expressions of the elements of the Spin group it is then easy to compute the well known formulas in a change of observers (Th.Physics 3.2.3). They hold : to go from one tetrad of an observer to the tetrad of another observer, located at the same point, for the components of vectors expressed in the tetrads. No assumption of "inertial motion" or the speed of light is necessary. As expected the formulas involve the spatial speed $\overrightarrow{v}$. The Spin group belongs to the Clifford algebra $Cl(3,1)$ which leads to introduce Clifford bundles, and gives a simpler way to make all these computations.

The unified theory is based on the existence of a copy of the Clifford Algebra $Cl(\mathbb{C},4)$ at each point, in which the states ψ of particles are represented. There is a hermitian product in $Cl(\mathbb{C},4)$ and a unitary group U which is a subset of $Cl(\mathbb{C},4)$ and gives the gauge for the measure of ψ. The copy of U at each point defines a principal bundle P_U and the field is represented by a principal connection on P_U.

So there is a Clifford bundle $P_C(\Omega, Cl(\mathbb{C},4), \pi)$, and a principal bundle P_U, subbundle of P_C.

We introduce now these fiber bundles.

1.2 The fiber bundle P_{Cl}

From the orthonormal bases $(\varepsilon_i(m))_{i=0}^3$ we have a 4 dimensional vector space $F(m) = Span(\varepsilon_i(m))_{i=0}^3$ endowed with a scalar product with signature $(3,1)$ and a Clifford Algebra $Cl(m) \sim Cl(3,1)$ located at m with orthonormal basis $F_\alpha = \varepsilon_{j_1} \cdot ... \cdot \varepsilon_{j_p}$

i) The tetrads at each point define on Ω a vector bundle of orthonormal bases for the metric. Using the functor between the categories of vector bundles and Clifford bundles, we have a Clifford bundle $P_{Cl} = P(\Omega, Cl(\mathbb{R},3,1), \pi)$, that is at each point a copy of the Clifford bundle $Cl(\mathbb{R},3,1)$.

An orthonormal basis of $Cl(\mathbb{R},3,1)$ at m is the ordered product of vectors of a tetrad. It will be convenient to use the basis :

$Z = a + v_0\varepsilon_0 + v_1\varepsilon_1 + v_2\varepsilon_2 + v_3\varepsilon_3 + w_1\varepsilon_0 \cdot \varepsilon_1 + w_2\varepsilon_0 \cdot \varepsilon_2 + w_3\varepsilon_0 \cdot \varepsilon_3 + r_1\varepsilon_3 \cdot \varepsilon_2 + r_2\varepsilon_1 \cdot \varepsilon_3 + r_3\varepsilon_2 \cdot \varepsilon_1$

$+x_0\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 + x_1\varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_2 + x_2\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_3 + x_3\varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_1 + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

and to represent a vector by the notation :

$Z = [a, v_0, v, w, r, x_0, x, b]$ in $Cl(\mathbb{R},3,1)$ with the 4 scalars a, v_0, x_0, b and the 4 vectors $v, w, r, x \in \mathbb{R}^3$.

Transposition, graded involution and scalar product are extended from the Clifford algebra to the Clifford bundle, pointwise.

$(a, v_0, v, w, r, x_0, x, b)^t = (a, v_0, v, -w, -r, -x_0, -x, b)$

The adjoint map is the Clifford morphism : $Ad_g : Cl(\mathbb{R},3,1) \to Cl(\mathbb{R},3,1) :: Ad_g Z = g \cdot Z \cdot g^{-1}$ defined for any invertible element $g \in Cl(\mathbb{R},3,1)$.

$Ad_{g \cdot g'} = Ad_g \circ Ad_{g'}; Ad_1 = Id$

$Ad_g(X \cdot Y) = Ad_g X \cdot Ad_g Y$

The Spin group $Spin(3,1)$ is the subset of $Cl(\mathbb{R},3,1)$ whose elements can be written as the product $g = u_1 \cdot ... \cdot u_{2p}$ of an even number of vectors of norm $\langle u_k, u_k \rangle = 1$. The adjoint map preserves the scalar product if g belongs to the orthogonal group, equal to the product $\mathbb{R} \times Spin(3,1)$. Thus $\forall g \in Spin(3,1) : g^t \cdot g = 1$

ii) By restriction of P_{Cl} to the Spin group we have a principal bundle $P_G(\Omega, Spin(3,1), \pi)$. It can be equivalently defined by the standard gauge $\mathbf{p}(m)$ equal at $1 \in Cl(\mathbb{R},3,1)$ located at m and associated to the gauge of the standard observer and his tetrad on one hand, and the right action $\rho : P_G \times Spin(3,1) \to P_G$ defined by the product point wise $p(m) = \mathbf{p}(m) \cdot \sigma$ on the other hand. $\mathbf{p} \in \mathfrak{X}(P_G)$ is a section of P_G defined by $\mathbf{p}(m) = 1$ at each point.

From the standard observer one defines any other observer, located at the same point, by $\sigma \in Spin(3,1)$ and another tetrad $(\widetilde{\varepsilon}_i(m))_{i=0}^3$ deduced from $(\varepsilon_i(m))_{i=0}^3$ by the action of $\sigma : \widetilde{\varepsilon}_i(m) = \sum_{j=0}^3 [Ad_\sigma]_i^j \varepsilon_j(m)$.

This is equivalent to define another observer by the gauge $p(m) = \rho(\mathbf{p}(m), \varkappa) \in P_G$ with the right action.

iii) All geometric measures, such as the motion, are done in P_{Cl} at a point, with respect to some tetrad associated to an observer. The measures are equivariant : we have the relation of equivalence :

$(p(m), Z(m)) \sim (\rho(p(m), \varkappa), Ad_{\varkappa^{-1}} Z(m))$.

The measures are done in the associated vector bundle $P_G[Cl(\mathbb{R},3,1), Ad]$.

In a change of gauge : $\mathbf{p}_G \to \widetilde{(\mathbf{p}_G)} = \rho(\mathbf{p}_G, \varkappa^{-1})$

the basis $F_\alpha = \varepsilon_{j_1} \cdot ... \cdot \varepsilon_{j_p} \to \widetilde{F}_\alpha = \widetilde{\varepsilon}_{j_1} \cdot ... \cdot \widetilde{\varepsilon}_{j_p} = Ad_\varkappa F_\alpha$

$Z(m) = (\mathbf{p}_G(m), Z) \sim (\rho(\mathbf{p}_G, \varkappa^{-1}), Ad_\varkappa Z)$

1.3 The unitary group U

1.3.1 The Clifford algebra $Cl(\mathbb{C},4)$

$Cl(\mathbb{C},4)$ is the Clifford algebra on $\mathbb{C}^4$ endowed with the bilinear symmetric form $\langle Z, Z' \rangle = \sum_{j=0}^3 Z^j Z'^j$.

The basis $(F_a)_{a=1}^{16}$ of $Cl(\mathbb{C},4)$ is the product of vectors of an orthonormal basis of $\mathbb{C}^4$ so, formally, it is the same as the basis of $Cl(\mathbb{R},3,1)$ with complex components. A vector of $Cl(\mathbb{C},4)$ is represented with the notation :

$Z = (a, v_0, v, w, r, x_0, x, b)$ with the 4 complex scalars a, v_0, x_0, b and the 4 vectors $v, w, r, x \in \mathbb{C}^3$.

The product is in $Cl(\mathbb{C},4)$:

$(a, v_0, v, w, r, x_0, x, b) \cdot (a', v_0', v', w', r', x_0', x', b') = (A, V_0, V, W, R, X_0, X, B)$

$A = aa' + v_0 v_0' + v^t v' - w^t w' - r^t r' - x_0' x_0 - x^t x' + bb'$

$V_0 = av_0' + v_0 a' - v^t w' + w^t v' - r^t x' - x^t r' + x_0 b' - bx_0'$

$V = av' + a'v + v_0 w' - v_0' w + x_0' r + x_0 r' + b'x - bx' + j(v) r' + j(r) v' - j(w) x' + j(x) w'$

$W = aw' + a'w + v_0 v' - v_0' v + b'r + br' + x_0' x - x_0 x' - j(v) x' + j(w) r' + j(r) w' + j(x) v'$

$R = ar' + a'r - x_0' v - x_0 v' + b'w + bw' + v_0' x + v_0 x' - j(v) v' + j(w) w' + j(r) r' + j(x) x'$

$X_0 = ax_0' + a'x_0 + v_0 b' - bv_0' - v^t r' - r^t v' + w^t x' - x^t w'$

$X = ax' + a'x + b'v - bv' - x_0' w + x_0 w' + v_0 r' + v_0' r + j(v) w' - j(w) v' + j(r) x' + j(x) r'$

$B = ab' + a'b + v_0 x_0' - v_0' x_0 + v^t x' - x^t v' - w^t r' - r^t w'$

Notation 7 *In the expression above j is the operator : $j : \mathbb{C}^3 \to L(\mathbb{C},3)$ ::*
$j(z) = \begin{bmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{bmatrix}$

It has many algebraic properties and is very convenient (See Th.Physics, Annex).

1.3.2 Real structure on $Cl(\mathbb{C},4)$

There is a Clifford morphism $C : Cl(3,1) \to Cl(\mathbb{C},4)$:

$j = 1,2,3 : C(\varepsilon_j) = \varepsilon_j$

$C(\varepsilon_0) = i\varepsilon_0$

$Cl(3,1)$ is a real form of $Cl(\mathbb{C},4) : Cl(\mathbb{C},4) = C(Cl(3,1)) \oplus iC(Cl(3,1))$

$\forall Z \in Cl(\mathbb{C},4), Z = \operatorname{Re} Z + i \operatorname{Im} Z$ with $\operatorname{Re} Z = C(Z_1), \operatorname{Im} Z = C(Z_2)$

The basis $(F_a)_{a=1}^{16}$ of $Cl(\mathbb{C},4)$ is the product of vectors of an orthonormal basis of $Cl(\mathbb{R},3,1)$ but a vector $\operatorname{Re} Z$ or $\operatorname{Im} Z$ *can have real or pure imaginary components* in the basis of $Cl(\mathbb{C},4)$:

$C(Cl(3,1)) = \{(a, iv_0, v, iw, r, x_0, ix, ib), a, v_0, v, w, r, x_0, x, b \in \mathbb{R}\} \subset Cl(\mathbb{C},4)$

and the complex conjugate of a vector of $Cl(\mathbb{C},4)$ is :

$CC(\operatorname{Re} Z + i \operatorname{Im} Z) = \operatorname{Re} Z - i \operatorname{Im} Z$

$$CC(a, v_0, v, w, r, x_0, x, b) = \left(\overline{(a)}, -\overline{(v_0)}, \overline{(v)}, -\overline{(w)}, \overline{(r)}, \overline{(x_0)}, -\overline{(x)}, -\overline{(b)}\right) \quad (4)$$

The complex conjugate of a map $f \in \mathcal{L}(Cl(\mathbb{C},4); Cl(\mathbb{C},4))$ is $CC(f)$ such that $CC(f)(Z) = CC(f(CC(Z)))$. The map is real if $CC(f) = f$.

$CC(Ad_g) = Ad_{CC(g)}$

The adjoint map is real if g is real.

1.3.3 Hermitian scalar product

There is a scalar product on $Cl(\mathbb{C},4)$, bilinear, symmetric, complex valued form denoted $\langle Z, Z' \rangle_{Cl(\mathbb{C},4)}$.

On $Cl(\mathbb{C},4)$ one defines the hermitian scalar product :

$\langle Z, Z' \rangle_H = \langle CC(Z), Z' \rangle_{Cl(\mathbb{C},4)}$

$$\langle (a, v_0, v, w, r, x_0, x, b), (a', v_0', v', w', r', x_0', x', b') \rangle_H \quad (5)$$

$$= \overline{(a)}a' - \overline{(v_0)}v_0' + \overline{(v)}^t v' - \overline{(w)}^t w' + \overline{(r)}^t r' + \overline{(x_0)}x_0' - \overline{(x)}^t x' - \overline{(b)}b' \quad (6)$$

The Clifford morphisms on $Cl(\mathbb{C},4)$ which preserves the hermitian product are necessarily of the form Ad_g where g is the product of an even number of vectors of $\operatorname{Re} Cl(\mathbb{C},4)$. The unitary group of $Cl(\mathbb{C},4)$ is then defined as :

$$U = \left\{g \in Cl(\mathbb{C},4) : CC\left(g^t\right) \cdot g = 1\right\} \quad (7)$$

and the Clifford morphisms which preserve the hermitian scalar product are Ad_u with $u \in \mathbb{C} \times U(Cl(\mathbb{C},4))$.

Because $\sigma \in Spin(3,1) \Rightarrow \sigma^t \cdot \sigma = 1, C(\sigma) \in U$

1.3.4 Lie algebra

i) The group U is a 16 dimensional real Lie group with real Lie algebra : $T_1U = \{T \in Cl(\mathbb{C},4) : CC(T^t) + T = 0\}$

ii) In the usual orthonormal basis of $Cl(\mathbb{C},4)$:

$T_1U = \left\{(iA, V_0, iV, iW, R, X_0, iX, B), A, V_0, X_0, B \in \mathbb{R}, V, W, R, X \in \mathbb{R}^3\right\}$

T_1U has a basis : $\kappa_a = \zeta_a F_a$ where $\zeta_a = i$ or 1 so that in this basis :

$$\begin{array}{c} Z \in T_1U :: Z = \sum_{a=1}^{16} z^a \kappa_a, z^a \in \mathbb{R} \\ Z = (A, V_0, V, W, R, X_0, X, B), A, V_0, X_0, B \in \mathbb{R}, V, W, R, X \in \mathbb{R}^3 \end{array} \tag{8}$$

The real basis of T_1U is :

$(\kappa_a)_{a=1}^{16} = i, \varepsilon_0, i\varepsilon_1, i\varepsilon_2, i\varepsilon_3, i\varepsilon_0 \cdot \varepsilon_1, i\varepsilon_0 \cdot \varepsilon_2, i\varepsilon_0 \cdot \varepsilon_3, \varepsilon_3 \cdot \varepsilon_2, \varepsilon_1 \cdot \varepsilon_3, \varepsilon_2 \cdot \varepsilon_1, \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3, i\varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_2, i\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_3, i\varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_1, \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3.$

In the real basis κ_a the coordinates of $CC(Z^t)$ for $Z \in T_1U$ are just $-z^a$.

iii) The basis (κ_a) is orthonormal for the hermitian product, with $\langle \kappa_\beta, \kappa_\gamma \rangle_H = \langle F_\beta, F_\gamma \rangle_H = \eta_{\beta\gamma}$ with the matrix $[\eta]$ of the scalar product on $Cl(3,1)$.

Then :

$$\left\langle \sum_{a=1}^{16} X^a \kappa_a, \sum_{a=1}^{16} Y^a \kappa_a \right\rangle_H = \sum_{a=1}^{16} \eta_{aa} X^a Y^a \tag{9}$$

iv) We have a useful identity :$\forall X, Y, Z \in T_1U : \langle X, [Z,Y] \rangle_H = \langle [X,Z], Y \rangle_H$

Proof. Let $X, Y \in T_1U$

$\forall Z \in T_1U : \frac{d}{d\tau}\left(Ad_{\exp \tau Z} X\right) = Ad_{\exp \tau Z}[Z, X]$

$\langle Ad_{\exp \tau Z} X, Ad_{\exp \tau Z} Y \rangle_H = \langle X, Y \rangle_H$

By taking the derivative with respect to τ :

$\langle Ad_{\exp \tau Z}[Z,X], Ad_{\exp \tau Z} Y \rangle_H + \langle Ad_{\exp \tau Z} X, Ad_{\exp \tau Z}[Z,Y] \rangle_H = 0$

$\langle [Z,X], Y \rangle_H + \langle X, [Z,Y] \rangle_H = 0$ ■

1.3.5 Chart on U

i) The property $CC(u^t) \cdot u = 1$ implies relations between the coordinates of u

$u = (a, v_0, v, w, r, x_0, x, b), a, v_0, x_0, b \in \mathbb{C}, v, w, r, x \in \mathbb{C}^3$

expressed in the usual basis of $Cl(\mathbb{C},4)$:

$\operatorname{Im}(av_0 + bx_0 + v^t w + x^t r) = 0$

$\operatorname{Im}(ax_0 + bv_0 + v^t r + x^t w) = 0$

$\operatorname{Im}(ab + v_0 x_0 + v^t x + w^t r) = 0$

$\operatorname{Im}(x_0 v + bw + ar + v_0 x) + j(\operatorname{Re} v)\operatorname{Im} v - j(\operatorname{Re} w)\operatorname{Im} w + j(\operatorname{Re} r)\operatorname{Im} r - j(\operatorname{Re} x)\operatorname{Im} x = 0$

$\operatorname{Re}(av + v_0 w - x_0 r - bx + j(x) w - j(v) r) = 0$

$\operatorname{Re}(v_0 v + aw - br - x_0 x + j(r) w - j(v) x) = 0$

$\operatorname{Re}(-bv + x_0 w - v_0 r + ax + j(v) w - j(x) r) = 0$

$a\overline{(a)} - v_0\overline{(v_0)} + v^t\overline{(v)} - w^t\overline{(w)} + r^t\overline{(r)} + x_0\overline{(x_0)} - x^t\overline{(x)} - b\overline{(b)} = \langle u, u \rangle_H = 1$

U is not a vector space but a real manifold embedded in $Cl(\mathbb{C},4)$. There are several charts on U.

ii) Its Lie algebra provides a chart of the manifold :

$\forall \phi \in \mathbb{R} : u \in U \Rightarrow e^{i\phi} u \in U$

The center of U is $U(1)$ which is a normal subgroup.

$T_1 U(1) = (iA, 0, 0, 0, 0, 0, 0, 0)$

$U_0 = U/U(1)$ is a Lie group, there is a unique $A \in \mathbb{R} : u \in U : u = e^{iA} \cdot \gamma, \gamma \in U/U(1)$

Its Lie algebra is :

$T_1 U_0 = (0, V_0, iV, iW, R, X_0, iX, B)$

It has a real and imaginary part :

$(0, V_0, iV, iW, R, X_0, iX, B) = (0, 0, 0, iW, R, X_0, iX, 0) \oplus (0, V_0, iV, 0, 0, 0, 0, B)$

$(0, 0, 0, iW, R, X_0, iX, 0)$ is the real part of $T_1 U_0$, this is a real Lie algebra

$T_r = (0, 0, 0, 0, R, 0, 0, 0)$ is the Lie algebra of $C(Spin(3)) = \left(\cos \mu_r, 0, 0, 0, \frac{\sin \mu_r}{\mu_r} R, 0, 0, 0\right)$

$\exp T_r = \exp(0, 0, 0, 0, R, 0, 0, 0) = \cos \mu_r + \frac{\sin \mu_r}{\mu_r}(T_r)$ with $\mu_r^2 = R^t R = -T_r \cdot T_r$

$(0, 0, 0, iW, R, 0, 0, 0)$ is the Lie algebra of the Real part of $Spin(\mathbb{C}, 4)$

$\exp T_w = \exp(0, 0, 0, iW, 0, 0, 0, 0) = \cosh \mu_w + \frac{\sinh \mu_w}{\mu_w}(T_w)$ with $\mu_w^2 = W^t W = T_w \cdot T_w$

$\operatorname{Re} Spin(\mathbb{C}, 4) = \{(a, 0, 0, iw, r, 0, 0, ib)\} = \{\exp T_r \cdot \exp T_w\}$

$a = \cosh \mu_w \cos \mu_r$

$w = \frac{\sinh \mu_w}{\mu_w}\left(\cos \mu_r - \frac{\sin \mu_r}{\mu_r} j(R)\right) W$

$r = \cosh \mu_w \frac{\sin \mu_r}{\mu_r} R$

$b = -\frac{\sinh \mu_w}{\mu_w} \frac{\sin \mu_r}{\mu_r}(W^t R)$

$(0, 0, 0, iW, R, 0, 0, 0) \oplus (0, 0, 0, 0, 0, X_0, iX, 0)$ is the real part of $T_1 U_0$

$T_x = (0, 0, 0, 0, 0, X_0, iX, 0)$, $T_x \cdot T_x = \left(-X_0^2 + X^t X, 0, 0, 0, 0, 0, 0, 0\right)$

$\exp T_x = \cosh \mu_x + \frac{\sinh \mu_x}{\mu_x} T_x$ with $\mu_x^2 = -X_0^2 + X^t X =$ and one can check that $(\exp T_x)^t \cdot \exp T_x = 1$

$T_v = (0, V_0, iV, 0, 0, 0, 0, B), T_v \cdot T_v = \left(V_0^2 - V^t V + B^2, 0, 0, 0, 0, 0, 0, 0\right) \in Cl(3,1)_R$

$\exp T_v = \cosh \mu_v + \frac{\sinh \mu_v}{\mu_v} T_v = \left(\cosh \mu_v, \frac{\sinh \mu_v}{\mu_v} V_0, \frac{\sinh \mu_v}{\mu_v} iV, 0, 0, 0, 0, \frac{\sinh \mu_v}{\mu_v} B\right)$

$= (\cosh \mu_v, 0, 0, 0, 0, 0, 0, 0) + iC\left(\left[0, -\frac{\sinh \mu_v}{\mu_v} V_0, \frac{\sinh \mu_v}{\mu_v} V, 0, 0, 0, 0, -\frac{\sinh \mu_v}{\mu_v} B\right]\right)$

with $\mu_v^2 = T_v \cdot T_v = V_0^2 - V^t V + B^2$

Then the elements of the group read :

$U = \left\{e^{iA} g, A \in \mathbb{R}, g = (a, v_0, v, w, r, x_0, x, b), a, v_0, x_0, b \in \mathbb{C}, v, w, r, x \in \mathbb{C}^3\right\}$

$\Rightarrow CC(g^t) \cdot g = 1$

Thus we have the chart of U :

$$\varphi_u : T_1 U \to U :: \varphi_u\left(\sum_{a=1}^{16} z^a \kappa_a\right) = \varphi_u(A, V_0, V, W, R, X_0, X, B) \quad (10)$$

$$= u = e^{iA} \exp T_r \cdot \exp T_w \cdot \exp T_x \cdot \exp T_v \quad (11)$$

where $z^a \in \mathbb{R}$

iii) The elements of the unitary group U are necessarily the product of an even number of vectors of $Span\left(\varepsilon_j\right)_{j=0}^3$, so they can be expressed as homogeneous elements, and then $Z^{-1} = Z^t / \langle Z, Z\rangle$ with the bilinear scalar product on $Cl\left(\mathbb{C}, 4\right)$.

$Z^{-1} = Z^t / \langle Z, Z\rangle = (a, v_0, v, -w, -r, -x_0, -x, b) / \langle Z, Z\rangle$

and they are unitary if :

$Z^{-1} = CC\left(Z^t\right) = Z^t / \langle Z, Z\rangle \Leftrightarrow CC\left(Z\right) = Z / \langle Z, Z\rangle$

$CC\left(a, v_0, v, w, r, x_0, x, b\right) = \left(\overline{(a)}, -\overline{(v_0)}, \overline{(v)}, -\overline{(w)}, \overline{(r)}, \overline{(x_0)}, -\overline{(x)}, -\overline{(b)}\right) = (a, v_0, v, w, r, x_0, x, b) / \langle Z, Z\rangle$

We have the relations :

$\overline{(a)} = a / \langle Z, Z\rangle ; \overline{(v)} = v / \langle Z, Z\rangle ; \overline{(r)} = r / \langle Z, Z\rangle ; \overline{(x_0)} = x_0 / \langle Z, Z\rangle ;$

$\overline{(v_0)} = -v_0 / \langle Z, Z\rangle ; \overline{(w)} = -w / \langle Z, Z\rangle ; \overline{(x)} = -x / \langle Z, Z\rangle ; \overline{(b)} = -b / \langle Z, Z\rangle$

The condition : $\overline{(a)} = a / \langle Z, Z\rangle \Leftrightarrow a = \overline{(a)} \langle Z, Z\rangle$ reads :

$\operatorname{Re} a + i \operatorname{Im} a = (\operatorname{Re} a - i \operatorname{Im} a)\left(\operatorname{Re}\langle Z, Z\rangle + i \operatorname{Im}\langle Z, Z\rangle\right)$

$\operatorname{Re} a = \operatorname{Re} a \operatorname{Re}\langle Z, Z\rangle + \operatorname{Im} a \operatorname{Im}\langle Z, Z\rangle$

$\operatorname{Im} a = \operatorname{Re} a \operatorname{Im}\langle Z, Z\rangle - \operatorname{Im} a \operatorname{Re}\langle Z, Z\rangle$

$\operatorname{Re} a\left(1 - \operatorname{Re}\langle Z, Z\rangle\right) - \operatorname{Im} a \operatorname{Im}\langle Z, Z\rangle = 0$

$\operatorname{Re} a \operatorname{Im}\langle Z, Z\rangle - \operatorname{Im} a\left(\operatorname{Re}\langle Z, Z\rangle + 1\right) = 0$

The condition for $a \neq 0$ is

$$\det \begin{bmatrix} 1 - \operatorname{Re}\langle Z, Z\rangle & -\operatorname{Im}\langle Z, Z\rangle \\ \operatorname{Im}\langle Z, Z\rangle & -\left(\operatorname{Re}\langle Z, Z\rangle + 1\right) \end{bmatrix} = \left(\operatorname{Re}\langle Z, Z\rangle\right)^2 + \left(\operatorname{Im}\langle Z, Z\rangle\right)^2 - 1 = 0$$

and similarly for v, r, x_0

The condition : $\overline{(b)} = -b / \langle Z, Z\rangle \Leftrightarrow b = -\overline{(b)} \langle Z, Z\rangle$ reads :

$\operatorname{Re} b + i \operatorname{Im} b = -(\operatorname{Re} b - i \operatorname{Im} b)\left(\operatorname{Re}\langle Z, Z\rangle + i \operatorname{Im}\langle Z, Z\rangle\right)$

$\operatorname{Re} b = -\operatorname{Re} b \operatorname{Re}\langle Z, Z\rangle - \operatorname{Im} b \operatorname{Im}\langle Z, Z\rangle$

$\operatorname{Im} b = -\operatorname{Re} b \operatorname{Im}\langle Z, Z\rangle + \operatorname{Im} b \operatorname{Re}\langle Z, Z\rangle$

$\operatorname{Re} b\left(1 + \operatorname{Re}\langle Z, Z\rangle\right) + \operatorname{Im} b \operatorname{Im}\langle Z, Z\rangle = 0$

$\operatorname{Re} b \operatorname{Im}\langle Z, Z\rangle + \operatorname{Im} b\left(-\operatorname{Re}\langle Z, Z\rangle + 1\right) = 0$

The condition for $b \neq 0$ is

$$\det \begin{bmatrix} 1 + \operatorname{Re}\langle Z, Z\rangle & \operatorname{Im}\langle Z, Z\rangle \\ \operatorname{Im}\langle Z, Z\rangle & -\operatorname{Re}\langle Z, Z\rangle + 1 \end{bmatrix} = 1 - \left(\operatorname{Re}\langle Z, Z\rangle\right)^2 - \left(\operatorname{Im}\langle Z, Z\rangle\right)^2 = 0$$

and similarly for v_0, w, x

So we have necessarily $\langle Z, Z\rangle = e^{i\phi}, \phi \in \mathbb{R}$.

Writing $a = \rho_a e^{i\phi_a}$ then we have the relation :

$\overline{(a)} = a / \langle Z, Z\rangle \Leftrightarrow a = \overline{(a)} \langle Z, Z\rangle$

$\rho_a e^{i\phi_a} = \rho_a e^{i(\phi - \phi_a)}$

$\phi_a = \phi - \phi_a$

$\phi_a = \frac{1}{2}\phi + k\pi$

We can write :

$a = e^{\frac{1}{2}i\phi}\rho_a$

and similarly :

$b = -\overline{(b)} \langle Z, Z\rangle \Leftrightarrow \rho_b e^{i\phi_b} = -\rho_b e^{i(\phi - \phi_b)}$

$\phi_b = \phi - \phi_b + k\pi$

$\phi_b = \frac{1}{2}\phi + \frac{1}{2}k\pi$

or, with $a, v_0, x_0, b, \phi \in \mathbb{R}, v, w, r, x \in \mathbb{R}^3$

$\overline{(a)} = a/\langle Z,Z\rangle ; \overline{(v)} = v/\langle Z,Z\rangle ; \overline{(r)} = r/\langle Z,Z\rangle ; \overline{(x_0)} = x_0/\langle Z,Z\rangle ;$
$\overline{(v_0)} = -v_0/\langle Z,Z\rangle ; \overline{(w)} = -w/\langle Z,Z\rangle ; \overline{(x)} = -x/\langle Z,Z\rangle ; \overline{(b)} = -b/\langle Z,Z\rangle$
$Z = e^{\frac{1}{2}i\phi}\left(a, v_0 e^{ik_v\frac{\pi}{2}}, v, we^{ik_w\frac{\pi}{2}}, r, x_0, xe^{ik_x\frac{\pi}{2}}, be^{ik_b\frac{\pi}{2}}\right)$
Then :
$\langle Z,Z\rangle = e^{i\phi}\left(a^2 + v^t v - v_0^2 - w^t w + r^t r + x_0^2 - x^t x - b^2\right)$
$= e^{i\phi}$
$\Rightarrow a^2 + v^t v - v_0^2 - w^t w + r^t r + x_0^2 - x^t x - b^2 = 1$
The elements of the group U are then of the form :
$Z = e^{i\phi}\{a + v_1\varepsilon_1 + v_2\varepsilon_2 + v_3\varepsilon_3 + r_1\varepsilon_3\varepsilon_2 + r_2\varepsilon_1\varepsilon_3 + r_3\varepsilon_2\varepsilon_1 + x_0\varepsilon_1\varepsilon_2\varepsilon_3\}$
$+e^{i\phi}e^{ik\frac{\pi}{2}}\{v_0\varepsilon_0 + w_1\varepsilon_0\varepsilon_1 + w_2\varepsilon_0\varepsilon_2 + w_3\varepsilon_0\varepsilon_3 + x_1\varepsilon_0\varepsilon_3\varepsilon_2 + x_2\varepsilon_0\varepsilon_1\varepsilon_3 + x_3\varepsilon_0\varepsilon_2\varepsilon_1 + b\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\}$

$$\begin{array}{c} U \\ Z = e^{\frac{1}{2}i\phi}\left(a, v_0 e^{ik_v\frac{\pi}{2}}, v, we^{ik_w\frac{\pi}{2}}, r, x_0, xe^{ik_x\frac{\pi}{2}}, be^{ik_b\frac{\pi}{2}}\right) \\ a, v_0, x_0, b, \phi \in \mathbb{R}, v, w, r, x \in \mathbb{R}^3 \\ a^2 + v^t v - v_0^2 - w^t w + r^t r + x_0^2 - x^t x - b^2 = 1 \end{array} \tag{12}$$

1.3.6 The left action ϑ

The group U acts on the Clifford algebra $Cl(\mathbb{C},4)$ by the left action :

$$\vartheta : U \times Cl(\mathbb{C},4) :: \vartheta(u)(Z) = e^{iA}Ad_u Z \tag{13}$$

where A is the unique $A \in \mathbb{R} : u \in U : u = e^{iA}\cdot\gamma, \gamma \in U/U(1)$.

This action preserves the hermitian scalar product, U is a unitary group and $(Cl(\mathbb{C},4),\vartheta)$ a unitary representation of U.

1.4 The fiber bundles P_C, P_U

i) Using the functor between the category of real Clifford bundles $E(\Omega, Cl(\mathbb{R},3,1),\pi)$ and complex Clifford bundles $E(\Omega, Cl(\mathbb{C},4),\pi)$ the Clifford bundle P_{Cl} defines a complex Clifford bundle $P_C(\Omega, Cl(\mathbb{C},4),\pi)$ where the vector space $\mathbb{C}^4$ is endowed with its usual bilinear form : $\langle\varepsilon_i,\varepsilon_j\rangle = \delta_{ij}$.

The morphism $C : Cl(3,1) \to Cl(\mathbb{C},4)$ is extended pointwise to a morphism $P_{Cl} \to P_C$ and P_{Cl} is a real form of $P_C(\Omega, Cl(\mathbb{C},4),\pi)$.

$\forall Z \in P_C(\Omega, Cl(\mathbb{C},4),\pi), Z = \operatorname{Re} Z + i\operatorname{Im} Z$ with $\operatorname{Re} Z = C(Z_1), \operatorname{Im} Z = C(Z_2)$

$$P_C = P_{Cl} \oplus iP_{Cl} \tag{14}$$

Its real structure is consistent with the specific status of the time vector ε_0. Spatial vectors map to real vectors and time vectors to imaginary vectors :
$j = 1,2,3 : C(\varepsilon_j(m)) = \varepsilon_j(m)$
$C(\varepsilon_0(m)) = i\varepsilon_0(m)$

A change of observer in P_{Cl} is given by a map Ad_σ with $\sigma \in Spin(3,1)$ which is a morphism, it has for image $C(Ad_\sigma) = Ad_{C(\sigma)} = Ad_\sigma$ that is a real map, which is extended to the Clifford bundles, and the distinction real

/ imaginary stays the same : it does not depend on the observer. Moreover $C(Spin(3,1)) \subset U$ and the hermitian scalar product is preserved.

The graded involution, transposition and complex conjugation can be extended point wise on P_C, as well as the hermitian scalar product.

ii) By restriction of P_C to the elements of $U(Cl(\mathbb{C},4))$ we have the fiber bundle $P_U = \{Z \in P_C : CC(Z^t) \cdot Z = 1\}$. By taking $\mathbf{p}_u(m) = 1 \in \pi(m)^{-1}(1)$ with the natural right action $\rho_u : P_U \times U \to P_U :: p_U(m) = \mathbf{p}_u(m) \cdot u$ we have a principal fiber bundle $P_U(\Omega, U, \pi)$.

$\mathbf{p}_u(m)$ represents the gauge used by an observer, for physical measures done on the state of a particle.

1.5 Tangent bundles

i) For any Lie group $G \in Cl(3,1)$ the left invariant vector fields are $Z(\tau) = \exp \tau T$ with $T \in T_1 G$. The tangent space at $g \in G$ to the manifold G is $T_g G = \{g \cdot T, T \in T_1 G\}$

The tangent bundle TG is a group, isomorphic to the group product : $TG \simeq G \times (T_1 G, +)$

$(g, \kappa) \times (g', \kappa') = (gg', \kappa + \kappa')$

$(g, \kappa) \times (1, 0) = (g, \kappa)$

$(g, \kappa)^{-1} = (g^{-1}, -\kappa)$

ii) A vector of the tangent bundle TP_{Cl} reads :

$v_p = \sum_{\alpha \in A} v_m^\alpha \partial m_\alpha + \sum_{i \in I} Z_u^a \mathbf{F}_a(m)$

where $\partial m_\alpha = \varphi'_{P_{Cl} m}(m, Z) \partial \xi_\alpha, \pi'(p) \partial m_\alpha = \partial \xi_\alpha$

The vertical bundle $VP_{Cl} = \ker \pi'$ is isomorphic to $Cl(3,1)$

A vector field $Y \in \mathfrak{X}(TP_{Cl})$ is projectable if $\pi'(p)(Y(p)) = y(\pi(p))$ is a vector field on TM. The condition is that the component v_m^α depends only on m.

iii) The fundamental vector fields on TP_G are defined, for any fixed κ in $T_1 Spin(3,1)$, by :

$\zeta : T_1 Spin(3,1) \to VP_G :: \zeta(\kappa)(\rho(\mathbf{p}, \sigma)) = \rho'_g(\mathbf{p}, \sigma) L'_\sigma 1(\kappa)$

$= \rho(\mathbf{p}, \sigma) \cdot \kappa = p \cdot \kappa$

In a change of gauge $\mathbf{p}(m) \to \widetilde{(\rho)}(m) = \rho(\mathbf{p}_a(m), \varkappa(m))$ they change as : $\rho'_p(p, \varkappa)(\zeta(\kappa)(p)) = \zeta(Ad_{\varkappa^{-1}} \kappa)(\rho(p, \varkappa))$

A vector of the tangent bundle TP_G can be written :

$W = V_{mg} + \zeta(\kappa)(p)$ where V_{mg} does not depend on the trivialization : $\pi'_G(p_G)(V_{mg}) = v_m \in T_m M$

iv) Similarly :

The tangent bundle TU is a real group, isomorphic to the semi direct group product : $TU \simeq U \propto_{Ad} (T_1 U, +)$

A vector of the tangent bundle TP_C at $p = \varphi_{P_c}(m, Z)$ reads :

$v_p = \sum_{\alpha \in A} v_m^\alpha \partial m_\alpha + \sum_{i \in I} Z_u^a \mathbf{F}_a(m)$

where $\partial m_\alpha = \varphi'_{P_C m}(m, Z) \partial \xi_\alpha, \pi'(p) \partial m_\alpha = \partial \xi_\alpha$

The vertical bundle $VP_C = \ker \pi'$ is isomorphic to $Cl(\mathbb{C}, 4)$

The fundamental vector fields on TP_U are defined, for any fixed κ in T_1U, by :

$$\zeta_u : T_1U \to VP_U :: \zeta_u(\kappa)(\rho_u(\mathbf{p}_u,\sigma)) = \rho'_{ug}(\mathbf{p}_u,\sigma)\, L'_\sigma 1(\kappa) = p_u(m)\cdot\kappa \quad (15)$$

A vector of the tangent bundle TP_U can be written :

$$Y = Y_{mg} + \zeta_u(\kappa)(p_u) \quad (16)$$

where V_{mg} does not depend on the trivialization : $\pi'_u(p_u)(Y_{mg}) = y \in T_mM$

1.6 About the change of the observer and gauge

It is useful to review some subtle definitions about the observer and the gauge.

1) A standard observer is a construct based on :

i) A chart, that is a map : $\varphi_o : \mathbb{R}^4 \to \Omega :: m = \varphi\left(ct,\xi^1,\xi^2,\xi^3\right)$ usually built from the choice of a spatial domain $\Omega_3(0)$, a spatial chart of this domain $x = \varphi_\Omega\left(\xi^1,\xi^2,\xi^3\right)$ then $m = \varphi_o(ct,x)$. This chart provides a holonomic basis $(\partial\xi_\beta)_{\beta=0..3}$ at each point : the vectors are local translations along the coordinates. The observer has no choice for $\partial\xi_0 = c\partial t$.

ii) The metric defines a time vector field ε_0 on Ω through $\varepsilon_0(m) = \frac{1}{c}gradf'_o(m)$. The 3 dimensional hypersurfaces $\Omega_3(t)$ are diffeomorphic by the flow of the vector ε_0..

iii) The choice, by the observer, of a spatial orthonormal basis $(\varepsilon_i)_{i=1;2;3}$ at each point $m\in\Omega$, which, completed by ε_0, gives the tetrad $(\varepsilon_i)_{i=0..3}$. The tetrads are not necessarily diffeomorphic by the flow of ε_0 : the spatial basis can change arbitrarily with time. Their vectors can be measured in the holonomic basis $(\partial\xi_\beta)_{\beta=0..3}$ by P_i^α and we have necessarily $[P] = \begin{bmatrix} 1 & 0 \\ 0 & [Q]_{3\times3}\end{bmatrix}$ with $\partial\xi_0 = c\partial t$. And conversely the components of the metric are defined by $[g] = [P']^t[\eta][P']$, which is summed up in the euclidean metric $[g_3] = [Q']^t[Q']$.

iv) The tetrad defines automatically the vectors $(F_a)_{a=1}^{16}$ of a basis of $Cl(\mathbb{C},4)$ at each point, and so the Clifford bundle $P_C(M,Cl(\mathbb{C},4),\pi)$, from which P_U is deduced by reduction.

v) The motion (spatial speed and instantaneous rotation) of a material body are measured by the observer in his spatial tetrad, and from there in $Cl(\mathbb{C},4)$.

vi) It is assumed that, by physical experiments, which go further than the measure of the motion, the observer can measure the state ψ of a particle, that is the components of a vector in $P_C(M,Cl(\mathbb{C},4),\pi)$. The standard gauge $\mathbf{p}_u$ is naturally the gauge coming from the tetrad.

vii) Using this mathematical framework to each property of a physical object is then assigned, in a model, a variable whose values are the measures of the property.

2. So what a change of gauge or a change of observer does mean ?

i) For the same observer a change of tetrad is restricted : he has no choice for the vector ε_0. The change of tetrad is then given by an element of $Spin(3)$.

However it is assumed that he can proceed to measures, in U, which go beyond the measures of rotation and translation, he is then not limited to $Spin(3)$.

ii) For another observer the whole construct above should be done. In particular the area Ω is usually not the same : $\Omega_3(0)$ must be orthogonal to the vector ε_0. In Relativity the definition of a system is observer dependent. So it is assumed that, in a change of observer, the comparison can be done on the intersection ω of the areas, if they are large enough.

iii) The change of chart is represented by a transition map : there is a diffeomorphism $f : \varphi_o^{-1}(\omega) \to \varphi_{o'}^{-1}(\omega)$ which gives the coordinates η of the same point for the new observer, with respect to the coordinates ξ of the standard observer. The components of tensors, representing physical quantities, change according to the usual rules, with the jacobian $J = \left[\frac{\partial \eta^\lambda}{\partial \xi^\mu}\right]$ and its inverse K.

However the assumptions above impose some constraints to the change of chart. We have necessarily :

$$\widetilde{[g]} = \begin{bmatrix} -1 & 0 \\ 0 & \widetilde{[g]}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \widetilde{[Q']}^t \widetilde{[Q']} \end{bmatrix}$$

Whoever the observer, expressed in his chart : $g_{00}(m) = -1, \alpha = 1,2,3$: $g_{0\alpha}(m) = g_{\alpha 0}(m) = 0$

This is the consequence of the fact that the observer cannot choose its time vector. However, in the picture above the euclidean metric g_3 depends on t, in accordance with the fact, commonly assumed, that the "Universe is expanding".

So a change of standard chart is actually subject to some conditions, that we can express by looking at the formulas for the matrix of the metric. A change of chart is expressed through the jacobian, and with

$$[K] = [J]^{-1} = \begin{bmatrix} K_0^0 & [K^0]_{1\times 3} \\ [K_0]_{3\times 1} & [k]_{3\times 3} \end{bmatrix}$$

$$\widetilde{[g]} = [K]^t [g] [K] = \begin{bmatrix} K_0^0 & [K_0]^t \\ [K^0]^t & [k]^t \end{bmatrix}^t \begin{bmatrix} -1 & 0 \\ 0 & [g_3] \end{bmatrix} \begin{bmatrix} K_0^0 & [K^0] \\ [K_0] & [k] \end{bmatrix}$$

$$\widetilde{[g]} = \begin{bmatrix} -\left(K_0^0\right)^2 + [K_0]^t [g_3] [K_0] & -K_0^0 [K^0] + [K_0]^t [g_3] [k] \\ -K_0^0 [K^0]^t + [k]^t [g_3] [K_0] & -[K^0]^t [K^0] + [k]^t [g_3] [k] \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & [\widetilde{g}_3] \end{bmatrix}$$

The change of standard chart is then subject to the conditions :

$-\left(K_0^0\right)^2 + [K_0]^t [g_3] [K_0] = -1 \Leftrightarrow \langle \partial\eta_0, \partial\eta_0 \rangle = 0$

$-K_0^0 [K^0] + [K_0]^t [g_3] [k] = 0 \Leftrightarrow \langle \partial\eta_0, \partial\eta_\alpha \rangle = 0$

These conditions are met if ε_0 is preserved : we have then a simple change of spatial basis.

iv) The new basis of $Cl(\mathbb{C}, 4)$ at each point is deduced from the old basis by the action of some $u \in U$. So this is a change of gauge.

v) In any model, using the relations in a change of gauge, an observer can compute what another observer could measure. The fact that the comparison can be physically done is the way to prove experimentally the validity of the construct. But of course it does not mean that the standard observer must change his own gauge.

3. We will usually assume that the domain Ω followed by an observer is a fibered manifold $\Omega(\mathbb{R},\pi_o)$ with a map π_o which is a surjective submersion :$\forall m\in\Omega,\exists t\in\mathbb{R}:\pi_o(m)=t$. With a trivialization : $\varphi_o:\mathbb{R}\times\Omega_3(0)\to\Omega::m=\varphi_o(ct,x)$ we have a fiber bundle $\Omega(\mathbb{R},\Omega_3(0),\pi_o)$: the fibers $\pi_o^{-1}(t)$ over t are diffeomorphic, but *not identical*, to $\Omega_3(0)$

If $E(\Omega,V,\pi_E)$ is a fiber bundle with Ω as base then $\forall p\in E,\pi_E(p)=m,\pi_o(m)=t$ and $\pi_R=\pi_o\circ\pi_E$ is a surjective submersion and we have a fibered manifold $E_R(\mathbb{R},\pi_R)$. However the fiber $\pi_R^{-1}(t)$ is not diffeomorphic to V but to $\Omega_3(0)\times V$. With a trivialization

$\varphi_E:\Omega\times V\to E::p=\varphi_E(m,v)=\varphi_E(\varphi_o(ct,x),v)$

so we have a fiber bundle $E_R(\mathbb{R},\Omega_3(0)\times V,\pi_R)$

A section S over E is defined by :

$S(m)=\varphi_E(m,s(m))$

$s(m)=\varphi_V(\varphi_o(ct,x))$

A section over E_R is defined by :

$S(t)=\varphi_{ER}(t,(x(t),v(t)))$

Physically a map $v:\mathbb{R}\to V::v(t)$ gives the evolution of v, and a section of E_R gives both a trajectory and the evolution of v.

There is no canonical morphism : $\mathfrak{X}(E)\to\mathfrak{X}(E_R)$.

2 Particles

2.1 State of a particle

i) The state of a particle ψ at any point is a variable valued in the Clifford bundle P_C. Along its trajectory $q:\mathbb{R}\to M$ followed by an observer the state is a variable $\psi:\mathbb{R}\to P_C::\psi(t)$

ii) There is the left action ϑ of the group U on $Cl(\mathbb{C},4)$:

$$\vartheta:U\to\mathcal{L}(Cl(\mathbb{C},4);Cl(\mathbb{C},4))::\vartheta(u)(\psi)=\vartheta\left(e^{iA}\gamma\right)(\psi)=e^{iA}Ad_\gamma\psi \quad (17)$$

where

$$u\in U:u=e^{iA}\cdot\gamma,\gamma\in U/U(1) \quad (18)$$

It preserves the hermitian product.

So the Clifford bundle has the U structure given by the left action ϑ, and P_U is the unique principal bundle such that P_C has the structure of associated bundle $P_U[Cl(\mathbb{C},4),\vartheta]$.

At each point m an element of $P_U[Cl(\mathbb{C},4),\vartheta]$ is the class of equivalence

$P_U[Cl(\mathbb{C},4),\vartheta]:(\mathbf{p}_u,\psi)\sim\left(\rho_u(\mathbf{p}_u,u),\vartheta\left(u^{-1}\right)\psi\right)$

which can be represented in a trivialization by $(\mathbf{p}_u,\psi)$.

iii) The fundamental assumption for the particles is the following :

Proposition 8 *Elementary particles are characterized by a vector* $\psi_0 \in Cl(\mathbb{C},4)$ *such that* $\psi = (\mathbf{p}_u, \psi_0) \sim \left(\rho_u(\mathbf{p}_u, u), \vartheta\left(u^{-1}\right)\psi_0\right)$

ψ_0 is the state of the particle in the standard gauge, and it changes with it as $\psi_0 \to \vartheta\left(u^{-1}\right)\psi_0$

2.1.1 Antiparticles

$(Cl(\mathbb{C},4), \vartheta)$ is a representation if the group U, and the fundamental states ψ_0 for the known particles are deduced from their charges under the action of the different parts of the force field. But antiparticles do exist and are stable, and we need to accommodate their representation in the model.

The contragredient representation $\left(Cl(\mathbb{C},4), \overline{(\vartheta)}\right)$ of U provides an inequivalent representation of U, which is mathematically acceptable. The conjugate $\overline{(\vartheta)}$ of the map ϑ is defined by $\overline{(\vartheta)}(u)(\psi) = \overline{(\vartheta(u))\overline{(\psi)}}$ so this is equivalent to represent the state ψ^c of a particle by its conjugate. The Lie algebra of the field, as well as the fundamental vectors ψ_0 have both a real and an imaginary part. The contragredient representation sums up to swap the real and the imaginary parts : $\psi^c = \operatorname{Im}\psi + i\operatorname{Re}\psi = iCC(\psi)$. In the process C' replace C and in the Hermitian form $-\eta$ replace η. The 2 representations are acceptable, experiments show the existence of an antiparticle associated to each particle. The CPT conservation principle says that in a "time inversion" particles transform in anti-particles. So we can state :

Proposition 9 *Antiparticles are represented in the contragredient representation* $\left(Cl(\mathbb{C},4), \overline{(\vartheta)}\right)$ *of* U*. In the standard representation* $(Cl(\mathbb{C},4), \vartheta)$ *of* U *the fundamental state* ψ_0^c *of the antiparticle associated to* ψ_0 *is* $\psi_0^c = \overline{(\psi_0)}$*.*

Or, in other words : the distinction between particles / antiparticles is equivalent to the choice of a signature $(3,1)$ or $(1,3)$ for the metric. In any model we need to make the choice of a signature. It is arbitrary but we need to live with our choice. So, whenever we deal with an antiparticle, in an equation which involve ψ (or the charges as we will see) and the time t we must simultaneously make the change $t \to -t$ and the velocity reads $V = c\varepsilon_0 + \overrightarrow{v} \to V^c = c\varepsilon_0 - \overrightarrow{v}$. Which is seen usually as antiparticles "moving backward in time", which is not physically true, but the operation is necessary to work in a unique mathematical model.

2.2 Geometric Motion of a particle

The first characteristic of the state of a particle is its geometric motion, which is represented in the fiber bundle P_{Cl}.

i) The particle travels on a world line $q(\tau)$ with proper time τ and velocity $\frac{dq}{d\tau} \in TM$ such that $\left\langle \frac{dq}{d\tau}, \frac{dq}{d\tau} \right\rangle = -c^2$.

The velocity $\frac{dq}{dt}$ of the particle reads for an observer in the basis of the standard chart : $V = \frac{dq}{dt} = \sum_{\alpha=0}^{3} V^{\alpha}\partial\xi_{\alpha} = c\varepsilon_0 + \overrightarrow{v}$ where $\overrightarrow{v}$ is the spatial speed with respect to the observer. So $V = \frac{dq}{dt} = \frac{dq}{d\tau}\frac{d\tau}{dt}$ and

$\langle V, V\rangle = \|\overrightarrow{v}\|^2 - c^2 = \left\langle \frac{dq}{d\tau}, \frac{dq}{d\tau}\right\rangle \left(\frac{d\tau}{dt}\right)^2 = -c^2\left(\frac{d\tau}{dt}\right)^2$

$\frac{d\tau}{dt} = \sqrt{1 - \frac{\|\overrightarrow{v}\|^2}{c^2}} = \frac{1}{c}\sqrt{-\langle V, V\rangle}.$

ii) It is assumed that a tetrad $(e_i(\tau))_{i=0}^{3}$ is attached to the particle and measured by the observer in P_G by $s(t) \in Spin(3,1)$:

$e_i(q(\tau)) = Ad_{s(t)}\varepsilon_i(q(\tau))$

Its vector e_0 is such that $\frac{dq}{d\tau} = ce_0$:

$V = \frac{d\tau}{dt}ce_0 = \sqrt{-\langle V, V\rangle}e_0 = \sqrt{-\langle V, V\rangle}Ad_s\varepsilon_0$

Moreover :

$\langle V, \varepsilon_0\rangle_{TM} = -c = \sqrt{-\langle V, V\rangle}\,\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{TM}$

$\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{TM}$ is the scalar product in the tetrad, so

$\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{TM} = \langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}$

$\sqrt{-\langle V, V\rangle} = -\frac{c}{\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}}$

So the geometric state of a particle is represented, at each point of its trajectory, by $s \in Spin(3,1)$, $s(q(t)) \in P_{Cl}(q(t))$ such that :

$$\begin{gathered} e_i = Ad_s\varepsilon_i \\ V = \frac{dq}{dt} = c\varepsilon_0 + \overrightarrow{v} = -\frac{c}{\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}}Ad_s\varepsilon_0 \end{gathered} \tag{19}$$

$\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}$ does not depend on the metric. Notice that $Ad_s\varepsilon_0, \varepsilon_0$ are both vectors in the fixed vector space $\mathbb{R}^4$

In a change of gauge : $\mathbf{p}(m) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \varkappa(m)^{-1}$:

$s \to \widetilde{s} = \varkappa(m) \cdot s$

$(\mathbf{p}(m), e_i) \sim \left(\widetilde{\mathbf{p}}(m), Ad_{\chi(m)}e_i\right) = \left(\widetilde{\mathbf{p}}(m), Ad_{\chi(m)}Ad_s\varepsilon_i\right) = \left(\widetilde{\mathbf{p}}(m), Ad_{\widetilde{s}}\varepsilon_i\right)$

iii) The tetrad attached to the particle is measured in the tetrad of the observer, and the motion is defined by derivation with respect to a fixed observer. A continuous motion is such that the map : $s : \mathbb{R} \to Spin(3,1)$ with respect to the time t of the observer is smooth. From the definitions above (remember that the vectors are defined in a fixed vector space) :

$\forall i = 0..3 : e_i = Ad_s\varepsilon_i$

$\frac{de_i}{dt} = \frac{d}{dt}Ad_s\varepsilon_i = Ad_s\left[s^{-1} \cdot \frac{ds}{dt}, \varepsilon_i\right] = \left[\frac{ds}{dt} \cdot s^{-1}, Ad_s\varepsilon_i\right] = \left[\frac{ds}{dt} \cdot s^{-1}, e_i\right]$

$$\forall i = 0..3 : \frac{de_i}{dt} = \left[\frac{ds}{dt} \cdot s^{-1}, e_i\right] \tag{20}$$

$V = \sqrt{-\langle V, V\rangle}\mathbf{Ad}_s\varepsilon_0 = \sqrt{-\langle V, V\rangle}e_0 = -\frac{c}{\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}}e_0$

$\frac{dV}{dt} = \frac{c}{\left(\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}\right)^2}\frac{d}{dt}\left(\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}\right)e_0 - \frac{c}{\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)}}\frac{de_0}{dt}$

$= \left(\frac{1}{c}\left\langle\left[\frac{ds}{dt} \cdot s^{-1}, V\right], \varepsilon_0\right\rangle_{Cl(3,1)}\right)V + \left[\frac{ds}{dt} \cdot s^{-1}, V\right]$

$$\frac{dV}{dt} = \frac{V}{c}\left\langle \left[\frac{ds}{dt}\cdot s^{-1}, V\right], \varepsilon_0 \right\rangle_{Cl(3,1)} + \left[\frac{ds}{dt}\cdot s^{-1}, V\right] \tag{21}$$

iv) The geometric motion is measured by $s \in Spin(3,1)$ in P_G, with respect to the Clifford bundle $P_G[Cl(3,1), Ad]$. The state of the particle is measured in $P_U[Cl(\mathbb{C},4), \vartheta]$ by u with respect to a gauge $p \in P_U$. The quantities u, s are related.

U is a real manifold embedded in $Cl(\mathbb{C},4)$ with chart :

$\varphi_u : \mathbb{R}^{16} \to U : \varphi_u(A, V_0, V, W, R, X_0, X, B) = u = e^{iA}\sigma(W,R)\cdot\exp(T_x)\cdot\exp(T_v) \in Cl(\mathbb{C},4)$

where $\sigma(W,R) = C(s)$ and $s \in Spin(3,1)$ is the measure of the motion. W corresponds to a translational motion, R corresponds to a rotational motion.

$\sigma = \operatorname{Re} Spin(\mathbb{C},4) = (a,0,0,iw,r,0,0,ib) = \exp T_r \cdot \exp T_w$

$a = \cosh\mu_w \cos\mu_r$

$w = \frac{\sinh\mu_w}{\mu_w}\left(\cos\mu_r - \frac{\sin\mu_r}{\mu_r} j(R)\right) W$

$r = \cosh\mu_w \frac{\sin\mu_r}{\mu_r} R$

$b = -\frac{\sinh\mu_w}{\mu_w}\frac{\sin\mu_r}{\mu_r}(W^t R)$

$\mu_r^2 = R^t R = -T_r\cdot T_r$

$\mu_w^2 = W^t W = T_w \cdot T_w$

and :

$w^t r = -ab$

$a^2 - b^2 - w^t w + r^t r = 1 \Rightarrow b^2 + w^t w = a^2 + r^t r - 1$

$[Ad_\sigma] = \left(2a^2 + 2r^t r - 1\right) I_4$

$+2\begin{bmatrix} 1 & i(aw - br + j(w)r)^t \\ i(-aw + br + j(w)r) & aj(r) + bj(w) + j(r)j(r) + j(w)j(w) \end{bmatrix}$

$C(Ad_s\varepsilon_0) = Ad_{C(s)}C(\varepsilon_0) = iAd_s\varepsilon_0$

$\langle Ad_s\varepsilon_0, \varepsilon_0\rangle_{Cl(3,1)} = \langle C(Ad_s\varepsilon_0), C(\varepsilon_0)\rangle_{Cl(\mathbb{C},4)} = \langle iAd_\sigma\varepsilon_0, i\varepsilon_0\rangle_{Cl(\mathbb{C},4)} = -\langle Ad_\sigma\varepsilon_0, \varepsilon_0\rangle_{Cl(\mathbb{C},4)}$

$C(V) = \frac{c}{\langle Ad_s\varepsilon_0,\varepsilon_0\rangle_{Cl(\mathbb{C},4)}} iAd_s\varepsilon_0 \in Cl(\mathbb{C},4)$

The computation gives :

$Ad_\sigma\varepsilon_0 = \left(2(\cosh\mu_w)^2 + 1\right)\varepsilon_0 - 2i\frac{\cosh\mu_w\sinh\mu_w}{\mu_w} W$

$\langle Ad_\sigma\varepsilon_0, \varepsilon_0\rangle_{Cl(\mathbb{C},4)} = \left(2(\cosh\mu_w)^2 + 1\right) - 2i\frac{\cosh\mu_w\sinh\mu_w}{\mu_w}\langle W, \varepsilon_0\rangle$

$= 2(\cosh\mu_w)^2 + 1$

$C(V) = \frac{c}{\langle Ad_\sigma\varepsilon_0,\varepsilon_0\rangle_{Cl(\mathbb{C},4)}} iAd_\sigma\varepsilon_0 = ci\varepsilon_0 + c2\frac{\cosh\mu_w\sinh\mu_w}{\mu_w}\frac{1}{2(\cosh\mu_w)^2+1} W \in Cl(\mathbb{C},4)$

The 4 dimensional velocity of the particle is then :

$$V = c\left(\varepsilon_0 + 2\frac{\sinh\mu_w\cosh\mu_w}{1 + 2(\cosh\mu_w)^2}\sum_{a=6}^{9}\frac{W^{a-5}}{\mu_w}\varepsilon_j\right)$$

thus the spatial speed is : $v = 2\frac{\sinh\mu_w\cosh\mu_w}{1+2(\cosh\mu_w)^2}\frac{c}{\mu_w}W$. The component W of u in the chart of U depends only on the spatial speed.

The components of the velocity, expressed in the holonomic basis of a chart, are then :

$$V^{\alpha}=\sum_{j=0}^{3}V^{j}P_{j}^{\alpha}=cP_{0}^{\alpha}+2\frac{\sinh\mu_{w}\cosh\mu_{w}}{1+2\left(\cosh\mu_{w}\right)^{2}}\sum_{j=1}^{3}\frac{W^{j+5}}{\mu_{w}}P_{j}^{\alpha} \tag{22}$$

By computing :

$v^{t}v=4\left(\frac{\sinh\mu_{w}\cosh\mu_{w}}{1+2(\cosh\mu_{w})^{2}}\right)^{2}\frac{c^{2}}{\mu_{w}^{2}}W^{t}W=4c^{2}\left(\frac{\sinh\mu_{w}\cosh\mu_{w}}{1+2(\cosh\mu_{w})^{2}}\right)^{2}$

$\frac{\sinh\mu_{w}\cosh\mu_{w}}{1+2(\cosh\mu_{w})^{2}}=\frac{1}{2}\left\|\frac{\overrightarrow{v}}{c}\right\|$

$v=\left\|\frac{\overrightarrow{v}}{c}\right\|\frac{c}{\mu_{w}}W$

$$\frac{W}{\mu_{w}}=\frac{v}{\|\overrightarrow{v}\|} \tag{23}$$

and for $\left\|\frac{\overrightarrow{v}}{c}\right\|\ll 1$:

$\left\|\frac{\overrightarrow{v}}{c}\right\|=\frac{\sinh 2\mu_{w}}{2+\cosh 2\mu_{w}}\simeq\frac{\frac{1}{2}\left(1+2\mu_{w}+2\mu_{w}^{2}-\left(1-2\mu_{w}+2\mu_{w}^{2}\right)\right)}{1+(1+2\mu_{w}+2\mu_{w}^{2}+(1-2\mu_{w}+2\mu_{w}^{2}))}=2\frac{\mu_{w}}{4\mu_{w}^{2}+3}\simeq\frac{2}{3}\mu_{w}$

$$W\simeq\frac{2}{3}\frac{\overrightarrow{v}}{c} \tag{24}$$

v) The arrangement of the tetrad of the particle with respect to the tetrad of the observer is $\sigma=\exp T_{r}\cdot\exp T_{w}$ the product of a spatial rotation ($\exp T_{r}\in$ $Spin\left(3\right)$ is a rotation which leaves invariant the time vector) and a translation. With the convention above, relating W,v the sign of W is fixed with respect to the speed of the particle. The instantaneous rotational motion (spatial) is then represented by T_{r}. In this representation there is no need for a “spin” number : the values $+T_{r}$ and $-T_{r}$ represents opposite rotational motions (the usual “spin up / spin down”). The spin number is actually necessary in an euclidean representation, where the rotational motion is represented by a vector (the axis of rotation) belonging to the Lie algebra $SO(3)$.

2.3 Momentum

The existence of a fundamental state ψ_{0} can be efficiently represented in the formalism of jet bundles, which encompasses the case of discontinuous processes.

The first jet bundle $J^{1}P_{U}$ has for elements $(u,\delta_{\alpha}\kappa^{a},\alpha=0...3,a=1...16)$ where $\delta_{\alpha}\kappa\in T_{1}U$. For a smooth section $\delta_{\alpha}\kappa=u^{-1}\cdot\partial_{\alpha}u$. The first jet bundle $J^{1}P_{C}$ has for elements $(\psi,\delta_{\alpha}\psi^{a},\alpha=0...3,a=1...16)$ where $\delta_{\alpha}\psi\in Cl\left(\mathbb{C},4\right)$. Because of the relation $\psi=\vartheta\left(u\right)\psi_{0}$ the elements of $J^{1}P_{C}$ representing the state of the particles can be written $\left(\vartheta\left(u\right)\psi_{0},\vartheta\left(u\right)\vartheta'\left(1\right)\left(\delta_{\alpha}\kappa\right)\psi_{0}\right)$.

The existence of a fundamental state is equivalent to the existence, for each particle, of a linear differential operator :

$$\mathfrak{M}:J^{1}P_{U}\rightarrow J^{1}P_{C}::\mathfrak{M}\left(\left(u,\delta_{\alpha}\kappa,\alpha=0...3\right)\right)=\left(\vartheta\left(u\right)\psi_{0},\vartheta\left(u\right)\vartheta'\left(1\right)\left(\delta_{\alpha}\kappa\right)\psi_{0}\right) \tag{25}$$

$\left(u, u^{-1} \cdot \delta_\alpha \kappa, \alpha = 0...3\right)$ is the generalized motion of the particle, and $\left(\vartheta\left(u\right)\psi_0, \vartheta\left(u\right)\vartheta'\left(1\right)\left(\delta_\alpha \kappa\right)\psi_0\right)$ is its generalized momentum.

2.4 The interaction field / particle

Measures ψ of the state are done with respect to a gauge. The gauge $u = 1$ is associated to a state where there is no action of the field.

The evolution of the state of a particle along its trajectory can be seen in two, complementary, ways, which correspond to physical situations.

2.4.1 Identification of a particle

The state of a particle is measured with respect to a gauge, defined by the action ϑ of the group U on $Cl\left(\mathbb{C}, 4\right)$ which is associated to experiments done by the physicist and represented by known values $u\left(q\left(t\right)\right)$.

The state of the particle is measured in the associated bundle $P_U\left[Cl\left(\mathbb{C}, 4\right), \vartheta\right]$, represented by a section $\psi \in \mathfrak{X}\left(P_U\left[Cl\left(\mathbb{C}, 4\right), \vartheta\right]\right)$ and from the couple of measures at each time $\left\{p_u\left(q\left(t\right)\right) = \rho_u\left(\mathbf{p}_u\left(q\left(t\right)\right), u\left(q\left(t\right)\right)\right), \psi\left(q\left(t\right)\right)\right\}$ the physicist estimates, by a statistical procedure, a vector ψ_0 such that

$\left(\rho_u\left(\mathbf{p}_u, u\right), \vartheta\left(u^{-1}\right)\psi\right) \sim \left(\mathbf{p}_u, \psi_0\right) \Leftrightarrow \psi\left(q\left(t\right)\right) = \vartheta\left(u\left(q\left(t\right)\right)\right)\psi_0$

Or equivalently that the state of the particle is constant on a trajectory on the associated bundle. This trajectory is projected on M by $q(t)$ and the tangent to the trajectory on $P_U\left[Cl\left(\mathbb{C}, 4\right), \vartheta\right]$ is a projectable vector Y, which does not depend on the observer.

2.4.2 Measure of the field

The measure of the state, for a known particle, is $\psi = \vartheta\left(u\right)\psi_0$, then u can be seen as a measure of the field. A particle is never immobile and follows a path in M. In the absence of field its state would stay constant. The value of the field changes with the location along the trajectory and by interacting with the particle. So, in the measure of the field from the state of a known particle, what is actually measured is the change of the state along the trajectory. The state $\psi\left(t\right) \in P_C$ and its derivative with respect to t is a vector Y which has 2 components :

$Y = Y_m + Y_u$ where $Y_m = \varphi'_{P_C m}\left(m, \psi\right) y, \pi'\left(p\right) Y_m = y$ is associated to the trajectory y in M and Y_u to the change δu of u.

Because of the relation $\psi = \vartheta\left(u\right)\psi_0$ and using the identity $\lambda'_g\left(g, p\right) = \lambda'_g\left(1, \lambda\left(g, p\right)\right) R'_{g^{-1}} g$:

$Y = \vartheta\left(u\right)\vartheta'\left(1\right)\left(\delta T\right)\psi_0$ where $\delta T \in T_1 U$

The vector δT :

- depends on the trajectory of the particle : usually the field is not isotropic and its variation depends on the direction on M

- should vary in a consistent way as $\delta\psi$ in a change of gauge by $\varkappa \in P_U$

The simplest, and common, assumption meeting these conditions is that there is a principal connection $\grave{\mathbf{A}} \in \Lambda_1\left(TP; VU\right)$ on P_U and $\zeta\left(\delta T\right)\left(u\left(t\right)\right)$ is the covariant derivative $\nabla_y u$ of $u \in \mathfrak{X}(P_U)$ along the trajectory on M, with tangent y.

$\nabla_y u = \zeta\left(L'_{u^{-1}}(u)\left(\frac{du}{dt} + R'_u(1)\sum_\alpha \grave{A}_\alpha(m)\, y^\alpha\right)\right)(u(q(t)))$

$= \zeta\left(u^{-1}\cdot\frac{du}{dt} + Ad_{u^{-1}}\grave{A}(y)\right)(u)$

So that :

$\delta T = u^{-1}\cdot\frac{du}{dt} + Ad_{u^{-1}}\grave{A}(y) \in T_1 U$

$Y = \vartheta(u)\,\vartheta'(1)\left(u^{-1}\cdot\frac{du}{dt} + Ad_{u^{-1}}\grave{A}(y)\right)\psi_0 = \nabla_y\psi$

In these 2 physical processes the field appears through 2 different mathematical objects : an element u of the unitary group located at m, and a principal connection $\grave{\mathbf{A}}$ on P_U. The group U acts *on the state* ψ through ϑ. The connection $\grave{\mathbf{A}}$ acts *on the variation of the state* through the covariant derivative. u and its derivative are linked to particles : physically they defined the momentum of the particle. Meanwhile the field exists in the vacuum, where there is no particle (mathematically the universe is the vacuum "almost everywhere"). So the physical field is represented by the connection.

3 The field

The force field is the third object of Physics. Its properties are the following :

- it exists everywhere

- it interacts with material bodies according to their characteristics : its value is changed and the state of the particles is changed

- it propagates in the vacuum : its value changes with the location, even in the vacuum.

The existence everywhere of the field is a necessity to meet the Principle of Locality, and avoid the idea of interaction at a distance. Material bodies "act" on each others through the force field.

For a given observer the field does not exist in the future, but it has existed at any point in the past. As a consequence, for a given observer, the field propagates on the 3 dimensional hypersurfaces $\Omega_3(t)$: it propagates at the spatial speed c, the universal constant introduced previously. This is the logical consequence of the properties of the geometry and of the field.

The interaction between the field and material bodies depends on the characteristics of the latter, summed in their charges. There are different types of charges, the gravitational charges (equal to the inertial mass), the Electromagnetic (EM) charge, the charges for the weak and strong interactions. Their value is at the base of the distinction of different elementary particles, and conversely of the different types of fields. Material bodies are composed of elementary particles assembled in systems, nuclei, atoms, molecules, crystals,... whose existence and properties depend themselves of the internal interaction particles / field.

The representation of the force field is directly related to the representation of the state of particles : this is a mathematical object which acts on the other mathematical object representing the particle, and their common ground is the group U.

3.1 Connexion

3.1.1 Definition

Proposition 10 *The field is represented by a principal connection* $\grave{\mathbf{A}}$ *on the principal bundle* P_U.

i) The vertical space at p to the fiber bundle P_U is : $V_pP_U = \ker \pi'_U(p)$ and generated by the fundamental vector fields $\zeta(\kappa)(p)$. A connection on P_U is a projection acting on vectors of the tangent bundle TP_U and valued in the vertical bundle VP_U. This is a tensor $\grave{\mathbf{A}} \in \Lambda_1(P_U; VP_U)$ such that

$\grave{\mathbf{A}}(p) : TP_U \to VP_U :: \grave{\mathbf{A}}(p)(V_m + \zeta_u(\kappa)(p)) = \zeta_u(\kappa + \Gamma(p)V_m)(p)$

is a projection. The definition is geometric : it does not depend on the trivialization, but is defined by a vector of T_1U.

The connexion form is $\widehat{\grave{\mathbf{A}}} : TP_U \to T_1U :: \grave{\mathbf{A}}(p)(V_p) = \zeta_u\left(\widehat{\grave{\mathbf{A}}}(p)V_p\right)(p)$.

ii) A principal connection is a connection which is invariant under the right action : $\rho_u : \rho_u(p,g)^* \grave{\mathbf{A}}(p) = \rho'_p(p,g)\grave{\mathbf{A}}(p)$. Then it is defined by a family of maps, the potential, $\grave{A} \in \Lambda_1(\Omega; T_1U)$:

$\Gamma(\rho_u(\mathbf{p}_u, u)) = Ad_{u^{-1}}\grave{A}(m) \Rightarrow \Gamma(\mathbf{p}_u(m)) = \grave{A}(m)$

$$\grave{\mathbf{A}}(\rho_u(\mathbf{p}_u,u))(V_{mu} + \zeta_u(\kappa)(\rho_u(\mathbf{p}_u,u))) = \zeta_u\left(\kappa + Ad_{u^{-1}}\grave{A}(m)V_m\right)(\rho_u(\mathbf{p}_u,u)) \tag{26}$$

In a change of gauge on $P_U : \mathbf{p}_U \to \widetilde{(\mathbf{p}_U)} = \rho_u(\mathbf{p}_U, \varkappa)$ the function $\grave{A}$ changes with an *affine* law : $\grave{A}(m) \to \widetilde{\grave{A}}(m) = Ad_{\varkappa}\left(\grave{A}(m) - L'_{\varkappa^{-1}}\varkappa(\varkappa')\right)$ with the derivative of $\varkappa : M \to U :: \varkappa(m)$. But in a global change of gauge ($\varkappa$ does not depend on m) $\widetilde{\grave{A}}(m) = Ad_{\varkappa}\grave{A}(m)$ and $\grave{A}$ belongs to the representation $(\Lambda_1(\Omega; T_1U), Ad)$ of U.

The fundamental vectors are invariant by a principal connexion.

iii) The horizontal bundle is $HP_U = \ker \grave{\mathbf{A}}(p)$ and the horizontal form is $\chi_H(p) = Id - \grave{\mathbf{A}}(p)$:

$\chi_H(p)(V_{mu} + \zeta_u(\kappa)(p_u)) = V_{mu} + \zeta_u(\kappa)(p_u) - \zeta_u(\kappa + \Gamma(p)V_{mu})(p_u) = V_{mu} - \zeta_u(\Gamma(p)V_{mu})(p_u)$

$\chi_H(p)(Y) = 0$ if Y is vertical, that is $V_{mu} = 0$.

A vector field $V \in \mathfrak{X}(TM)$ can be lifted by a connection as a projectable vector field :

$\chi_H : \mathfrak{X}(TM) \to \mathfrak{X}(HP_U) :: \chi_H(p)(V) = (V, -\zeta_u(\Gamma(p)V)(p_u)) \in HP_U$

3.1.2 Covariant derivative

Proposition 11 *The action of the field on an elementary particle is represented by the covariant derivative of the connection* $\grave{\mathbf{A}}$

i) The connection acts on *vectors* of the fiber bundle TP_U, the covariant derivative acts on *sections* of P_U :

$\nabla : \mathfrak{X}(P_U) \to \mathfrak{X}(VP_U \otimes TM^*) :: \nabla S = S^* \grave{\mathbf{A}}$

The covariant derivative of a section $S \in \mathfrak{X}(P_U)$ along a vector field $V \in \mathfrak{X}(TM)$ is a fundamental vector field on TP_U :

$$S(m) = \rho_u(\mathbf{p}_u(m), u(m)) \to \tag{27}$$

$$\nabla_V S = \grave{\mathbf{A}}(S(m)) S'(m) V = \zeta\left(u^{-1} \cdot u'(V) + Ad_{u^{-1}} \grave{A}(m)(V)\right)(S(m)) \tag{28}$$

that we can write, in a continuous motion along the trajectory with tangent $V : \nabla_V S = \zeta\left(\widehat{\nabla}_V u\right)(S(m))$ with :

$\widehat{\nabla}_V : \mathfrak{X}(P_U) \to T_1U : \widehat{\nabla}_V u = \left(u^{-1} \cdot \frac{du}{dt} + Ad_{u^{-1}} \sum_{\alpha=0}^{3} \grave{A}_\alpha V^\alpha\right)(m)$

$\nabla_V S$ is *invariant* in a change of gauge but its expression with a map u varies the usual way for a section in a change of gauge.

A section of J^1P_U is $j^1u = (m, u, \delta_\beta u, \beta = 0..3)$ with $\delta_\beta u \in T_1U$ then $\widehat{\nabla}$ is the differential operator :

$$\widehat{\nabla} : \mathfrak{X}\left(J^1P_U\right) \to \Lambda_1(M; T_1U) : \widehat{\nabla} j^1 u = \left(\sum_{\beta=0}^{3}\left(u^{-1} \cdot \delta_\beta u + Ad_{u^{-1}} \grave{A}_\beta\right) \otimes d\xi^\beta\right)(m) \tag{29}$$

ii) The covariant derivative on P_U induces a covariant derivative on any associated bundle. Because $P_U[Cl(\mathbb{C},4), \vartheta]$ is a vector bundle $\nabla_y \psi$ can be seen as a section of $P_U[Cl(\mathbb{C},4), \vartheta]$.

$\vartheta'(1) \in \mathcal{L}(T_1U; \mathcal{L}(Cl(\mathbb{C},4); Cl(\mathbb{C},4)))$

$\sum_{\beta=0}^{3} \vartheta'(1)\left(\grave{A}_\beta(m)\right) V^\beta \in \mathcal{L}(Cl(\mathbb{C},4); Cl(\mathbb{C},4))$

$\psi(m) = (\mathbf{p}_u(m), \psi(m)) \sim \left(\rho_u(\mathbf{p}_u(m), u(m)), \vartheta\left(u^{-1}(m)\right)\psi(m)\right)$

$\nabla_V \psi = \left(\mathbf{p}_u(m), \sum_{\beta=0}^{3}\left(\partial_\beta \psi + \vartheta'(1)\left(\grave{A}_\beta(m)(\psi)\right)\right) V^\beta\right)$

For $\psi(m) = (\mathbf{p}_u(m), \vartheta(u(m))\psi_0)$:

$\nabla_V \psi = \sum_{\beta=0}^{3}\left(\partial_\beta \psi + \vartheta'(1)\left(\grave{A}_\beta(m)(\psi)\right)\right) V^\beta$

$= \sum_{\beta=0}^{3}\left(\vartheta'(u)(\partial_\beta u)\psi_0 + \vartheta'(1)\left(\grave{A}_\beta(m)(\vartheta(u)\psi_0)\right)\right) V^\beta$

$\vartheta'(u) = \vartheta(u)\vartheta'(1) L'_{u^{-1}} u$ (Maths.p.425)

$\partial_\beta \vartheta(u) = \vartheta(u)\vartheta'(1)\left(u^{-1} \cdot \partial_\beta u\right)$

$\vartheta'(1)\grave{A}_\beta(\vartheta(u)) = \vartheta'(u) R'_u 1 \grave{A}_\beta = \vartheta(u)\vartheta'(1) L'_{u^{-1}} u R'_u 1 \grave{A}_\beta = \vartheta(u)\vartheta'(1) Ad_{u^{-1}} \grave{A}_\beta$

$Ad_g = L'_g g^{-1} \circ R'_{g^{-1}} 1 = R'_{g^{-1}} g \circ L'_g 1$
$\nabla_V \psi = \sum_{\beta=0}^3 V^\beta \left(\vartheta(u) \vartheta'(1) \left(u^{-1} \cdot \partial_\beta u\right) + \vartheta(u) \vartheta'(1) Ad_{u^{-1}} \grave{A}_\beta \right) \psi_0$
$= \sum_{\beta=0}^3 V^\beta \vartheta(u) \vartheta'(1) \left(u^{-1} \cdot \partial_\beta u + Ad_{u^{-1}} \grave{A}_\beta \right) \psi_0$
Then the covariant derivative of a section ψ of P_C along a vector field $V \in \mathfrak{X}(TM)$ is :

$$\nabla_V \psi = \vartheta(u) \vartheta'(1) \left(\widehat{\nabla}_V u \right) \psi_0 \tag{30}$$

The action of the field is, as usual in Mechanics, represented by an action on the momentum. We have seen that the momentum of the particle can be represented by $\mathfrak{M} = (\vartheta(u)\psi_0, \vartheta(u)\vartheta'(1)(\delta_\alpha \kappa)\psi_0) \in J^1 P_C$. The covariant derivative can be seen as a differential operator between jet bundles, which enables to extend the operations to discontinuous processes.

The tangent V to the trajectory is fixed by u, in any motion :

$V^\alpha = \sum_{j=0}^3 V^j P_j^\alpha = c P_0^\alpha + 2 \frac{\sinh \mu_w \cosh \mu_w}{1+2(\cosh \mu_w)^2} \sum_{j=1}^3 \frac{W^{j+5}}{\mu_w} P_j^\alpha$ where W is the component of u in the chart of U.

and for the particle :
$\nabla_V : J^1 P_C \to J^1 P_C :: \nabla_V (\vartheta(u)\psi_0, \vartheta(u)\vartheta'(1)(\delta_\alpha \kappa)\psi_0)$
$= \left(\vartheta(u)\psi_0, \vartheta(u)\vartheta'(1) \left(\widehat{\nabla}_V j^1 u \right) \psi_0 \right)$

3.2 The strength of the field

The value of the field is measured by its action on particles, which is expressed through the potential $\grave{A}$. To be consistent we assume that $\grave{A}$ represents the value of the field, and that it is continuous. Then to model the propagation of the field we need a derivative. The connection form $\widehat{\grave{A}} \in \Lambda_1(TP_U; T_1U)$ is a tensor on the tangent bundle to the manifold P_U, valued in the fixed vector space T_1U. The most general procedure to define the derivative of a tensor is by the Lie derivative.

3.2.1 Definition

i) The Lie derivative is defined for a vector field $Y \in \mathfrak{X}(TU)$. Along the integral curve $\Phi_Y(s, p_u)$ we take around a given point $p_u \in P_U$ the pull back of $\widehat{\grave{A}}$ from $\Phi_Y(\tau, p_u)$ to p_u : $(\Phi_Y(\tau, .))^* \widehat{\grave{A}}(p_u) = \widehat{\grave{A}}(\Phi_Y(\tau, p_u)) \circ \Phi'_{Yp}(\tau, p_u)$ which is expressed in the holonomic basis at p_u and we can compute : $\Delta_R(\tau) \widehat{\grave{A}}(p_u) = \frac{1}{\tau} \left((\Phi_Y(\tau, .))^* \widehat{\grave{A}}(p_u) - \widehat{\grave{A}}(p_u) \right)$

Equivalently we can pull back $\widehat{\grave{A}}$ from $\Phi_Y(-\tau, p_u)$ to p_u
$(\Phi_Y(-\tau, .))^* \widehat{\grave{A}}(p_u) = \widehat{\grave{A}}(\Phi_Y(-\tau, p_u)) \circ \Phi'_{Yp}(-\tau, p_u)$
and compute :
$\Delta_L(\tau) \widehat{\grave{A}}(p_u) = \frac{1}{\tau} \left(\widehat{\grave{A}}(p_u) - (\Phi_Y(-\tau, .))^* \widehat{\grave{A}}(p_u) \right)$

The quantities $\Delta_R(\tau)\widehat{\hat{A}}(p_u), \Delta_L(\tau)\widehat{\hat{A}}(p_u)$ are maps : $\mathbb{R} \to T_1U$. And

$\lim_{\tau\to 0}\Delta_R(\tau)\widehat{\hat{A}}(p_u) = \frac{d}{d\tau}(\Phi_Y(\tau,.))^*\widehat{\hat{A}}(p_u)|_{\tau=0+}$

$\lim_{\tau\to 0}\Delta_L(\tau)\widehat{\hat{A}}(p_u) = -\frac{d}{d\tau}(\Phi_Y(-\tau,.))^*\widehat{\hat{A}}(p_u)|_{\tau=0-}$

are the right and left derivatives. If they exist and are equal one says that $\widehat{\hat{A}}(p_u)$ is Lie differentiable along the vector field Y and the Lie derivative is defined as :

$\pounds_Y\widehat{\hat{A}}(p_u) = \frac{d}{d\tau}(\Phi_Y(\tau,.))^*\widehat{\hat{A}}(p_u)|_{\tau=0}$

$\pounds_Y\widehat{\hat{A}}_u \in \Lambda_1(TP_U;T_1U)$ is defined by a difference, and changes with the adjoint map in a change of gauge (the affine map becomes a linear one) and can be seen as a one form on TP_U valued in the adjoint bundle $P_U[T_1U, Ad]$.

ii) The Lie derivative and the exterior differential are related : $\pounds_Y\widehat{\hat{A}} = d\left(i_Y\widehat{\hat{A}}\right) + i_Y\left(d\widehat{\hat{A}}\right)$

We want to compute the derivative along a vector field on TM (a change of gauge in the computation of the derivative has no interest here), so we take the horizontalisation of Y, using the horizontal form : $\chi \in \Lambda_1(P_U;HP_U)$:

$\pounds_{\chi(Y)}\widehat{\hat{A}} = d\left(i_{\chi(Y)}\widehat{\hat{A}}\right) + i_{\chi(Y)}\left(d\widehat{\hat{A}}\right)$

$\chi(Y)$ is horizontal : $i_{\chi(Y)}\widehat{\hat{A}} = 0$

$i_{\chi(Y)}\left(d\widehat{\hat{A}}\right) = \chi^* d\widehat{\hat{A}}(Y) = \nabla_{eY}\widehat{\hat{A}}$ is the exterior covariant derivative of the connection (see Annex) along Y.

$\nabla_e\widehat{\hat{A}} = \widehat{\Omega}$ is the curvature form of the connection. This is a 2 form on TP_U, valued in T_1U.

$\pounds_{\chi(Y)}\widehat{\hat{A}} = \widehat{\Omega}(Y)$

$\pounds\widehat{\hat{A}}$ is a 2 form on TP_U, valued in T_1U and $\pounds\widehat{\hat{A}} \circ \chi(Y) = \widehat{\Omega}(Y)$

iii) To keep the relation with vectors on TM we use a section on P_U. The standard gauge $\mathbf{p}_u(m)$ is a section : $\mathbf{p}_u : M \to P_U$ and we can pull back the curvature from P_U to M

$\mathbf{p}_u^*\left(\widehat{\Omega}\right) : TM \to T_1U :: \mathbf{p}_u^*\left(\widehat{\Omega}\right)(y) = \widehat{\Omega}(\mathbf{p}_u(m))\,\mathbf{p}_u'(m)(y)$

$\mathbf{p}_u^*\left(\widehat{\Omega}\right) = -\mathcal{F}$ where $\mathcal{F}$ is the strength of the field. So $\mathcal{F}$ is related to the curvature $\widehat{\Omega}$, which is commonly used in the Theory of Fields. This is, up to the sign, the same quantity but evaluated in the standard gauge, which makes sense because there is no need to involve a change of gauge in the computation of the derivative. In a change of gauge $\mathcal{F}$ changes as $Ad_{\varkappa^{-1}}$, there is no longer an affine map, so it can be seen as a 2 form on TM valued in the adjoint bundle $P_U[T_1U, Ad]$.

For any vector field y on TM the vector $\mathbf{p}_u'(m)(y)$ is a horizontal vector, and $\chi(\mathbf{p}_u(m))\,\mathbf{p}_u'(m)(y) = \mathbf{p}_u'(m)(y)$

$\mathbf{p}_u^*\left(\pounds\widehat{\hat{A}}\right)(y) = -\mathcal{F}(y)$

This holds for any vector field on TM and we write :$\mathcal{F} = -\mathbf{p}_u^*\left(\pounds\widehat{\hat{A}}\right)$

$\Leftrightarrow -\mathcal{F}(\partial\xi_\alpha) = -\mathbf{p}_u^*\left(\pounds\sum_{\beta=0}^3\widehat{\grave{A}}_\beta d\xi^\beta\right)(\partial\xi_\alpha) \Leftrightarrow \mathcal{F}_{\alpha\beta} = -\mathbf{p}_u^*\left(\pounds\widehat{\grave{A}}_\beta\right)(\partial\xi_\alpha)$

$\mathcal{F}\in\Lambda_2(M;T_1U)$ and its components are expressed in the holonomic basis of TM and the basis $\kappa_a = \zeta_a F_a$ of T_1U :

$$\begin{gathered}\mathcal{F} = -\mathbf{p}_u^*\left(\pounds\widehat{\grave{A}}\right)\in\Lambda_2(M;T_1U)\\ \mathcal{F}(m) = \sum_{\{\alpha,\beta\}=0..3}\sum_{a=1}^{16}\mathcal{F}_{\alpha\beta}^a d\xi^\alpha\wedge d\xi^\beta\otimes\kappa_a\\ \mathcal{F}_{\alpha\beta}^a = \partial_\alpha\grave{A}_\beta^a - \partial_\beta\grave{A}_\alpha^a + \left[\grave{A}_\alpha,\grave{A}_\beta\right]^a\end{gathered} \tag{31}$$

The scalar component $a=1$ is a scalar 2 form : $\mathcal{F}_{\alpha\beta}^1 = \partial_\alpha\grave{A}_\beta^1 - \partial_\beta\grave{A}_\alpha^1$ as expected for the EM field.

3.2.2 Matricial representation of the strength

i) For any scalar valued 2 form $\mathcal{F}$ on M it is convenient to write $\mathcal{F} = \mathcal{F}_r + \mathcal{F}_w$ with

$\mathcal{F}_r = \mathcal{F}_{32}d\xi^3\wedge d\xi^2 + \mathcal{F}_{13}d\xi^1\wedge d\xi^3 + \mathcal{F}_{21}d\xi^2\wedge d\xi^1$

$\mathcal{F}_w = \mathcal{F}_{01}d\xi^0\wedge d\xi^1 + \mathcal{F}_{02}d\xi^0\wedge d\xi^2 + \mathcal{F}_{03}d\xi^0\wedge d\xi^3$

and we will denote the 1×3 row matrices :

$$[\mathcal{F}_r] = \begin{bmatrix}\mathcal{F}_{32} & \mathcal{F}_{13} & \mathcal{F}_{21}\end{bmatrix};[\mathcal{F}_w] = \begin{bmatrix}\mathcal{F}_{01} & \mathcal{F}_{02} & \mathcal{F}_{03}\end{bmatrix} \tag{32}$$

$\mathcal{F}\wedge K = -\left([\mathcal{F}_r][K_w]^t + [\mathcal{F}_w][K_r]^t\right)d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$

ii) Any 2-form can also be written in matrix form :

$$[\mathcal{F}_{\alpha\beta}]_{\beta=0...3}^{\alpha=0...3} = \begin{bmatrix}0 & \mathcal{F}_{01} & \mathcal{F}_{02} & \mathcal{F}_{03}\\ \mathcal{F}_{10} & 0 & \mathcal{F}_{12} & \mathcal{F}_{13}\\ \mathcal{F}_{20} & \mathcal{F}_{21} & 0 & \mathcal{F}_{23}\\ \mathcal{F}_{30} & \mathcal{F}_{31} & \mathcal{F}_{32} & 0\end{bmatrix}_{4\times4} = \begin{bmatrix}0 & [\mathcal{F}_w]_{1\times3}\\ -([\mathcal{F}_w])_{3\times1}^t & j([\mathcal{F}_r])_{3\times3}\end{bmatrix}$$

$[\mathcal{F}]^t = -[\mathcal{F}]$

iii) The split in the two parts $\mathcal{F}_r,\mathcal{F}_w$ does not change in a change of spatial basis (the vectors $(\partial\xi_\alpha)_{\alpha=1}^3$), that is for a given observer, but changes for another observer who has not the same vector time ε_0. In a change of chart :

$[\mathcal{F}]\to\left[\widetilde{\mathcal{F}}\right]$ with $\left[\widetilde{\mathcal{F}}\right] = [K]^t[\mathcal{F}][K]$ where $[K]$ is the inverse of the jacobian $[K] = [J]^{-1},[J] = \left[\frac{\partial\eta^\alpha}{\partial\xi^\beta}\right]$

$$[K] = \begin{bmatrix}K_0^0 & \left[K^0\right]_{1\times3}\\ [K_0]_{3\times1} & [k]_{3\times3}\end{bmatrix}$$

$\left[\widetilde{\mathcal{F}}_r\right] = [\mathcal{F}_r]\left([k]^{-1}\right)^t\det[k] + [\mathcal{F}_w][k]\,j\left(\left[K^0\right]\right)$

$\left[\widetilde{\mathcal{F}}_w\right] = -[\mathcal{F}_r]\,j([K_0])[k] + [\mathcal{F}_w]\left([K]_0^0[k] - [K_0]\left[K^0\right]\right)$

$$\left[\widetilde{\mathcal{F}}_r,\widetilde{\mathcal{F}}_w\right] = [\mathcal{F}_r,\mathcal{F}_w]\begin{bmatrix}\left([k]^{-1}\right)^t\det[k] & [k]\,j\left(\left[K^0\right]\right)\\ -j([K_0])[k] & [K]_0^0[k] - [K_0]\left[K^0\right]\end{bmatrix}$$

If the vector ε_0 is preserved,

$[K]=\begin{bmatrix}1 & 0\\ 0 & [k]_{3\times 3}\end{bmatrix}$

$\left[\widetilde{\mathcal{F}}_r\right]=[\mathcal{F}_r]\left([k]^{-1}\right)^t\det[k]$

$\left[\widetilde{\mathcal{F}}_w\right]=[\mathcal{F}_w][K]_0^0[k]$

iv) The strength of the field is valued in the Lie algebra. And we will denote similarly :

$[\mathcal{F}_r]_{16\times 3}=[\mathcal{F}_r^a]^{a=1..16}=\begin{bmatrix}\mathcal{F}_{32}^1 & \mathcal{F}_{13}^1 & \mathcal{F}_{21}^1\\ ... & ... & ...\\ \mathcal{F}_{32}^{16} & \mathcal{F}_{13}^{16} & \mathcal{F}_{21}^{16}\end{bmatrix}$

$[\mathcal{F}_w]_{16\times 3}=[\mathcal{F}_w^a]^{a=1..16}=\begin{bmatrix}\mathcal{F}_{01}^1 & \mathcal{F}_{02}^1 & \mathcal{F}_{03}^1\\ ... & ... & ...\\ \mathcal{F}_{01}^{16} & \mathcal{F}_{02}^{16} & \mathcal{F}_{03}^{16}\end{bmatrix}$

$$[\mathcal{F}]_{16\times 6}=\left[\mathcal{F}_{\alpha\beta}^a\right]=\begin{bmatrix}\mathcal{F}_r & \mathcal{F}_w\end{bmatrix} \tag{33}$$

3.2.3 Chern-Weil Theorem

The definition of the strength of the connection is a bit complicated and one can guess that there are some identities. The Chern-Weil theorem tells that it must meet some conditions, which actually depends only on the principal bundle. This theorem is purely mathematical, no assumption is done about the connection (Maths.2118). For a 2 form on M it reads :

- for any Lie group G on a field $K=\mathbb{R},\mathbb{C}$

- for any principal bundle $P(M,G,\pi)$ and its adjoint bundle $P[T_1G,Ad]$

- for any principal connection with strength $\mathcal{F}$ on P

- for any symmetric n linear map $L\in\mathcal{L}^n(T_1G;K)$ invariant by G

the map :

$\widehat{L}(\mathcal{F}):(T_1M)^{2n}\to K::\widehat{L}(\mathcal{F})(X_1,...,X_{2n})$

$=\frac{1}{2n!}\sum_{\sigma\in\mathfrak{S}(2n)}L\left(\mathcal{F}\left(X_{\sigma(1)},X_{\sigma(2)}\right),...,\mathcal{F}\left(X_{\sigma(2n-1)},X_{\sigma(2n)}\right)\right)$

belongs to $\Lambda_{2n}(TM;K)$, it is such that $d\widehat{L}(\mathcal{F})=0$ and for any 2 principal connections with strength $\mathcal{F}_1,\mathcal{F}_2$ there is some form $\lambda\in\Lambda_{2n}(TM;K)$ such that $\widehat{L}(\mathcal{F}_1-\mathcal{F}_2)=d\lambda$.

For the 4 dimensional manifold M and the principal bundle P_U the only case of interest is $n=2$. A symmetric, bilinear map $L\in\mathcal{L}^2(T_1G;K)$ can be expressed as a 2 form : $L_2(X,Y)=\sum_{a;b}L_{ab}X^aY^a$ with $L_{ab}=L_{ba}$. This is a scalar product which is preserved by the adjoint map :

$\forall g\in U:L_2(Ad_gX,Ad_gY)=L_2(X,Y)\Leftrightarrow[X]^t[Ad_g]^t[L][Ad_g][Y]=[X]^t[L][Y]\Leftrightarrow[Ad_g]^t[L][Ad_g]=[L]$

$[Ad_g]^t[L][Ad_g]=[L]\Leftrightarrow[\eta]\left[Ad_{g^{-1}}\right][\eta][L][Ad_g]=[L]$

$[\eta][L][Ad_g]=[Ad_g][\eta][L]$

The strength reads $\mathcal{F}_{\alpha\beta}^a$ and, with notations as above, a straightforward computation gives :

$\widehat{L}(\mathcal{F})=-\frac{1}{3}\left(\langle\mathcal{F}_{01},\mathcal{F}_{32}\rangle+\langle\mathcal{F}_{02},\mathcal{F}_{13}\rangle+\langle\mathcal{F}_{03},\mathcal{F}_{21}\rangle\right)d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$

$=-\frac{1}{3}\sum_{p=1}^3[\mathcal{F}_w]_p^t[L][\mathcal{F}_r]_p\,d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$

$= -\frac{1}{3}\sum_{p=1}^{3}\sum_{a,b=1}^{16}[\mathcal{F}_w]^p_a [L]^a_b [\mathcal{F}_r]^b_p d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$
$= -\frac{1}{3}Tr\left([\mathcal{F}_w]^t [L] [[\mathcal{F}_r]]\right) d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

$\widehat{L}(\mathcal{F})$ is a 4 form, as well as $\lambda \in \Lambda_4(TM;K)$. So $\widehat{L}(\mathcal{F}_1 - \mathcal{F}_2) = d\lambda = 0$. The quantity $\widehat{L}(\mathcal{F})$ does not depend on the connection, only on the map L and the existence of the principal bundle P_U. The hermitian scalar product on T_1U is preserved by the adjoint map, thus :

Theorem 12 $\langle \mathcal{F}, \mathcal{F} \rangle_H = Tr\left([\mathcal{F}_w]^t [\eta] [[\mathcal{F}_r]]\right)$ *does not depend on the connection.*

This is a purely mathematical result, but nonetheless this condition should be met, at any point, in any physical system.

3.3 The propagation of the field

One of the properties which define the field is that it propagates in the vacuum : its value changes from one point to another in the vacuum, that is without any interaction with particles : it is assumed that the field interacts with itself. This self interaction should follow the Principle of Conservation of Energy, which is modelled through a lagrangian and the implementation of the Principle of Least Action. But this does not tell all the story. The phenomenon of propagation of the field raises several issues.

3.3.1 The field propagates on Killing curves

The speed of light

One of the strongest results of Physics is that "light propagates at constant spatial speed c". We have introduced previously the universal constant c in the representation of the Geometry, without any reference to a physical experiment or to the field. But this result is a logical consequence of the properties of the force field and of the observer : for any observer his hypersurfaces $\Omega_3(t)$ are the border between the past region, where the field has already acted and its value is fixed, and the future, where it has not yet acted and has not a value, so the field changes as the motion of the hypersurfaces $\Omega_3(t)$ with respect to the time, that is at the speed c. And this holds whoever is the observer. But there is more to it.

It is useful to come back on the experiments which "measure the speed of light". Their principle is that a small variation of the field (a signal) occurs at some point, it is detected at different points with some delay, from which one can compute an apparent spatial speed. The conclusion comes from the facts :

i) the signal can be acknowledged : it can be attenuated, or distorted (by the Doppler effect for instance), but it is recognizable.

ii) the signal follows different curves in the 4 dimensional manifold, whose spatial length can be computed.

iii) there is a constant relation between the spatial length of the curve and the time delay, even when the observers are in motion : the velocity $V = \frac{dq}{d\tau}$

of the propagation of the signal on the curve is such that the Lorentz scalar product is null : $\langle V, V \rangle = 0$.

More generally for any observer the location of a material body is done through a signal coming from the body, which is necessarily the propagation of the interaction of the field with the body (there is no action at a distance, even an acknowledgment). The fact that we can locate precisely an object, that is, in the 3 dimensional space, the direction of the incoming signal, means that the change in the value of the field due to its interaction with a material body propagates along a curve in the 4 dimensional manifold which is unique and has a tangent such that $\langle V, V \rangle = 0$. Which raises the existence of "preferred" curves for the propagation of the field.

For the EM field these features are a consequence of the wave equation and the Maxwell's laws, which can be established by the implementation of the Principle of Least Action in Special Relativity. The extension to General Relativity is a difficult problem, and it does not seem that the answer could come from the Principle of Least Action alone.

The First Principle of Optics says that "light propagates in straight lines". In General Relativity it is usually assumed that light propagates along geodesics, that is curves such that the covariant derivative of its tangent, using the connection of the gravitational field, is null. This is a natural choice when the connection is defined uniquely on the tangent bundle, and seen as the generalization of the idea that "light propagates along curves of shortest length", but this last feature holds only if the connection is special (the Levy-Civita connection).

In a Unified Theory of Field there is no reason to privilege the gravitational field. The connection is itself defined over the fiber bundle P_U, the covariant derivative acts on sections $S \in \mathfrak{X}(P_U)$, there is no obvious candidate for such a section and, moreover, the variables involved with the field are forms.

In all the cases above the general idea is that a tensor (representing the signal) is transported along some special family of curve. This mathematical concept is well known and not limited to geodesics. Tensors fields can be transported by diffeomorphisms, there is a bijective correspondence between diffeomorphisms and vector fields on a manifold : a tensor on TM can be transported along the integral curve of any vector field, using its flow (see Annex). The issue is then to figure what are these curves. We have seen from the Chern-Weil theorem that the scalar product $\langle \mathcal{F}, \mathcal{F} \rangle_H$ does not depend on the connection, but on the existence of the principal bundle P_U itself. It is built from the metric : whatever the system of interacting fields and particles, the value of $\langle \mathcal{F}, \mathcal{F} \rangle_H$ depends only on the metric. Diffeomorphisms which preserve the metric on a manifold are isometries, and the associated vector fields are Killing vector fields, which constitute a Lie algebra of dimension at most ten on a 4 dimensional manifold : they represent the symmetries of the metric, which is the physical part of the Geometry of the universe. The integral curves of Killing vector fields are Killing curves, there are infinitely many Killing curves going through a given point. Killing vector fields have a constant length and, locally, there is always a Killing curve, not necessarily unique, joining two points, with a tangent of fixed > 0 length. Isometries can be extended to morphisms on the tangent bundle, as well as the fiber bundle P_U (see Annex).

For these reasons we state :

Proposition 13 *The field propagates along Killing curves.*

This is a general assumption and we will see later how it can be implemented. An important point to understand the physical meaning of this property : the propagation on a Killing curve between the points A, B does not fix the value of the field at B, it tells only that there is a relation between the value of the field at A and the value of the field at B. This is the meaning of a signal.

Killing vectors fields and Killing curves

Killing vector fields V are characterized by the condition, with respect to the metric g :

$\pounds_V g = 0 \Leftrightarrow \alpha, \beta = 0...3 : \sum_{\gamma=0}^{3} V^{\gamma} [\partial_{\gamma} g]^{\alpha}_{\beta} + [g]^{\beta}_{\gamma} [\partial_{\alpha} V]^{\gamma} + [g]^{\alpha}_{\gamma} [\partial_{\beta} V]^{\gamma} = 0$

We have a set of 16 linear PDE. A Killing vector field is defined by its value and of at least 6 partial derivatives at a point.

For a standard observer, we have necessarily

$[g] = \begin{bmatrix} -1 & 0 \\ 0 & [g_3]_{3\times 3} \end{bmatrix}$

The equations read :

$$\begin{gathered} \partial_0 V^0 = 0 \\ \alpha, \beta = 1, 2, 3 : \\ \partial_{\beta} V^0 = \sum_{\gamma=1}^{3} g_{\beta\gamma} \partial_0 V^{\gamma} \\ V^0 \partial_0 g_{\alpha\beta} + \sum_{\gamma=1}^{3} V^{\gamma} \partial_{\gamma} g_{\alpha\beta} + g_{\beta\gamma} \partial_{\alpha} V^{\gamma} + g_{\alpha\gamma} \partial_{\beta} V^{\gamma} = 0 \end{gathered} \tag{34}$$

Null future oriented Killing vectors

Killing curves associated to the propagation of the field must be future oriented, their tangent must have a null length : $\langle V, V \rangle = 0 \Leftrightarrow \sum_{\alpha,\beta=1}^{3} g_{\alpha\beta} V^{\alpha} V^{\beta} = \left(V^0\right)^2$, and the component along ε_0 is equal to c :

$$\begin{gathered} V = c\varepsilon_0 + v \\ v\left(\varphi_o\left(ct, x\right)\right) \in T_x \Omega_3\left(t\right) : \langle v, v \rangle_3 = c^2 \end{gathered} \tag{35}$$

They do not constitute any longer a vector space but an embedded manifold, with specific properties.

The equations read :

$g_{11} \left(V^1\right)^2 + g_{22} \left(V^2\right)^2 + g_{33} \left(V^3\right)^2 + 2g_{32} V^3 V^2 + 2g_{13} V^1 V^3 + 2g_{21} V^1 V^2 = c^2$

$g_{11} \partial_0 V^1 + g_{21} \partial_0 V^2 + g_{13} \partial_0 V^3 = 0$

$g_{21} \partial_0 V^1 + g_{22} \partial_0 V^2 + g_{32} \partial_0 V^3 = 0$

$g_{13} \partial_0 V^1 + g_{32} \partial_0 V^2 + g_{33} \partial_0 V^3 = 0$

$c\partial_0 g_{11} + V^1 \partial_1 g_{11} + V^2 \partial_2 g_{11} + V^3 \partial_3 g_{11} + 2g_{11} \partial_1 V^1 + 2g_{21} \partial_1 V^2 + 2g_{13} \partial_1 V^3 = 0$

$c\partial_0 g_{22} + V^1 \partial_1 g_{22} + V^2 \partial_2 g_{22} + V^3 \partial_3 g_{22} + 2g_{21} \partial_2 V^1 + 2g_{22} \partial_2 V^2 + 2g_{32} \partial_2 V^3 = 0$

$c\partial_0 g_{33} + V^1\partial_1 g_{33} + V^2\partial_2 g_{33} + V^3\partial_3 g_{33} + 2g_{13}\partial_3 V^1 + 2g_{32}\partial_3 V^2 + 2g_{33}\partial_3 V^3 = 0$

$c\partial_0 g_{32} + V^1\partial_1 g_{32} + V^2\partial_2 g_{32} + V^3\partial_3 g_{32} + g_{32}\partial_2 V^2 + g_{32}\partial_3 V^3 + g_{13}\partial_2 V^1 + g_{21}\partial_3 V^1 + g_{22}\partial_3 V^2 + g_{33}\partial_2 V^3 = 0$

$c\partial_0 g_{13} + V^1\partial_1 g_{13} + V^2\partial_2 g_{13} + V^3\partial_3 g_{13} + g_{13}\partial_1 V^1 + g_{13}\partial_3 V^3 + g_{11}\partial_3 V^1 + g_{32}\partial_1 V^2 + g_{21}\partial_3 V^2 + g_{33}\partial_1 V^3 = 0$

$c\partial_0 g_{21} + V^1\partial_1 g_{21} + V^2\partial_2 g_{21} + V^3\partial_3 g_{21} + g_{21}\partial_1 V^1 + g_{21}\partial_2 V^2 + g_{11}\partial_2 V^1 + g_{22}\partial_1 V^2 + g_{32}\partial_1 V^3 + g_{13}\partial_2 V^3 = 0$

We can use the equation $\langle V, V\rangle = 0$ to express V^3 with respect to V^1, V^2, we are left with 6 equations for 6 partial derivatives and these Killing vector fields are defined by a point O and the tangent $v(O)$ at this point.

The field propagates towards the future, so we require from the Killing vectors V that they are future oriented. $V = c\varepsilon_0 + v$ is future oriented if v is the limit of a sequence of vectors $g_3(m)(v_n, v_n) < c^2$. With a given v, both vectors $V = c\varepsilon_0 + v$ or $V' = c\varepsilon_0 - v$ are null and future oriented. If $V = c\varepsilon_0 + v$ is a Killing vector field then $V' = c\varepsilon_0 - v$ is a Killing vector field if the metric does not depend on t as one can check on the equations. But, if $g(v(O), v(O)) = c^2$ there are Killing vector fields V, V' such that $V(O) = c\varepsilon_0 + v(O)$ and $V'(O) = c\varepsilon_0 - v(O)$, which are different.

The flow of a vector field is defined by the differential equation $\frac{\partial}{\partial\tau}\Phi_V(\tau, m)|_{\tau=\theta} = V(\Phi_V(\theta, m))$ and it defines uniquely an oriented integral curve going through m. So there is unique null, future oriented Killing curve going through m with a given tangent at m. If the metric does not depend on t, both $V = c\varepsilon_0 + v, V' = c\varepsilon_0 - v$ are Killing vector fields, but their flow are different.

Similarly the vector v is usually not a Killing vector field for the metric on $\Omega_3(t)$.

Notation 14 *The set of null, future oriented Killing vectors fields* $\mathfrak{X}(K)$

The first 4 equations read $[g]\partial_0 V$ and $\partial_0 V = 0$: the components of V in the standard chart do not depend on t :

$$\beta = 0...3 : V^\beta(\varphi_o(ct, x)) = V^\beta(\varphi_o(c(t+\theta), x)) \tag{36}$$

Let $V = c\varepsilon_0 + v \in \mathfrak{X}(K), m = \varphi_o(ct, x) \to q(\tau) = \Phi_V(\tau, m) = \varphi_o(T(\tau), y(\tau))$

$\frac{dq}{d\tau}|_{\tau=\theta} = V(\Phi_V(\theta, m)) = \varphi'_{ot}(T(\tau), y(\tau))T'(\theta) + \varphi'_x(T(\theta), y(\theta))y'(\theta) = c\varepsilon_0(q(\theta)) + v(q(\theta))$

$c\varepsilon_0(q(\theta)) = \varphi'_{ot}(T(\tau), y(\tau))T'(\theta) = \varepsilon_0(q(\theta))T'(\theta) \Rightarrow T'(\theta) = c$

$$\forall V \in \mathfrak{X}(K) : \Phi_V(\theta, \varphi_o(ct, x)) = \varphi_o(c(t+\theta), y(\theta, x)) \tag{37}$$

Φ_V is a diffeomorphism from $\Omega_3(t)$ to $\Omega_3(t+\theta)$

The commutator of the vector fields $V \in \mathfrak{X}(K), \varepsilon_0$ is null :

$[V, \varepsilon_0]^\alpha = \sum_{\beta=0}^3 \left(V^\beta\partial_\beta\varepsilon_0^\alpha - \varepsilon^\beta\partial_\beta V^\alpha\right) = -\sum_{\beta=0}^3 \varepsilon^\beta\partial_\beta V^\alpha = -\partial_0 V^\alpha = 0$

then (Maths.1386):

$\forall\tau, \theta : \Phi_V(\tau, \Phi_{c\varepsilon_0}(\theta, m)) = \Phi_{c\varepsilon_0}(\theta, \Phi_V(\tau, m))$

$$\Phi_{c\varepsilon_0 *}(V) = V; \Phi_{V*}(\varepsilon_0) = \varepsilon_0 \tag{38}$$

The time vector ε_0 is preserved by the push forward by V.

If there is always, locally, a Killing curve joining 2 points, this is no longer true for null Killing curves.

Given a point $O = \varphi_o(ct, x)$, for any $v(O)$ such that $g(v(O), v(O)) = c^2$ there is a Killing vector field $V = c\varepsilon_0 + v$ with integral curve : $q(\tau) = \Phi_V(\tau, O)$ and $q(\tau) = \varphi_o(c(t+\tau), y(\tau)), \frac{dy}{d\tau} = v(y(\tau)), \beta = 1,2,3 : v^\beta(y(\tau)) = v^\beta(y(0))$.

$\langle V, V\rangle = 0 = -c^2 + g_3(q(\tau))(v(y(\tau)), v(y(\tau))) = -c^2 + g_3(O)(v(O), v(O))$

The curves originating from O generate a cone, with apex O and axis $\varphi_o(c(t+\tau), x) = \Phi_{c\varepsilon_0}(\tau, O)$.

For a fixed θ the cone intersects $\Omega_3(t+\theta)$ on a 2 dimensional submanifold $S(O,\theta)$ of $\Omega_3(t+\theta)$ similar to a sphere with center $\varphi_o(c(t+\theta), x)$ and radius $c\theta$. For any point $q(\theta)$ of $S(O,\theta)$ there is a unique null Killing curve joining O to $q(\theta)$. Conversely the points of $S(O,\theta)$ are the only ones of $\Omega_3(t+\theta)$ which can be joined to O by a Killing curve.

For a given vector field

$$\forall V \in \mathfrak{X}(K) : \Phi_V(\theta, \varphi_o(ct, x)) = \varphi_o(c(t+\theta), y(\theta, x)) \in S(\varphi_o(c(t+\theta), x), c\theta) \tag{39}$$

We can have an idea of $S(O,\theta)$ in Special Relativity, with a frame based in A : the propagation curves are straight lines with a fixed vector $V = c\varepsilon_0 + v$:

$\overrightarrow{AO(t)} = ct\varepsilon_0$

$\overrightarrow{AO(t+\theta)} = c(t+\theta)\varepsilon_0$

$\overrightarrow{O(t)\,q(\theta)} = (c\varepsilon_0 + v)\theta$

$\overrightarrow{Aq(\theta)} = ct\varepsilon_0 + (c\varepsilon_0 + v)\theta$

$\overrightarrow{O(t+\theta)\,q(\theta)} = -c(t+\theta)\varepsilon_0 + ct\varepsilon_0 + (c\varepsilon_0 + v)\theta = v\theta$

$S(O,\theta)$ is the sphere centered in $O(t+\theta)$ with radius $c\theta$, the spatial vector v is orthogonal to the sphere.

3.3.2 Discontinuities

The propagation is naturally represented through a derivative and, because the connection is a tensor, this is a Lie derivative. In the computation above, when $\tau \to 0$

$(\Phi_Y(\tau,.))^* \widehat{\grave{A}}(p_u) - \widehat{\grave{A}}(p_u) \to 0$

$\widehat{\grave{A}}(p_u) - (\Phi_Y(-\tau,.))^* \widehat{\grave{A}}(p_u) \to 0$

because the field is assumed to be continuous but

$\Delta_R \widehat{\grave{A}}(p_u) = \frac{1}{\tau}\left((\Phi_Y(\tau,.))^* \widehat{\grave{A}}(p_u) - \widehat{\grave{A}}(p_u)\right), \Delta_L \widehat{\grave{A}}(p_u)$

$= \frac{1}{\tau}\left(\widehat{\grave{A}}(p_u) - (\Phi_Y(-\tau,.))^* \widehat{\grave{A}}(p_u)\right)$

can have limits which are not equal.

Physically it could happen. The interaction between a continuous field and a particle represented by a point raises an issue [2]. To be rigorous we should introduce an "in" and an "out" field which do not have necessarily the same value. The discontinuity can be smeared out over some area, but one cannot exclude the possibility that it propagates itself. In Classic Electromagnetism the solution is the introduction of similar but distinct, continuous, variables to account for the perturbation of a medium, assumed to be smeared out at a large scale. But we cannot avoid the problem in Particle Physics.

This is at the origin of bosons, which will be studied with discontinuous processes.

The expression of $\mathcal{F}$ involves the partial derivatives of the function $\grave{A}^a_\alpha$. To study in full rigor the discontinuities the right formalism is the jet formalism. An element of the first jet extension $J^1\Lambda_1(TM;T_1U)$ reads $j^1\grave{A} = \left(m, \grave{A}^a_\alpha, \grave{A}^a_{\beta\alpha}, a = 1..16, \alpha, \beta = 0..3\right)$ where $\sum_{\alpha=0}^3 \sum_{a=1}^{16} \grave{A}^a_{\beta\alpha} d\xi^\alpha \otimes \kappa_a$ are 4 independent one form (Maths.6.2). If $j^1\grave{A}$ is the prolongation of a section then $\grave{A}^a_{\beta\alpha} = \partial_\beta \grave{A}^a_\alpha$.

The strength can then be seen as a differential operator :

$\mathcal{F} : J^1\Lambda_1(TM;T_1U) \to \Lambda_2(TM;T_1U) ::$

$\mathcal{F}\left(j^1\grave{A}\right) = \sum_{\{\alpha,\beta\}=0..3} \sum_{a=1}^{16} \left(\grave{A}^a_{\alpha\beta} - \grave{A}^a_{\beta\alpha} + \left[\grave{A}_\alpha, \grave{A}_\beta\right]^a\right) d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a$

This is a 2 form valued in $T_1U : \mathcal{F}(m) = \sum_{\{\alpha,\beta\}=0..3} \sum_{a=1}^{16} \mathcal{F}^a_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a$ and $\partial_\alpha \grave{A}^a_\beta - \partial_\beta \grave{A}^a_\alpha$ is replaced by $\grave{A}^a_{\alpha\beta} - \grave{A}^a_{\beta\alpha}$.

In the following we will use, to alleviate the notations, the expression with partial derivatives, but we will specify when it is specifically required that it is so.

We will come back to these issues in the following.

4 Energy

4.1 Principles

The concept of Energy is fundamental in Physics. To each physical object is assigned a scalar, representing the energy that the object exchanges with other objects in a system. According to the principle of locality this quantity is computed at each point of M : particles exchange energy through the field or in collisions. What is measured is *the flow of energy which is exchanged*. The process can be continuous or discontinuous, so the quantity representing the flow of energy involves a variation, not necessarily continuous, of the variables and its mathematical framework is jets.

The system is composed of elementary particles and the field, observed during the period $[0, t_0]$ by the standard observer, over an area $\Omega \subset M$. Each object

[2] Actually even if the particle had some volume we still would have the issue of the border.

of the system is represented by variables denoted collectively z, belonging to fiber bundles E. A change of the object is then represented by a section of the jet prolongation of E. Here, because only the first derivatives are involved, it is a section $j^1 z = \left(m, z^i, z^i_\alpha, \alpha = 0...3, i = 1...n\right)$ of E. To each object of a system is assigned a real scalar function $L\left(j^1 z\right)$, the scalar lagrangian, representing the energy exchanged by the object at each point of Ω. The lagrangian is then the product of $L\left(j^1 z\right)$ by a volume form.

In General Relativity the metric is a variable, it is measured by the observer through the vectors P^α_i of the tetrad, which are themselves variables.

The energy depends on the observer, but the equilibrium should hold for any observer. Meanwhile the scalar lagrangians depend on the observer, their product with the form should be invariant, either in a change of gauge, or in a change of chart, defining another observer. This gives strong constraints for the formulation of the lagrangians. It can be proven (see Th.Physics 6.1) that the scalar lagrangian must involve only the variables $\psi, \nabla_\alpha \psi, \mathcal{F}, P^i_\alpha, V^\alpha$. In particular :

- the potential should appear explicitly only through the covariant derivative and the strength

- the velocity of the particle must be expressed in the coordinates of a chart (and not in a tetrad), and its derivative should not be present

- the derivative of P^i_α, or equivalently the derivatives $\partial_\gamma g_{\alpha\beta}$ of the metric should not appear.

Moreover are tensors the partial derivatives of the scalar lagrangian L_p for particles and L_F for the field :

$\frac{\partial L_p}{\partial \psi^a} F^a, \frac{\partial L_p}{\partial \nabla_\beta \psi^a} F^a \otimes \partial \xi_\beta, \frac{\partial L}{\partial P^\alpha_i} \varepsilon^i \otimes \partial \xi_\alpha, \frac{\partial L_F}{\partial \grave{A}^a_\beta} F^a \otimes \partial \xi_\beta, \frac{\partial L_p}{\partial V^\beta} d\xi^\beta$

with the components ε^i, F^a of the dual of the vector space $Cl\left(\mathbb{C}, 4\right)$.

This construct has a physical meaning if the scalar lagrangians represent the *flow of energy* which is really exchanged : there is no place for potential energy, or even kinetic energy, and even less for "energy at rest". The variables must represent the change in the states of the objects : for particles it will be the covariant derivative, and for the field the strength.

4.2 Lagrangian for the particles

4.2.1 Scalar Lagrangian

The hermitian scalar product is defined on $P_U\left[Cl\left(\mathbb{C}, 4\right), \vartheta\right]$. Along the trajectory of a particle :

$\left\langle \left(p, \psi\left(\tau\right)\right), \left(p, \psi\left(\tau\right)\right)\right\rangle_H = \left\langle \psi\left(\tau\right), \psi\left(\tau\right)\right\rangle_H = \left\langle \vartheta\left(u\left(\tau\right)\right)\psi_0, \vartheta\left(u\left(\tau\right)\right)\psi_0\right\rangle_H = \left\langle \psi_0, \psi_0\right\rangle_H = Ct$

the state changes as the covariant derivative

$\Rightarrow \nabla_V \left\langle \psi\left(\tau\right), \psi\left(\tau\right)\right\rangle_H = \left\langle \nabla_V \psi\left(\tau\right), \psi\left(\tau\right)\right\rangle_H + \left\langle \psi\left(\tau\right), \nabla_V \psi\left(\tau\right)\right\rangle_H = 0$

and $\left\langle \psi, \nabla_V \psi\right\rangle_H \in i\mathbb{R}$.

A section of $J^1 P_U$ is $j^1 u = \left(m, u, u^{-1} \cdot \delta_\beta u, \beta = 0..3\right)$ with $\delta_\beta u \in T_1 U$

For any motion we have a map :

$\widehat{\nabla}:\mathfrak{X}\left(J^{1}P_{U}\right)\rightarrow\Lambda_{1}\left(M;T_{1}U\right):\widehat{\nabla}j^{1}u=\left(\sum_{\alpha=0}^{3}\left(u^{-1}\cdot\delta_{\alpha}u+Ad_{u^{-1}}\grave{A}_{\alpha}\right)\otimes d\xi^{\alpha}\right)(m)$

The tangent V to the trajectory is fixed by u, in any motion :

$V^{\alpha}=\sum_{j=0}^{3}V^{j}P_{j}^{\alpha}=cP_{0}^{\alpha}+2\frac{\sinh\mu_{w}\cosh\mu_{w}}{1+2(\cosh\mu_{w})^{2}}\sum_{j=1}^{3}\frac{W^{j+5}}{\mu_{w}}P_{j}^{\alpha}$ where W is the component of u in the chart of U.

So we have :

$$\widehat{\nabla}_{V}j^{1}u=\left(\sum_{\alpha=0}^{3}\left(u^{-1}\cdot\delta_{\alpha}u+Ad_{u^{-1}}\grave{A}_{\alpha}\right)V^{\alpha}\left(u\right)\right)(m) \qquad (40)$$

and we state :

Proposition 15 *The lagrangian for the elementary particles is :*

$$L_{p}\left(j^{1}u\right)=\frac{1}{i}\left\langle\psi,\nabla_{V}\psi\right\rangle_{H}=\frac{1}{i}\left\langle\psi_{0},\vartheta'\left(1\right)\left(\widehat{\nabla}_{V}j^{1}u\right)\psi_{0}\right\rangle_{H} \qquad (41)$$

The lagrangian depends only on the covariant derivative, as it is required to respect the equivariance. It does not depend on the choice of gauge or observer.

4.2.2 Charges

The map : $\vartheta'(1):T_{1}U\rightarrow\mathcal{L}\left(Cl\left(\mathbb{C},4\right);Cl\left(\mathbb{C},4\right)\right)::\vartheta'\left(1\right)\left(\delta\kappa\right)\left(\psi_{0}\right)=\delta\kappa^{1}\psi_{0}+\left[\delta\kappa,\psi_{0}\right]$ with $\delta\kappa=\sum_{a=1}^{16}\delta\kappa^{a}F_{a}\in T_{1}U\subset Cl\left(\mathbb{C},4\right)$

Denoting :

$\delta\kappa=\left\{\left(iT_{A},T_{V_{0}},iT_{V},iT_{W},T_{R},T_{X_{0}},iT_{X},T_{B}\right),T_{A},T_{V_{0}},T_{X_{0}},T_{B}\in\mathbb{R},T_{V},T_{W},T_{R},T_{X}\in\mathbb{R}^{3}\right\}$ in the usual orthonormal basis of $Cl\left(\mathbb{C},4\right)$, that is $\delta\kappa=\sum_{a=1}^{16}T^{a}\kappa_{a}$ in the real basis of $T_{1}U$

$\psi_{0}=\left(a,v_{0},v,w,r,x_{0},x,b\right)$ with complex components in the usual orthonormal basis of $Cl\left(\mathbb{C},4\right)$

$\frac{1}{i}\left\langle\psi_{0},\vartheta'\left(1\right)\left(\delta\kappa\right)\left(\psi_{0}\right)\right\rangle_{H}$ reads :

$\frac{1}{i}\left\langle\psi_{0},\vartheta'\left(1\right)\left(\delta\kappa\right)\left(\psi_{0}\right)\right\rangle_{H}=Q_{A}T_{A}+Q_{V_{0}}T_{V_{0}}+Q_{V}^{t}T_{V}+Q_{W}^{t}T_{W}+Q_{R}^{t}T_{R}+Q_{X_{0}}T_{X_{0}}+Q_{X}^{t}T_{X}+Q_{B}T_{B}$

with the charges

$Q_{A}=\left\langle\psi_{0},\psi_{0}\right\rangle_{H}=\overline{(a)}a-\overline{(v_{0})}v_{0}+\overline{(v)}^{t}v-\overline{(w)}^{t}w+\overline{(r)}^{t}r+\overline{(x_{0})}x_{0}-\overline{(x)}^{t}x-\overline{(b)}b$

$Q_{V_{0}}=4\operatorname{Im}\left(v^{t}w+bx_{0}\right)$

$Q_{V}=4\operatorname{Re}\left(v_{0}w-bx+j\left(r\right)v\right)$

$Q_{W}=4\operatorname{Re}\left(-v_{0}v+x_{0}x+j\left(r\right)w\right)$

$Q_{R}=4\left(-j\left(\operatorname{Re}v\right)\operatorname{Im}v+j\left(\operatorname{Re}w\right)\operatorname{Im}w-j\left(\operatorname{Re}r\right)\operatorname{Im}r+j\left(\operatorname{Re}x\right)\operatorname{Im}x\right)$

$Q_{X_{0}}=4\operatorname{Im}\left(bv_{0}+x^{t}w\right)$

$Q_{X}=4\operatorname{Re}\left(bv-x_{0}w+j\left(x\right)r\right)$

$Q_{B}=4\operatorname{Im}\left(v_{0}x_{0}+v^{t}x\right)$

The charges are real scalars. The variation of the energy of a particle is then a real linear function of $\delta\kappa$. We have a formulation which is a simple extension of the common formulation of classic Theory of Fields.

Antiparticles are represented in the contragredient representation $\left(Cl\left(\mathbb{C},4\right),\overline{(\vartheta)}\right)$. The fundamental state ψ_{0}^{c} of the antiparticle associated to ψ_{0} is $\psi_{0}^{c}=\overline{(\psi_{0})}$ and we can check with the relations above that the charges take the opposite value $Q^c = -Q$. And no fermion is its own antiparticle.

We can give a simple expression to the lagrangian. By construct L_p does not depend on the choice of a gauge. $\widehat{\nabla}_V j^1 u = \sum_{a=1}^{16} T^a \kappa_a \in T_1 U$ where the quantities T^a are the components of a vector of the adjoint representation $P_U [T_1 U, Ad]$ and in a change of gauge κ_a changes as $Ad_{\varkappa^{-1}}$.

Let $Q = \sum_{a=1}^{16} Q^a \kappa_a$ with $Q^a = (Q_A, -Q_{V_0}, Q_V, -Q_W, Q_R, Q_{X_0}, -Q_X, -Q_B)$, κ_a the vectors of the basis of $T_1 U$ in the adjoint representation $P_U [T_1 U, Ad]$ thus $Q \in P_U [T_1 U, Ad]$.

Charges of elementary particles (the index $a = 1, 2, 3$ is for each generation of particle)

Electrons :

$\psi_0 = (a^e, v_0^e + i v_0^{ae}, v^{ae} + i v^e, 0, 0, v_0^e, i v^e, a^e)$

$Q_e = ((v_0^e)^2 - (v_0^{ae})^2 + (v^{ae})^t v^{ae} + 2 (v^e)^t v^e, 0, 0, 4 (v_0^{ae} v^e + v_0^e v^{ae}), 4j (v^e) v^{ae},$
$4 a^{er} v_0^{ae}, -4 a^e v^{ae}, -4 \left(v_0^{ae} v_0^e + (v^{ae})^t v^e \right))$

Neutrinos :

$\psi_0 = (a^\nu, v_0^\nu + i v_0^{a\nu}, v^{a\nu} + i v^\nu, 0, 0, v_0^\nu, i v^\nu, a^\nu)$

$Q_\nu = ((v_0^\nu)^2 - (v_0^{a\nu})^2 + (v^{a\nu})^t (v^{a\nu}) + 2 (v^\nu)^t v^\nu, 0, 0, 4 (v_0^{a\nu} v^\nu + v_0^\nu v^{a\nu}), 4j (v^\nu) v^{a\nu},$
$4 a^\nu v_0^{a\nu}, -4 a^\nu v^{a\nu}, -4 \left(v_0^{a\nu} v_0^\nu + (v^{a\nu})^t v^\nu \right))$

with $(v_0^\nu)^2 - (v_0^{a\nu})^2 + (v^{a\nu})^t (v^{a\nu}) + 2 (v^\nu)^t v^\nu = 0$

Quarks u (the index c is for the color red, blue, green):

$\psi_0 = (a^u, v_0^u + i v_0^{au}, v^{au} + i v^u, 0, i r^c, i x_0^c, 0, a^u)$

$Q_u = (-(v_0^u)^2 - (v_0^{au})^2 + (v^{au})^t v^{au} + (v^u)^t v^u + (r^c)^t r^c - (x_0^c)^2, -4 a^u x_0^c, 4j (v^u) r^c,$
$-4 (v_0^{au} v^u - v_0^u v^{au}), 4j (v^u) v^{au}, 4 a^u v_0^{au}, -4 a^u v^{au}, 4 v_0^u x_0^c)$

Quarks d (the index c is for the color red, blue, green)::

$\psi_0 = \left(a^d, v_0^d + i v_0^{ad}, v^{ad} + i v^d, 0, i r^c, i x_0^c, 0, a^d\right)$

$Q_d = (-\left(v_0^d\right)^2 - \left(v_0^{ad}\right)^2 + \left(v^{ad}\right)^t v^{ad} + \left(v^d\right)^t v^d + (r^c)^t r^c - (x_0^c)^2, -4 a^d x_0^c, 4j \left(v^d\right) r^c,$
$-4 \left(v_0^{ad} v^d - v_0^d v^{ad}\right), 4j \left(v^d\right) v^{ad}, 4 a^d v_0^{ad}, -4 a^d v^{ad}, 4 v_0^d x_0^c)$

The charges can be expressed by taking as unit the EM charge of the electron $Q_A = \langle \psi_0, \psi_0 \rangle_H = -1$ with a universal constant.

Then :

$\langle Q, T \rangle_H = \left\langle \sum_{a=1}^{16} Q^a \kappa_a, \sum_{a=1}^{16} T^a \kappa_a \right\rangle_H = \sum_{a=1}^{16} \eta_{aa} Q^a T^a = L_p$

$\left\langle Q, \widehat{\nabla}_y j^1 u \right\rangle_H$ is a real scalar which does not depend on the gauge.

And the scalar lagrangian reads for the particles :

$$L_p \left(j^1 u\right) = \left\langle Q, \widehat{\nabla}_V j^1 u \right\rangle_H = \left\langle Q, \left(\sum_{\alpha=0}^{3} \left(u^{-1} \cdot \delta_\alpha u + Ad_{u^{-1}} \grave{A}_\alpha \right) V^\alpha (u) \right) \right\rangle_H \tag{42}$$

4.2.3 Lagrangian

For a given particle the section j^1u is a map :

$\mathbb{R} \to J^1U :: j^1u(t) = \left(q(t), u(t), u^{-1} \cdot \delta_\beta u(t), \beta = 0..3\right)$

the lagrangian is :

$$\int_0^{t_0} L_P\left(j^1u(t)\right) dt = \int_0^{t_0} \left\langle Q, \left(\sum_{\alpha=0}^{3}\left(u^{-1} \cdot \delta_\alpha u + Ad_{u^{-1}} \grave{A}_\alpha(q(t))\right) V^\alpha(u)\right)\right\rangle_H dt \tag{43}$$

The tangent V to the trajectory is defined in the tetrad through the relations :

$V = c\varepsilon_0 + 2\frac{\sinh \mu_w \cosh \mu_w}{1+2(\cosh \mu_w)^2} \sum_{j=1}^3 \frac{W^j}{\mu_w} \varepsilon_j$ where $W^j = W^{a-5}$ is the component of $u(t)$ in the chart of U

$$V^\alpha = cP_0^\alpha + 2\frac{\sinh \mu_w \cosh \mu_w}{1+2(\cosh \mu_w)^2} \sum_{j=1}^{3} \frac{W^j}{\mu_w} P_j^\alpha$$

So the lagrangian depends on the value $\grave{A}_\alpha(q(t))$ of the potential and of the tetrad P_j^α at its location $q(t)$.

4.3 Lagrangian for the field

The field propagates in the vacuum, by interacting with itself and the exchange of energy in the process involves the potential and its derivative $\mathcal{F}$. The volume form is ϖ_4 given by the metric.

4.3.1 Scalar product for r forms on TM

On any n dimensional real manifold endowed with a non degenerate metric g there is a scalar product, denoted G_r for scalar r-forms $\lambda \in \Lambda_r(M;\mathbb{R})$ (Maths.19.1.2). G_r is a bilinear symmetric form, which does not depend on a chart, is non degenerate and definite positive if g is Riemannian.

$G_r(\lambda, \mu) = \sum_{\{\alpha_1...\alpha_r\}\{\beta_1...\beta_r\}} \lambda_{\alpha_1...\alpha_r} \mu_{\beta_1...\beta_r} \det\left[g^{-1}\right]^{\{\alpha_1...\alpha_r\}}_{\{\beta_1...\beta_r\}}$

G_r defines an isomorphism between r and $n-r$ forms. The Hodge dual $*\lambda$ of a r form λ is a $n-r$ form such that :

$\forall \mu \in \Lambda_{n-r}(M) : *\lambda \wedge \mu = G_r(\lambda, \mu)\varpi_n$

where ϖ_n is the volume form deduced from the metric. The Hodge dual has the property, on M, that : $**\lambda_r = (-1)^{r^2} \lambda_r$

For 2 forms on M : $\forall \lambda, \mu \in \Lambda_2(M;\mathbb{R}) : *\lambda \wedge \mu = G_2(\lambda, \mu)\varpi_4$

The Hodge dual $*\mathcal{F}$ of a scalar 2-form $\mathcal{F} \in \Lambda_2(M, \mathbb{R})$ is a 2 form whose expression, with the Lorentz metric, is simple when a specific ordering is used. Writing $\mathcal{F} = \mathcal{F}_r + \mathcal{F}_w$ with :

$\mathcal{F}_r = \mathcal{F}_{32} d\xi^3 \wedge d\xi^2 + \mathcal{F}_{13} d\xi^1 \wedge d\xi^3 + \mathcal{F}_{21} d\xi^2 \wedge d\xi^1$

$\mathcal{F}_w = \mathcal{F}_{01} d\xi^0 \wedge d\xi^1 + \mathcal{F}_{02} d\xi^0 \wedge d\xi^2 + \mathcal{F}_{03} d\xi^0 \wedge d\xi^3$

then : $*\mathcal{F} = *\mathcal{F}_r + *\mathcal{F}_w$

$$*\mathcal{F}_r = -\left(\mathcal{F}^{01} d\xi^3 \wedge d\xi^2 + \mathcal{F}^{02} d\xi^1 \wedge d\xi^3 + \mathcal{F}^{03} d\xi^2 \wedge d\xi^1\right) \sqrt{|\det g|}$$
$$*\mathcal{F}_w = -\left(\mathcal{F}^{32} d\xi^0 \wedge d\xi^1 + \mathcal{F}^{13} d\xi^0 \wedge d\xi^2 + \mathcal{F}^{21} d\xi^0 \wedge d\xi^3\right) \sqrt{|\det g|}$$
$$\mathcal{F}^{\alpha\beta} = \sum_{\lambda\mu=0}^{3} g^{\alpha\lambda} g^{\beta\mu} \mathcal{F}_{\lambda\mu} \tag{44}$$

The components of the parts are exchanged and the indices are raised with the metric g .Notice that the Hodge dual is a 2 form : even if the notation uses raised indexes, they refer to the basis $d\xi^\alpha \wedge d\xi^\beta$.

The computation of $\mathcal{F}^{\alpha\beta}$ is then easier with the previous notations (Th.Physics 2.4) , in the standard chart :

$$[*\mathcal{F}_r] = [\mathcal{F}_w] [g_3]^{-1} \sqrt{\det g_3}; [*\mathcal{F}_w] = -[\mathcal{F}_r] [g_3] / \sqrt{\det g_3} \tag{45}$$

The scalar product of 2 scalar forms can then be expressed equivalently as :

$$G_2(\mathcal{F}, K) = -\frac{1}{\sqrt{|\det g|}} \left([*\mathcal{F}_w] [K_r]^t + [*\mathcal{F}_r] [K_w]^t \right) = \sum_{\{\alpha\beta\}} \mathcal{F}^{\alpha\beta} K_{\alpha\beta} = \frac{1}{2} \sum_{\alpha\beta=0}^{3} \mathcal{F}^{\alpha\beta} K_{\alpha\beta} \tag{46}$$

The same computation can be done for each component of $\mathcal{F}^a_{\alpha\beta}$ so the previous formulas holds for $\left[\mathcal{F}^a_{\alpha\beta}\right] = \left[\begin{array}{cc} \mathcal{F}_r & \mathcal{F}_w \end{array} \right]$.

4.3.2 Scalar product for the strength of the connection

We have to combine a scalar product of 2 forms and a scalar product in the Lie algebra T_1U.

For the latter, one can use the Killing form, which is invariant by the adjoint map, but is degenerate (it is null in the center). So we will use the hermitian product on $Cl(\mathbb{C}, 4)$, which is preserved by the adjoint map when $g \in U$.

$\mathcal{F} = \sum_{a=1}^{16} \sum_{\{\alpha,\beta\}=0}^{3} \mathcal{F}^a_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a$ where $\kappa_a = \zeta_a F_a$ with $\mathcal{F}^a_{\alpha\beta} \in \mathbb{R}$

Then :

$\langle \mathcal{F}, K \rangle = \left\langle \sum_{a=1}^{16} \sum_{\{\alpha,\beta\}=0}^{3} \mathcal{F}^a_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a, \sum_{b=1}^{16} \sum_{\{\alpha,\beta\}=0}^{3} K^b_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_b \right\rangle$

$= \sum_{a,b=1}^{16} \left\langle \sum_{\{\alpha,\beta\}=0}^{3} \mathcal{F}^a_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta, \sum_{\{\alpha,\beta\}=0}^{3} K^b_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \right\rangle \langle \kappa_a, \kappa_b \rangle_H$

$= \sum_{a=1}^{16} \left\langle \sum_{\{\alpha,\beta\}=0}^{3} \mathcal{F}^a_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta, \sum_{\{\alpha,\beta\}=0}^{3} K^a_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \right\rangle_{TM} \eta_{aa}$

$\langle \mathcal{F}, K \rangle = \sum_{a=1}^{16} \eta_{aa} G_2(\mathcal{F}^a, K^a) = \sum_{a=1}^{16} \eta_{aa} \sum_{\{\alpha\beta\}} \mathcal{F}^{a\alpha\beta} K^a_{\alpha\beta}$

$= -\frac{1}{\sqrt{|\det g|}} \sum_{a=1}^{16} \eta_{aa} \left([*\mathcal{F}^a_w] [K^a_r]^t + [*\mathcal{F}^a_r] [K^a_w]^t \right)$

$= -\frac{1}{\sqrt{|\det g|}} Tr \left([K_r]^t [\eta] [*\mathcal{F}_w] + [K_w]^t [\eta] [*\mathcal{F}_r] \right)$

Which can be expressed equivalently :

$$\langle \mathcal{F}, K\rangle = \sum_{a=1}^{16} \eta_{aa} \sum_{\{\alpha\beta\}} \mathcal{F}^{a\alpha\beta} K^a_{\alpha\beta} = \frac{1}{2}\sum_{a=1}^{16} \eta_{aa} \sum_{\alpha\beta} \mathcal{F}^{a\alpha\beta} K^a_{\alpha\beta}$$
$$\langle \mathcal{F}, K\rangle = -\frac{1}{\sqrt{|\det g|}} Tr\left([K_r]^t [\eta][*\mathcal{F}_w] + [K_w]^t [\eta][*\mathcal{F}_r]\right) \quad (47)$$

The lagrangian involves only $\mathcal{F}$, as requested by the equivariance.

4.3.3 Lagrangian for the field

The scalar lagrangian for the field is then $L_F = \langle \mathcal{F}, \mathcal{F}\rangle$ and with the volume form : $\varpi_4 = \sqrt{|\det g|} d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$ we state :

Proposition 16 *The action for the field is :*
$\int_\Omega \langle \mathcal{F}, \mathcal{F}\rangle \varpi_4 = -\int_\Omega Tr\left([\mathcal{F}_r]^t [\eta][*\mathcal{F}_w] + [\mathcal{F}_w]^t [\eta][*\mathcal{F}_r]\right) d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

The action does not depend on the gauge or the observer, so we can take the metric for the standard observer with :
$[g] = [P']^t [\eta][P'] = \begin{bmatrix} -1 & 0 \\ 0 & [g]_3 \end{bmatrix}$
which gives the action :
$\int_\Omega \langle \mathcal{F}, \mathcal{F}\rangle \varpi_4 =$
$-\int_\Omega Tr\left([\mathcal{F}_r]^t [\eta][\mathcal{F}_r][g_3](\det Q) + [\mathcal{F}_w]^t [\eta][\mathcal{F}_w][g_3]^{-1}(\det Q')\right) d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$
$\det Q = 1/\sqrt{\det g_3}; \det Q' = \sqrt{\det g_3}$

$$\int_\Omega \langle \mathcal{F}, \mathcal{F}\rangle \varpi_4 = \quad (48)$$
$$-\int_\Omega Tr\left([\mathcal{F}_r]^t [\eta][\mathcal{F}_r][g_3]\left(\sqrt{\det g_3}\right)^{-1} + [\mathcal{F}_w]^t [\eta][\mathcal{F}_w][g_3]^{-1}\sqrt{\det g_3}\right) d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \quad (49)$$

All these formulas hold when $\mathcal{F}$ is expressed with the 1st jet extension of $\grave{A}$.

4.4 The issue of the metric

The metric changes with the location. We need to represent this process.

4.4.1 Einstein's Theory of gravitation

It is a distinct part of his General Relativity (see Th.Physics 5.5). It is based on 3 physical assumptions.

i) Starting from the fact that the inertial mass is equal to the gravitational charge (as it is verified with a great accuracy) Einstein stated that actually the gravitational forces were just inertial forces felt by material body moving in a curved space time. But then we have to explain why material bodies do not move on straight lines, curved space or not. So the first assumption is that material bodies travel on geodesics.

ii) The most general definition of geodesics, in any metric set, is lines which have the shortest length. The general definition on manifolds endowed with a linear connection is curves such that the covariant derivative of their tangent y is null : $\nabla_y y = 0$. It happens that, on manifolds endowed with a metric, there is a unique linear connection which is both symmetric and preserves the scalar product, called the Lévi-Civita connection, and with this connection one can prove that geodesics are indeed the curves of shortest length. The second assumption is then that the geodesics are defined with this connection. It can be computed from the metric only.

iii) To explain the variation of the metric Einstein postulated an equation based on the scalar curvature R, which can be computed from the Lévy-Civita connection. Because in this theory there is no genuine gravitational field and its energy is not defined, the action for the metric is the Hilbert action $\int R\varpi_4$,to which actions for the particles and the EM field can be added in a way similar as above. Then the Einstein's equation can be derived in a more classic way using the Principle of Least Action.

This theory is consistent and beautiful. It is based entirely on the metric, but leads to very complicated computations (only one solution is known, for a special case). It is the basis of all Cosmological models, and of many models in Astrophysics, but there are unexplained facts, which lead to the introduction of "dark matter". Actually the choice of the Lévi-Civitta connection is controversial, and Einstein himself has explored more general connections (non symmetric, with torsion).

This theory can be expressed in the more general framework of gauge theories, in which it is a special case. The covariant derivative of a connection is symmetric if its Riemann curvature is null (Maths.2083), then $\mathcal{F} = 0$.

Even if this is not essential in the theory, it is assumed that "light propagates on geodesics".

4.4.2 The issue of the metric in a unified Theory of Fields

In a unified theory of fields, as it is presented here, this is the inertia which disappears : the mass is just one charge (accompanied by the inertial moment). The gravitational field is represented by the subalgebra :

$T_1U_G = \left\{(0,0,0,iW,R,0,0,0)\,, W,R \in \mathbb{R}^3\right\}$

and the charges are then :

$Q_W = 4\,\mathrm{Re}\left(-v_0 v + x_0 x + j\left(r\right) w\right)$

$Q_R = 4\left(-j\left(\mathrm{Re}\, v\right)\mathrm{Im}\, v + j\left(\mathrm{Re}\, w\right)\mathrm{Im}\, w - j\left(\mathrm{Re}\, r\right)\mathrm{Im}\, r + j\left(\mathrm{Re}\, x\right)\mathrm{Im}\, x\right)$

arranged in 3 generations for elementary particles.

The gravitational field is part in the exchange of energy, and is represented in $\langle \mathcal{F}, \mathcal{F} \rangle$.

According to the Principle of Locality the metric should depend on physical quantities at each point. In a system of particles and fields the metric is a variable whose value depends on the particles and the field, and in particular as they result from the balance of energy in the system. In a system defined over a compact area Ω we assume that the metric and the field have initial values at the border. We can expect to compute the metric and the field as they result from the internal processes of the system, and the initial conditions. If there is an "external field" its value should be known and added to the one which is computed. We do not know how to isolate a system from the gravitational field, however, in usual circumstances and certainly for elementary particle, the variations of the external gravitational field are very weak. But we cannot exclude strong variations of the field, gravitational or not, close to particles, and thus of the metric, which is a variable as the others.

The metric is measured by the observer through the tetrad (the variation of their components in the chart), so the variables are P_i^β. Their involvement in the equations for a system can be seen as the price to be paid, in terms of energy, to keep or change the metric, which can be measured through the energy momentum tensor.

Moreover the Chern-Weil theorem, even if it is purely mathematical, provides a condition (the quantity $\langle \mathcal{F}, \mathcal{F} \rangle_H$ does not depend on the field) which should be met in any physical system. It depends only on the existence of the principal bundle P_U which is defined through the metric. So we can say that, at any point, $\langle \mathcal{F}, \mathcal{F} \rangle_H$ depends on the metric only.

Part II
SYSTEMS OF PARTICLES / FIELD

All the systems that we will consider are composed of the field, elementary particles, followed by an observer over a relatively compact area during the period $[0, t_0]$.

5 Quantization

??

5.1 General results

The "Axioms of Quantum Mechanics" are actually Mathematical theorems which apply to models meeting some *precise* properties (Th.Physics 2).

In the picture above a state of a system, over an area Ω,followed by the observer on a period $[0, t_0]$, is characterized by maps :

$\psi : [0, t_0] \to P_C$ for each particle - we do not assume that their type is known, so they are vectorial quantities

$\widehat{\grave{A}} \in \Lambda_1 (TP_U; T_1U)$

$\mathcal{F} \in \Lambda_2 (TM; T_1U)$

$g \in \odot^2 (TM; \mathbb{R})$

They are valued in vector bundles, assumed to be differentiable, they belong to infinite dimensional Fréchet spaces (Maths.7.1) with a relatively compact support, and meet the necessary conditions for quantization. The fact that the strength $\mathcal{F}$ is computed from $\widehat{\grave{A}}$ does not matter here : derivatives are considered as independent variables (as in the r-jets formalism).

There are Hilbert spaces such that to each map $\grave{A}, g, \psi, \mathcal{F}$ is associated a vector of the Hilbert spaces $H_A, H_g, H_p, H_{\mathcal{F}}$:

$\Upsilon_p : C([0, t_0]; P_{Cl}) \to H_p$

$\Upsilon_A : \Lambda_1 (TP_U; T_1U) \to H_A$

$\Upsilon_{\mathcal{F}} : \Lambda_2 (TM; T_1U) \to H_{\mathcal{F}}$

$\Upsilon_g : \odot^2 (TM; \mathbb{R}) \to H_g$

and the maps $\Upsilon_p, \Upsilon_A, \Upsilon_{\mathcal{F}}, \Upsilon_g$ are linear isometries. For each variable X, and basis $(e_i)_{i \in I}$ of the vector space to which it belongs, there are unique families $(\varepsilon_i)_{i \in I}, (\phi_i)_{i \in I}$ of independent vectors of the associated Hilbert space H such that :

$X = \sum_{i \in I} \langle \phi_i, \Upsilon (X) \rangle_H e_i \Leftrightarrow \Upsilon (X) = \sum_{i \in I} \langle \phi_i, \Upsilon (X) \rangle_H \varepsilon_i$

$\varepsilon_i = \Upsilon (e_i), \langle \varepsilon_i, \phi_j \rangle_H = \delta_{ij}$

Each variable can be represented as a vector, with a basis which is the image by Υ^{-1}, of a basis of the Hilbert spaces.

Experiments provide only a finite number of data. The physicist represents each variable by a vector valued in a finite dimensional space, whose components can then be estimated by a statistical method, from a finite number of observations. These simplified representations (a "specification" in Statistics) are defined by choosing some vectors, which are image of vectors of the Hilbert spaces by Υ^{-1}, as basis. They are the usual observables of Quantum Mechanics. To any such observable X_J is associated a self adjoint, compact, trace class operator $\widehat{X}_J = \Upsilon^{-1} \circ X_J \circ \Upsilon$ in the Hilbert space, such that a measure of the observable is, if the system is in the state X :

$X_J(X) = \sum_{i \in J} \left\langle \phi_i, \widehat{X}_J(\Upsilon(X)) \right\rangle_H e_i$

This is just the statistical estimator from a finite batch of data.

In a global change of gauge (not depending on m) the variables represent the same state of the system. (P_C, ϑ) is a representation of the group U, as well as $(\Lambda_1(TP_U; T_1U), Ad)$, as a consequence $\left(H_p, \Upsilon_p \circ \vartheta \circ \Upsilon_p^{-1}\right), \left(H_A, \Upsilon_A \circ Ad \circ \Upsilon_A^{-1}\right),$

$\left(H_{\mathcal{F}}, \Upsilon_{\mathcal{F}} \circ Ad \circ \Upsilon_{\mathcal{F}}^{-1}\right)$ are unitary Hilbert representations of the group U. And the corresponding observables are finite dimensional irreducible unitary representations. Actually this is at the foundation of elementary particles : finite dimensional unitary representations are sums or products of fundamental representations, which correspond to elementary particles.

Whenever a scalar function (such as the energy) is added to the model, the Hilbert spaces split in subspaces corresponding to a value of the scalar.

ψ is valued in a normed vector space, if the evaluation map $\mathcal{E}(t) : C([0, t_0]; P_{Cl}) \rightarrow Cl(\mathbb{C}, 4) :: \mathcal{E}(t)(\psi) = \psi(t)$ is continuous, then there is :

a Hilbert space F_P, a map $\Theta_P : [0, t_0] \rightarrow \mathcal{L}(F_P; F_P)$ such that $\Theta_P(t)$ is unitary and $\Theta_P(t)(\psi(0)) = \psi(t)$,

for each t an isometry $\widehat{\Theta}_P(t) \in \mathcal{L}(H_P, F_P)$ such that $\widehat{\Theta}_P(t)\left(\Upsilon_P \widehat{\psi}\right) = \widehat{\psi}(t)$

Θ_P is an evolution operator. Moreover, by taking the projection from Ω to the fiber bundle $\Omega(\mathbb{R}, \Omega_3(0), \pi_R)$ the trajectories are themselves fixed by the initial conditions : $q(t) = \pi_R(\psi(t)) = \pi_R(\Theta_P(t)(\psi(0)))$.

We have nothing equivalent for the other variables. A section of a fiber bundle $E(\Omega, V, \pi)$ does not provide a section of $E_R([0, t_0], \Omega_3(0) \times V, \pi_R)$.

All these results (others can be added as we will see) depend only on the choice of the mathematical representation, and not of the physical state of the system (such as an equilibrium).

5.2 Systems without creation / annihilation of particles

Let us consider a system composed of N particles of unknown types interacting with the field, in a relatively compact area Ω over the time $[0, t_0]$ as above. It is assumed that there is no creation or annihilation of particles.

We can label the N particles, the state of the total system is the same whatever the permutation of the labels. Then the states of equilibrium are characterized by :

i) p integers $n_1 \leqslant n_2 ... \leqslant n_p : n_1 + n_2 + ... n_p = N$ which define a class of conjugacy of $\mathfrak{S}(N)$

ii) p distinct vectors θ_k of a basis of H_p which together define a Hilbert subspace $H_J \subset H$. To each of them is associated a map ψ_k.

iii) the state of the particles of the system can then be represented collectively by tensors X

either symmetric, belonging to $\odot^{n_1} H_J \odot^{n_2} H_J .... \odot^{n_p} H_J$

or antisymmetric, belonging to $\wedge^{n_1} H_J \wedge^{n_2} H_J .... \wedge^{n_p} H_J$

This is the basic representation of composite particles, which behave as a single particle whose state is represented by a tensor. The tensor is decomposable if the particles are not interacting, which is obviously not the case.

For distinct particles their collective state can be represented by a tensor in $\otimes^N H$. The integer p is the number of families of particles, a vector X_J *of the basis* of H_J is given by $X_{j_1...j_p} = \theta_{j_1} \otimes ... \otimes \theta_{j_p}$: each particle of the family $k = 1...p$ has the same state $\psi_k = \Upsilon^{-1}(\theta_{j_k})$.

A vector of a basis of $\otimes^{n_1} H_J \otimes^{n_2} H_J .... \otimes^{n_p} H_J$ is given by a tensorial product

$\otimes^{n_1}(X_{J_1}) \otimes^{n_2}(X_{J_2}) .... \otimes^{n_p}(X_{J_p}) \sim \otimes^{n_1}(\theta_{j_1}) \otimes^{n_2}(\theta_{j_{21}}) .... \otimes^{n_p}(\theta_{j_p})$

and corresponds to an equilibrium in which n_k particles are in a state $\psi_k = \Upsilon^{-1}(\theta_{j_k})$

The integers n_k are then the number of particles in the family k and a vector *of a basis* of $\otimes^{n_1} H_J \otimes^{n_2} H_J .... \otimes^{n_p} H_J$ corresponds to an equilibrium in which all particles of the same family are in the same state. The states of the particles are quantized. The class of conjugacy is fixed but the states of the system are represented by non decomposable tensors, linear combination of quantized states.

If the particles are all of the same type then the quantized states correspond to an equilibrium in which n_k particles are in the same state ψ_k. The class of conjugacy is not fixed, but in a continuous process it does not change.

The choice between symmetric / antisymmetric tensors depends on additional symmetries. Notably the action of the Spin group $Spin(3)$: particles are symmetric if their state does not change in the reverse orientation, in a global change of gauge. This is not the case for elementary particles (their spin is 1/2), so their systems are represented by antisymmetric tensors. If all the particles of the system are identical then the quantized states correspond to distinct values of the $\psi_k = \Upsilon^{-1}(\theta_{j_k})$.

6 Principles of conservation of Energy

6.1 Local equilibrium

The conservation of energy should hold at any point, for any observer. A deficit at a point cannot be compensated by an excess at another point or in the future. For the standard observer the flow of energy is measured at a point m through $\Omega_3(t)$ in a displacement $\delta\xi_0 = c\delta t$: $\left(\sum_{Particles} L_p + L_F\right) \varpi_3(m) \times c\delta t$

and the sum of the flows should be null along the axis $\partial\xi_0$:

$\left(L_F\left(j^1 z\right) + \sum_{Particles} L_p\left(j^1 z\right)\right) \varpi_3(m)$

$= i_{\partial\xi_0}\left(L_F\left(j^1 z\right) + \sum_{Particles} L_p\left(j^1 z\right)\right)\varpi_4\left(m\right) = 0$

This formulation is independent on the observer.

Denoting $L\left(m\right) = L_F\left(j^1 z\right) + \sum_{Particles} L_p\left(j^1 z\right)$ the condition is equivalent to $i_{\partial\xi_0} L\varpi_4 = 0$ which implies $L = 0$ at equilibrium.

So we have :

$$\left(\sum_{Particles}\left\langle Q, \widehat{\nabla}_V j^1 u\right\rangle_H + \left\langle\mathcal{F},\mathcal{F}\right\rangle\right)|_m = 0 \tag{50}$$

It gives equations which can be implemented in any process, including collisions. There is no totally discontinuous processes, and actually we have a transition between continuous processes, and the equation can be understood as an equality between the "in" and "out" states. Which gives a special interest to continuous processes.

In the vacuum it implies $\left\langle\mathcal{F},\mathcal{F}\right\rangle = 0$: the field does not exchange any energy with another object, and this results from the Principle of Conservation of Energy itself, and holds whatever the specification of the energy for the field. But this does not mean that the field does not "carry" energy from a point to another, only that the energy manifests itself in an interaction with a particle. All the theory is based on the existence and the unity of the Universe, which is not a collection of independent points. And the variables are continuous maps. So to get a meaningful picture we need to look at a definite domain, beyond a point. Which leads to the Principle of Least Action.

6.2 Principle of Least Action

The state of the system is represented by the 1st jet extension of maps, with coordinates $j^1 z = \left(z^a, z^a_\alpha\right)$. We can consider the conditions for a dynamic equilibrium, meaning the conditions that the maps $j^1 z$, and not only the values $j^1 z\left(m\right), j^1 z\left(t\right)$ at each point, must meet. In this formulation we focus on the global equilibrium over Ω, it provides conditions which *replace* those of a local equilibrium, except the Chern-Weil equation which is a general result, valid whatever the field. In particular in this context the quantity $\left\langle\mathcal{F},\mathcal{F}\right\rangle$ representing the energy of the field is not necessarily null in the vacuum : the field can transport energy in the vacuum.

The variables are the state ψ of each particle, defined over $\left[0, t_0\right]$, the connection, represented by the function $\grave{A}$, and the components P^α_i of the tetrad, representing the metric.

The total flow of energy for the system is :

$\ell = \ell_F\left(\grave{A}, P\right) + \sum_{Particles} \ell_p\left(\psi, \grave{A}, P\right)$

The conservation of energy is then represented the Principle of Least Action : at equilibrium the value of the variables must be such that the total action is stationary : any change entails either an increase or a decrease of the total action : $\ell_F + \sum \ell_P$. The variables are sections, and finding a solution is a problem of variational calculus, using variational derivatives, which can be extended to fiber bundles (Maths.7.6.1). It provides PDE for the sections, whose value is fixed by the initial conditions at $t=0$.

6.3 Variational derivatives

A functional $\ell : J^1E \to \mathbb{R}$ acting on sections $j^1\zeta = \left(\delta\zeta^i, \delta_\alpha\zeta^i\right)$ of the 1st jet prolongation J^1E of a vector bundle E has a variational derivative $\frac{\delta\ell}{\delta z}$ with respect to a variable z at z_0 is there is a linear functional $\frac{\delta\ell}{\delta z}$ such that :

$\forall\delta\zeta \in \mathfrak{X}(E) :: \lim_{\|\delta\zeta\|\to 0} \left|\ell\left(j^1z_0 + j^1\zeta\right) - \ell\left(j^1z_0\right) - \frac{\delta\ell}{\delta z}(\delta\zeta)\right| = 0$

$\frac{\delta\ell}{\delta z} : j^1E \to E'$: the functional derivative, is a *continuous linear map*, defined at each point of j^1E, acting on *sections* of E, which gives a scalar : it belongs to the dual of $\mathfrak{X}(E)$. This is a distribution. The key point is that in $\frac{\delta\ell}{\delta z}(\delta\zeta)$ only the values $\delta\zeta^i(m)$ appear and not the $\delta_\alpha\zeta^i$. The functional derivative "linearizes" the functional.

Not all functionals have a variational derivative. For integrals we have the linear functionals, acting on smooth sections with compact support $\delta z \in \mathfrak{X}_{c,\infty}(E)$:

$\frac{\delta\ell}{\delta z} = \frac{\delta}{\delta z}\int_0^{t_0} L\left(j^1z(t)\right)dt = \int_0^{t_0}\sum_{i=0}^n \left(\frac{\partial L}{\partial z^i} - \frac{d}{dt}\frac{\partial L}{\partial \frac{dz^i}{dt}}\right)e^i dt$

$\frac{\delta\ell}{\delta z} = \frac{\delta}{\delta z}\int_\Omega L\left(j^1z(m)\right)\varpi_4(m)$

$= \int_\Omega \sum_{i=0}^n \left(\frac{\partial}{\partial z^i}\left(L\sqrt{|\det g|}\right) - \sum_{\beta=0}^3 \frac{d}{d\xi^\beta}\frac{\partial}{\partial z^i_\beta}\left(L\sqrt{|\det g|}\right)\right)e^i d\xi$

where e^i is the dual of a holonomic basis of E and $d\xi = d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$

Notice that these expressions hold for any section of the first jet extension : the quantities z^i_β are not necessarily partial derivatives.

Then the states of equilibrium of the system are given by sections z_0 such that, for each variable, the variational derivative $\frac{\delta\ell}{\delta z}(z_0) = 0$. Because $\frac{\delta\ell}{\delta z}$ are distributions the solutions are not necessarily continuous.

The lagrangians are invariant in a change of gauge or chart. The variational derivatives are invariant in a *global* change of gauge.

The lagrangian represents the energy exchanged by the object and the action the sum of the energy exchanged during the process, so the variational derivative represents the infinitesimal exchange of energy with respect to a change of the variable, or equivalently the flow of energy along the process *with respect to this variable.* So variational derivatives have a physical meaning, which goes beyond their use in finding the conditions for an equilibrium.

We have seen in the quantization of a system that there are Hilbert spaces H_A, H_p, H_g associated to the variables. $\left(H_p, \Upsilon_p\circ\vartheta\circ\Upsilon_p^{-1}\right), \left(H_A, \Upsilon_A\circ Ad\circ\Upsilon_A^{-1}\right)$ are unitary Hilbert representations of the group U. By definition the variational derivatives are linear continuous maps, their kernel, corresponding to states of equilibrium, are closed. Their associated vector space in H_A, H_p are closed, thus Hilbert vector subspaces. If the kernel are finite dimensional the associated Hilbert subspaces have a basis composed of a finite number of vectors. To each of this vector is associated a state of equilibrium. Then the states of the system are quantized. The states of the system is a linear combination of these fundamental states, depending on the initial conditions. The "problem is well posed".

6.4 Energy Momentum tensor

The variables z are defined on M. If the solution $\widehat{z}$ is differentiable along a vector field y on M, the quantities $\pounds_y \widehat{z}(m)\,\delta\tau$ are well defined, and give the variation of z along y. In the neighborhood of the equilibrium the total action of the system changes by :

$\delta\ell_z = \frac{\delta\ell}{\delta z}\left(j^1\widehat{z}\right)\left(\pounds_y \widehat{z}(m)\,\delta\tau\right)$ or equivalently : $\frac{\delta\ell_z}{\delta\tau} = \frac{\delta\ell}{\delta z}\left(j^1\widehat{z}\right)\left(\pounds_y \widehat{z}(m)\right)$

Variational derivatives can be composed. Because the Lie derivative is linear with respect to y, $\frac{\delta\ell}{\delta\tau}$ is a scalar one form on TM. This is the variational Lie derivative of ℓ with respect to z and y :

$$\pounds_y \ell_z = \frac{\delta\ell}{\delta z}\left(j^1\widehat{z}\right)\left(\pounds_y \widehat{z}(m)\right) \in \Lambda_1\left(TM;\mathbb{R}\right) \tag{51}$$

It gives the resistance to a change of the variable z in the direction given by y. In Mechanics the momenta are the conjugate variables $p_j = \frac{\partial L}{\partial \dot{q}_j}$. Similarly the quantities $\pounds\ell_z$ can be seen as the variables conjugate to z. If $\frac{\delta\ell_z}{\delta z} = 0$ for some vector field y, the associated variations give equivalent equilibriums of the system. The sum, for all variables, $\sum_z \pounds_y \ell_z$ gives the resistance of the object to a change in the direction given by y, or equivalently the momentum of the system. In particular there is a momentum associated to the field, given by $\pounds_y \ell_F = \frac{\delta\ell}{\delta \grave{A}}\left(j^1\widehat{z}\right)\left(\pounds_y \grave{A}\right)$. The Lie derivative of the connection is the strength $\mathcal{F}$ so the momentum of the field is $\pounds\ell_F = \frac{\delta\ell}{\delta \grave{A}}\left(j^1\widehat{z}\right)\left(\mathcal{F}\right)$. We have assumed that the field propagates along Killing vectors fields, which then represent equivalent equilibriums of the system and $\pounds_y \ell_F = 0$ for $y \in \mathfrak{X}(K)$. The field does not oppose a resistance in the direction of its propagation.

Part III
CONTINUOUS PROCESSES

A continuous process, in a system of elementary particles interacting with the field is such that :

- there is no creation or annihilation of particles : the number of particles is constant, and they keep their characteristics ψ_0
- there is no collision

Then the states and location of particles can be represented through sections of the fiber bundle P_U.

Moreover we assume that the system is at equilibrium at $t = 0$, so the conditions for an equilibrium are consistent with the initial conditions. The choice of the origin $t = 0$ is then arbitrary, by definition the problem is well posed. There is an evolution operator which gives the value of the variables related the particle (their state and location) with respect to the initial conditions, at each time.

7 The model

7.1 Representation of the state and trajectories of particles

i) For particles the evolution operator gives, for any particle of the same family, the value of $\psi(t)$ with respect to $\psi(0)$. Because they share a common ψ_0, we can then consider, for each family of particles, a section ψ defined on Ω such that the state at t is $\psi(t) = \psi(q(t))$ or equivalently a section $\mathbf{u} \in \mathfrak{X}(P_U)$ such that $\psi(m) = \vartheta(\mathbf{u}(m))\psi_0$. There is a vector field $V \in \mathfrak{X}(TM)$ fixed by the value of $\mathbf{u}$ at each point, the trajectories of the particles of the family are then the integral curves of V : $q(t) = \Phi_V(t, q(0))$.

This is actually a symmetry on P_U.

For any section $\mathbf{u} = \rho_u(\mathbf{p}_u(m), u(m)) \in \mathfrak{X}(P_U)$ and vector field $V \in \mathfrak{X}(TM)$ we can define a vector field $\mathbf{V} \in \mathfrak{X}(TP_U)$ by following an integral curve of V :

$q(t) = \Phi_V(t, q(0)) \to \mathbf{u}(t) = \rho_u(\mathbf{p}_u(q(t)), u(q(t))) \in P_U \to \mathbf{V}(\mathbf{u}(t)) = \frac{d}{dt}\mathbf{u}(\Phi_V(t, q(0))) \in T_{\mathbf{u}(t)}P_U$

$\mathbf{V}(\mathbf{u}(t)) = \rho'_{up}(\mathbf{p}_u(q(t)), u(q(t)))V(q(t)) + \zeta(\kappa(t))(\mathbf{u}(t))$

$\kappa(t) = u^{-1}(q(t)) \cdot \frac{d}{dt}(u(t)) = u^{-1}(q(t)) \cdot u(q(t))' V = u^{-1}(q(t)) \cdot \sum_{\beta=0}^{3} V^\beta \partial_\beta u$

$\mathbf{V}$ is projectable on V : $\pi'_U(\mathbf{u}(t))\mathbf{V}(\mathbf{u}(t)) = V(q(t)) = V(\pi_U(\mathbf{u}(t)))$

The flow of the vector field $\mathbf{V}$ is a map : $\Phi_{\mathbf{V}}(\tau, .) : P_U \to P_U$ defined by the condition $\frac{\partial}{\partial \tau}\Phi_{\mathbf{V}}(\tau, u)|_{\tau=\theta} = \mathbf{V}(\Phi_{\mathbf{V}}(\theta, u))$ and by construct $\mathbf{u}(t) = \Phi_{\mathbf{V}}(t, \mathbf{u}(0)) = \mathbf{u}(\Phi_V(t, q(0))) \Rightarrow \frac{d\mathbf{u}}{dt} = \mathbf{V}(\Phi_V(t, q(0)))$

ii) The representation of particles is then based on the followings.

For each family k of particles :

there is a section $\mathbf{u}_k = \rho_u(\mathbf{p}_u(m), u_k(m)) \in \mathfrak{X}(P_U)$ and a vector field $V_k \in \mathfrak{X}(TM)$, from which is deduced $\mathbf{V}_k \in \mathfrak{X}(TP_U)$.

there is a section $\widehat{Q}_k \in \mathfrak{X}(P_U[T_1U, Ad])$ defined at any point $m = \varphi_o(ct, x)$ by $\widehat{Q}_k(m) = \widehat{Q}_k((\Phi_{V_k}(-t, m)))$ and at $x \in \Omega_3(0)$ by $\widehat{Q}_k(x) = \rho_u(\mathbf{p}_u(x), Q_k)$ where Q_k is the vector charges of the family k if there is such a particle at x, and $\widehat{Q}_k(x) = 0$ if not.

So $\mathbf{u}_k, \widehat{Q}_k$ are symmetric with respect to $V_k : \forall x \in \Omega_3(0) : \mathbf{u}_k(\Phi_{V_k}(t, x)) = \Phi_{\mathbf{V}_k}(t, \mathbf{u}_k(0)), \widehat{Q}_k(\Phi_{V_k}(t, x)) = \Phi_{\mathbf{V}_k}\left(t, \widehat{Q}_k(0)\right)$.

The vector field V_k is constraint by the relation, which can be seen as a continuity equation for a particle :

$V_k^\beta(m) = cP_0^\beta(m) + 2\frac{\sinh\mu_w \cosh\mu_w}{1+2(\cosh\mu_w)^2}\sum_{j=1}^3 \frac{W^{j+5}}{\mu_w} P_j^\beta(m)$ where W is the component of $\mathbf{u}_k(m)$ in the chart of U. Because it involves P_j^β it depends on the metric.

iii) The trajectories of particles of the family are integral curves of V_k : $q(t) = \Phi_{V_k}(t, q(0))$ and their state is $u(t) = \mathbf{u}_k(\Phi_{V_k}(t, q(0))) = \Phi_{\mathbf{V}_k}(t, u(0))$

$\frac{du}{dt} = \sum_{\beta=0}^3 \partial_\beta \mathbf{u}_k(q(t)) V_k^\beta(q(t))$

The exchange of energy along their trajectories is : $\int_0^{t_0} L_p(\Phi_{V_k}(t, q(0)))\, cdt$ with :

$L_p(\Phi_{V_k}(t, q(0))) = \left\langle \widehat{Q}_k, \sum_{\beta=0}^3 \left(\mathbf{u}_k^{-1} \cdot \partial_\beta \mathbf{u}_k + Ad_{\mathbf{u}_k^{-1}}\left(\grave{A}_\beta\right) V_k^\beta\right)\right\rangle_H | (\Phi_{V_k}(t, q(0)))$

and for the whole family :

$\ell_p^k = \int_{\Omega_3(0)} \left(\int_0^{t_0} L_p(\Phi_{V_k}(t, x))\, cdt\right) \varpi_3(x) = \int_\Omega L_p^k(m) \varpi_4$

iv) The metric is a variable : it is assumed that it depends on the particles and the field. It is involved through $\sqrt{|\det g|}$ in ϖ_4 as well as in the definition of the scalar lagrangian for the field. It is also involved in the definition of the trajectory of particles : V must be expressed in the basis of a chart, but it is related to the state of the particle. Moreover $g_{\alpha\beta} = \sum_{i,j=0}^3 P_\alpha^{\prime i} P_\beta^{\prime j}$ so the right variables should be P_j^α or equivalently $P_\alpha^{\prime i}$. The vectors of the tetrad are considered as free : a change of tetrad is a change of gauge, under which the lagrangians are invariant.

$\ell_p^k\left(\mathbf{u}_k, \grave{A}, P\right)$ is a functional acting on sections $\mathbf{u}_k \in \mathfrak{X}(P_U), \grave{A} \in \Lambda_1(TM; T_1U), P \in \Lambda_1\left(TM; \mathbb{R}^4\right)$.

7.2 Principle of least action

The flow of energy for the particles is : $\sum_{k=1}^p \ell_p^k\left(\mathbf{u}_k, \grave{A}, P\right)$

The field is represented by the connection form $\widehat{\grave{\mathbf{A}}} \in \Lambda_1(P_U; T_1U)$ valued in a fixed vector space. As usual we keep the standard gauge $\mathbf{p}_u$ and then $\mathbf{p}_u^*\widehat{\grave{A}}$ is defined by the real function :

$\grave{A} : \Omega \to T_1U :: \grave{A}(m) = \sum_{\beta=0}^3 \sum_{a=1}^{16} \grave{A}_\beta^a(m)\, d\xi^\beta \otimes \kappa_a$

The flow of energy for the field over Ω is given by the functional $\ell_F\left(\grave{A}, P\right) = \int_\Omega \langle \mathcal{F}, \mathcal{F}\rangle \varpi_4$ acting on sections $\grave{A} \in \Lambda_1(TM; T_1U), P \in \Lambda_1\left(TM; \mathbb{R}^4\right)$

The balance of energy over Ω is then : $\ell = \ell_F\left(\grave{A}, P\right) + \sum_{k=1}^p \ell_p^k\left(\mathbf{u}_k, \grave{A}, P\right)$

The state of the system over the area Ω is defined by sections $j^1 z : \Omega \to J^1 E$, and we can consider the conditions for a dynamic equilibrium, meaning the conditions that the maps $j^1 z$, and not only the values $j^1 z\,(m)$ at each point, must meet.

The implementation of the method of variational derivatives to the functional : $\ell = \ell_F\left(\grave{A}, P\right) + \sum_{k=1}^{p} \ell_p^k\left(\mathbf{u}_k, \grave{A}, P\right)$ for each of the variables $\grave{A}, P, \mathbf{u}_k$ (one for each family) gives the variational derivatives : $\frac{\delta\ell}{\delta\grave{A}}, \frac{\delta\ell}{\delta P}, \frac{\delta\ell}{\delta u}$ and the conditions for an equilibrium are that these distributions are null for values $\widehat{\grave{A}}, \widehat{P}, \widehat{\mathbf{u}}_k$ of the sections :

$$\frac{\delta}{\delta\grave{A}}\left(\ell_F\left(\grave{A}, P\right) + \sum_{k=1}^{p} \ell_p^k\left(\mathbf{u}_k, \grave{A}, P\right)\right) = 0$$

$$\frac{\delta}{\delta P}\left(\ell_F\left(\grave{A}, P\right) + \sum_{k=1}^{p} \ell_p^k\left(\mathbf{u}_k, \grave{A}, P\right)\right) = 0$$

$$\frac{\delta}{\delta u_k}\left(\ell_F\left(\grave{A}, P\right) + \sum_{k=1}^{p} \ell_p^k\left(\mathbf{u}_k, \grave{A}, P\right)\right) = 0$$

8 Variational derivatives

8.1 Variational derivatives for particles

The sections $\mathbf{u}_k \in \mathfrak{X}\left(P_U\right)$ are defined through the chart given previously :

$\varphi_U : T_1 U \to U :: \varphi_U\left(z^a, a = 1...16\right) = u$

where $z^a, a = 1...16 = \left\{A, V_0, X_0, B \in C\left(\Omega; \mathbb{R}\right), V, W, R, X \in C\left(\Omega; \mathbb{R}^3\right)\right\}$

$u = e^{iA} \exp T_r \cdot \exp T_w \cdot \exp T_x \cdot \exp T_v$

The tangent V to the trajectory is defined in the tetrad through the relations :

$V = c\varepsilon_0 + 2\frac{\sinh\mu_w \cosh\mu_w}{1 + 2\left(\cosh\mu_w\right)^2} \sum_{j=1}^{3} \frac{W^j}{\mu_w}\varepsilon_j$ where $W^j = W^{a-5}$ thus $y\left(z^6\left(m\right), z^7\left(m\right), z^8\left(m\right)\right)$

$V^\alpha = cP_0^\alpha + 2\frac{\sinh\mu_w \cosh\mu_w}{1 + 2\left(\cosh\mu_w\right)^2} \sum_{j=1}^{3} \frac{W^j}{\mu_w} P_j^\alpha.$

8.1.1 Partial derivatives

The partial derivatives of u with respect to z are

$u^{-1} \cdot \frac{\partial u}{\partial z_a} = i\frac{\partial A}{\partial z_a} + Ad_{\exp(-T_v)\cdot\exp(-T_x)\cdot\exp(-T_w)}\left(\exp\left(-T_r\right) \cdot \frac{\partial}{\partial z_a} \exp T_r\right)$

$+ Ad_{\exp(-T_v)\cdot\exp(-T_x)}\left(\exp\left(-T_w\right) \cdot \frac{\partial}{\partial z_a} \exp T_w\right) + Ad_{\exp(-T_v)}\left(\exp\left(-T_x\right) \cdot \frac{\partial}{\partial z_a} \exp\left(T_x\right)\right)$

$+ \exp\left(-T_v\right) \cdot \frac{\partial}{\partial z_a} \exp T_v\}$

Each exponential has a simple expression :

$\exp T = \cosh\mu + \frac{\sinh\mu}{\mu} T; \mu^2 = T \cdot T$

$\exp T_r = \cos\mu_r + \frac{\sin\mu_r}{\mu_r} T_r; \mu_r^2 = -T_r \cdot T_r$

And we get :

i) $a = 1 : z_1 = A : \frac{\partial u}{\partial z_1} \cdot u^{-1} = i$

ii) $a = 2, 3, 4, 5, 16 : T_v = \left(0, V_0, iV, 0, 0, 0, 0, B\right), \mu_v^2 = T_v \cdot T_v = V_0^2 - V^t V + B^2$

$u^{-1} \cdot \frac{\partial u}{\partial z^a} = \frac{\partial \mu_v}{\partial z^a} \frac{\sinh^2 \mu_v}{\mu_v} + \frac{\partial \mu_v}{\partial z^a} \left(\frac{\mu_v - \sinh \mu_v \cosh \mu_v}{\mu^2} \right) T_v + \frac{\sinh \mu_v \cosh \mu_v}{\mu_v} \frac{\partial T_v}{\partial z^a} -$
$\left(\frac{\sinh \mu_v}{\mu_v} \right)^2 T_v \cdot \frac{\partial T_v}{\partial z^a}$

iii) $a = 6, 7, 8 : T_w = (0, 0, 0, iW, 0, 0, 0, 0), \mu_w^2 = W^t W = T_w \cdot T_w$

$u^{-1} \cdot \frac{\partial u}{\partial z^a} =$

$Ad_{\exp(-T_v) \cdot \exp(-T_x)} \left(0, 0, 0, iW_a \left(\frac{\mu_w - \sinh \mu_w \cosh \mu_w}{\mu_w^3} \right) W + i \frac{\sinh \mu_w \cosh \mu_w}{\mu_w} \frac{\partial W}{\partial z^a}, 0, 0, 0, 0 \right)$

iv) $a = 9, 10, 11 : T_r = (0, 0, 0, 0, R, 0, 0, 0), \mu_r^2 = R^t R = -T_r \cdot T_r$

$u^{-1} \cdot \frac{\partial u}{\partial z^a} =$

$Ad_{\exp(-T_v) \cdot \exp(-T_x) \cdot \exp(-T_w)} \cdot \left(2R_a \left(\frac{\sin \mu_r}{\mu_r} \right)^2, 0, 0, 0, -R_a \frac{\mu_r - \cos \mu_r \sin \mu_r}{\mu_r^3} R + \frac{\sin \mu_r \cos \mu_r}{\mu_r} \frac{\partial R}{\partial z_a}, 0, 0, 0 \right)$

v) $a = 12, 13, 14 : T_x = (0, 0, 0, 0, 0, X_0, X, 0), \mu_x^2 = T_x \cdot T_x = -X_0^2 + X^t X$

$u^{-1} \cdot \frac{\partial u}{\partial z^a} =$

$Ad_{\exp(-T_v)} \left(\frac{\partial \mu_x}{\partial z^a} \frac{\sinh^2 \mu_x}{\mu_x} + \frac{\partial \mu}{\partial z^a} \left(\frac{\mu_x - \sinh \mu_x \cosh \mu_x}{\mu_x^2} \right) T_x + \frac{\sinh \mu_x \cosh \mu_x}{\mu_x} \frac{\partial T_x}{\partial z^a} - \left(\frac{\sinh \mu_x}{\mu_x} \right)^2 T_x \cdot \frac{\partial T_x}{\partial z^a} \right)$

Variational derivative with respect to u

$\ell_p^k \left(\mathbf{u}_k, \grave{A}, P \right) = \int_{\Omega_3(0)} \left(\int_0^{t_0} \left\langle \widehat{Q}_k, \sum_{\beta=0}^3 \left(\mathbf{u}_k^{-1} \cdot \partial_\beta \mathbf{u}_k + Ad_{\mathbf{u}_k^{-1}} \left(\grave{A}_\beta \right) V_k^\beta \right) \right\rangle_H cdt \right) \varpi_3(x)$

We compute first the derivative with respect to z. We drop the index k.

$\frac{\delta \ell_p}{\delta z^a} = \int_{\Omega_3(0)} \left(\int_0^{t_0} \left\langle \widehat{Q}, \sum_{a=1}^{16} \left(\frac{\partial T}{\partial z^a} - \frac{d}{dt} \frac{\partial T}{\partial \frac{dz^a}{dt}} \right) \right\rangle_H cdt \right) \varpi_3(x)$

with $T = u^{-1} \cdot \frac{du}{dt} + Ad_{u^{-1}} \sum_{\alpha=0}^3 \left(\grave{A}_\alpha V^\alpha \right)$

V depends on u, and thus on z.

$\frac{\partial T}{\partial \frac{dz^a}{dt}} = u^{-1} \cdot \frac{\partial}{\partial \frac{dz^a}{dt}} \left(\frac{du}{dt} \right) = u^{-1} \cdot \frac{\partial}{\partial \frac{dz^a}{dt}} \left(\sum_{b=1}^{16} \frac{\partial u}{\partial z^b} \frac{dz^b}{dt} \right) = u^{-1} \cdot \frac{\partial u}{\partial z^a}$

$\frac{d}{dt} \frac{\partial T}{\partial \frac{dz^a}{dt}} = -u^{-1} \cdot \frac{du}{dt} \cdot u^{-1} \cdot \frac{\partial u}{\partial z^a} + u^{-1} \cdot \frac{d}{dt} \frac{\partial u}{\partial z^a}$

$\frac{\partial T}{\partial z^a} = \frac{d}{dz^a} \left(u^{-1} \cdot \frac{du}{dt} \right) + \frac{d}{dz^a} \left(Ad_{u^{-1}} \grave{A}(V) \right)$

$\frac{d}{dz^a} \left(u^{-1} \cdot \frac{du}{dt} \right) = -u^{-1} \cdot \frac{\partial u}{\partial za} \cdot u^{-1} \cdot \frac{du}{dt} + u^{-1} \cdot \left(\sum_{b=1}^{16} \frac{\partial^2 u}{\partial z^a \partial z^b} \frac{dz_b}{dt} \right)$

$= -u^{-1} \cdot \frac{\partial u}{\partial z^a} \cdot u^{-1} \cdot \frac{du}{dt} + u^{-1} \cdot \frac{d}{dt} \frac{\partial u}{\partial z^a}$

$\frac{d}{dz^a} \left(Ad_{u^{-1}} \grave{A}(V) \right) = \frac{d}{dz^a} \left(u^{-1} \cdot \sum_{\alpha=0}^3 \grave{A}_\alpha V^\alpha \cdot u \right)$

$= -u^{-1} \cdot \frac{\partial u}{\partial z^a} \cdot u^{-1} \cdot \grave{A}(V) \cdot u + u^{-1} \cdot \sum_{\alpha=0}^3 \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \cdot u + u^{-1} \cdot \grave{A}(V) \cdot \frac{\partial u}{\partial z^a}$

$\frac{\partial T}{\partial z^a} = -u^{-1} \cdot \frac{\partial u}{\partial z^a} \cdot u^{-1} \cdot \frac{du}{dt} + u^{-1} \cdot \frac{d}{dt} \frac{\partial u}{\partial z^a} - u^{-1} \cdot \frac{\partial u}{\partial z^a} \cdot u^{-1} \cdot \grave{A}(V) \cdot u + u^{-1} \cdot$
$\sum_{\alpha=0}^3 \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \cdot u + u^{-1} \cdot \grave{A}(V) \cdot \frac{\partial u}{\partial za}$

$\frac{\partial T}{\partial z^a} - \frac{d}{dt} \frac{\partial T}{\partial \frac{dz^a}{dt}} = -u^{-1} \cdot \frac{\partial u}{\partial z^a} \cdot u^{-1} \cdot \frac{du}{dt} + u^{-1} \cdot \frac{d}{dt} \frac{\partial u}{\partial z^a} - u^{-1} \cdot \frac{\partial u}{\partial z^a} \cdot u^{-1} \cdot \grave{A}(V) \cdot$
$u + u^{-1} \cdot \sum_{\alpha=0}^3 \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \cdot u + u^{-1} \cdot \grave{A}(V) \cdot \frac{\partial u}{\partial z^a}$
$+ u^{-1} \cdot \frac{du}{dt} \cdot u^{-1} \cdot \frac{\partial u}{\partial z^a} - u^{-1} \cdot \frac{d}{dt} \frac{\partial u}{\partial z^a}$

$= \sum_{b=1}^{16} \left\{ \left[u^{-1} \cdot \frac{du}{dt}, u^{-1} \cdot \frac{\partial u}{\partial z^a} \right]^b + Ad_{u^{-1}} \left(\left[\grave{A}(V), \frac{\partial u}{\partial z^a} \cdot u^{-1} \right] + \sum_{\alpha=0}^3 \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \right)^b \right\} \kappa_b$

$\sum_{a=1}^{16} \left(\frac{\partial T}{\partial z^a} - \frac{d}{dt} \frac{\partial T}{\partial \frac{dz^a}{dt}} \right) =$

$\sum_{a,b=1}^{16} \left\{ \left[u^{-1} \cdot \frac{du}{dt}, u^{-1} \cdot \frac{\partial u}{\partial z^a} \right]^b + Ad_{u^{-1}} \left(\left[\grave{A}(V), \frac{\partial u}{\partial za} \cdot u^{-1} \right] + \sum_{\alpha=0}^3 \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \right)^b \right\} \kappa_b$

$\left\langle \widehat{Q}, \sum_{a=1}^{16} \left(\frac{\partial T}{\partial z^a} - \frac{d}{dt}\frac{\partial T}{\partial \frac{dz^a}{dt}} \right) \right\rangle_H$
$= \sum_{a=1}^{16} \left\langle \widehat{Q}, \sum_{b=1}^{16} \left\{ \left[u^{-1} \cdot \frac{du}{dt}, u^{-1} \cdot \frac{\partial u}{\partial z^a} \right]^b + Ad_{u^{-1}} \left(\left[\grave{A}(V), \frac{\partial u}{\partial z^a} \cdot u^{-1} \right] + \sum_{\alpha=0}^{3} \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \right)^b \right\} \kappa_b \right\rangle$
$= \sum_{a=1}^{16} \left\langle Ad_u \widehat{Q}, \left[\frac{du}{dt} \cdot u^{-1} + \grave{A}(V), \frac{\partial u}{\partial z^a} \cdot u^{-1} \right] + \sum_{\alpha=0}^{3} \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \right\rangle$
$\frac{\delta \ell_p}{\delta z^a} = \int_{\Omega_3(0)} \left(\int_0^{t_0} \sum_{a=1}^{16} \left\langle Ad_u \widehat{Q}, \left[\frac{du}{dt} \cdot u^{-1} + \grave{A}(V), \frac{\partial u}{\partial z^a} \cdot u^{-1} \right] + \sum_{\alpha=0}^{3} \grave{A}_\alpha \frac{dV^\alpha}{dz^a} \right\rangle \kappa^a c dt \right) \varpi_3(x)$

$$\frac{\delta \ell_p}{\delta z} = \int_\Omega \sum_{a=1}^{16} \left\langle Ad_u \widehat{Q}, \left[\frac{du}{dt} \cdot u^{-1} + \grave{A}(V), \frac{\partial u}{\partial z^a} \cdot u^{-1} \right] + \sum_{\alpha=0}^{3} \grave{A}_\alpha \frac{\partial V^\alpha}{\partial z^a} \right\rangle \kappa^a \varpi_4 \tag{52}$$

For $\delta z = \sum_{a=1}^{16} \delta z^a \kappa_a \in \mathfrak{X}(P_U) : \frac{\delta \ell_p}{\delta z}(\delta z)$
$= \int_\Omega \sum_{a=1}^{16} \left\langle Ad_u \widehat{Q}, \left[\frac{du}{dt} \cdot u^{-1}, \frac{\partial u}{\partial z^a} \cdot u^{-1} \right] \delta z^a + \left[\grave{A}(V), \frac{\partial u}{\partial z^a} \cdot u^{-1} \right] \delta z^a + \sum_{\alpha=0}^{3} \grave{A}_\alpha \frac{\partial V^\alpha}{\partial z^a} \delta z^a \right\rangle \kappa^a \varpi_4$
$= \int_\Omega \left\langle Ad_u \widehat{Q}, \left[\frac{du}{dt} \cdot u^{-1}, \delta u \cdot u^{-1} \right] + \left[\grave{A}(V), \delta u \cdot u^{-1} \right] + \sum_{\alpha=0}^{3} \grave{A}_\alpha \delta V^\alpha \right\rangle \varpi_4$

The variational derivative is the distribution, acting on sections $\delta u \in \mathfrak{X}(P_U)$ and valued in $\mathbb{R}$:

$$\frac{\delta \ell_p}{\delta u}(\delta u) = \int_\Omega \left\langle Ad_u \widehat{Q}, \left[\frac{du}{dt} \cdot u^{-1} + \grave{A}(V), \delta u \cdot u^{-1} \right] + \grave{A}(\delta V) \right\rangle_H \varpi_4 \tag{53}$$

with
$\delta u = \sum_{a=1}^{16} \frac{\partial u}{\partial z^a} \cdot u^{-1} \delta z^a$;
$\delta V^\alpha = \sum_{a=1}^{16} \frac{\partial V^\alpha}{\partial z^a} \delta z^a = \sum_{j=1}^{3} \left(2 \frac{\sinh \mu_w \cosh \mu_w}{1 + 2(\cosh \mu_w)^2} \sum_{j=1}^{3} \frac{\delta W^j}{\mu_w} \right) P_j^\alpha$ the derivatives are taken with P constant.

8.1.2 Variational derivative with respect to the potential of the field

The variational derivative of L_p with respect to $\grave{A}_\alpha^a$ is :
$\frac{\delta \ell_p}{\delta \grave{A}_\alpha^a} = \int_{\Omega_3(0)} \left(\int_0^{t_0} \left\langle \widehat{Q}, \left(\frac{\partial T}{\partial \grave{A}_\alpha^a} - \frac{d}{dt} \frac{\partial T}{\partial \frac{d \grave{A}_\alpha^a}{dt}} \right) \right\rangle_H c dt \right) \varpi_3$
$T = \left\langle \widehat{Q}, \left(u^{-1} \cdot \frac{du}{dt} + Ad_{u^{-1}} \grave{A}(V) \right) \right\rangle_H = \langle Q, u^{-1} \cdot \frac{du}{dt} \rangle + \left\langle Q, \left(Ad_{u^{-1}} \grave{A}(V) \right) \right\rangle$
$\left\langle Q, \left(Ad_{u^{-1}} \grave{A}(V) \right) \right\rangle_H = \left\langle \sum_{b=1}^{16} Q_b \kappa_b, Ad_{u^{-1}} \left(\sum_{\beta=0}^{3} \grave{A}_\beta V^\beta \right) \right\rangle_H$
$= \left\langle \sum_{b=1}^{16} Q_b \kappa_b, \sum_{c,d=1}^{16} [Ad_{u^{-1}}]_d^c \left(\sum_{\beta=0}^{3} \grave{A}_\beta^d V^\beta \right) \kappa_c \right\rangle_H$
$= \sum_{\beta=0}^{3} V^\beta \sum_{b,c,d=1}^{16} \eta_{bb} [Ad_{u^{-1}}]_d^c Q_b \left\langle \kappa_b, \grave{A}_\beta^d \kappa_c \right\rangle_H = \sum_{\beta=0}^{3} V^\beta \sum_{b,c,d=1}^{16} \eta_{bb} [Ad_{u^{-1}}]_d^b Q_b \grave{A}_\beta^d$
$\frac{\partial T}{\partial \grave{A}_\alpha^a} = \frac{\partial}{\partial \grave{A}_\alpha^a} \left(\sum_{\beta=0}^{3} V^\beta \sum_{b,c,d=1}^{16} [Ad_{u^{-1}}]_d^b Q_b \grave{A}_\beta^d \right) = V^\alpha \sum_{b=1}^{16} \eta_{bb} [Ad_{u^{-1}}]_a^b Q_b =$
$\langle V^\alpha Q, Ad_{u^{-1}} \kappa_a \rangle_H$
$\frac{\partial T}{\partial \frac{d \grave{A}_\alpha^a}{dt}} = 0$
$\frac{\delta \ell_p}{\delta \grave{A}_\alpha^a} = \int_{\Omega_3(0)} \left(\int_0^{t_0} \left\langle V^\alpha \widehat{Q}, Ad_{u^{-1}} \kappa_a \right\rangle_H c dt \right) \varpi_3$

We consider a variation $\delta \grave{A}=\sum_{\alpha=0}^{3}\sum_{a=1}^{16}\delta \grave{A}_{\alpha}^{a}d\xi^{\beta}\otimes\kappa_{a}\in\Lambda_{1c\infty}\left(TM;T_{1}U\right)$ defined over a compact support $\omega\subset\Omega$.

$\sum_{\alpha=0}^{3}\sum_{a=1}^{16}V^{\alpha}\sum_{b=1}^{16}\eta_{bb}\left[Ad_{u^{-1}}\right]_{a}^{b}\widehat{Q}_{b}\delta\grave{A}_{\alpha}^{a}=\sum_{b=1}^{16}\eta_{bb}\widehat{Q}_{b}\sum_{a=1}^{16}\left[Ad_{u^{-1}}\right]_{a}^{b}\left(\sum_{\alpha=0}^{3}\delta\grave{A}_{\alpha}^{a}V^{\alpha}\right)$

$\frac{\delta\ell_{p}}{\delta\grave{A}}\left(\delta\grave{A}\right)=\int_{\Omega_{3}(0)}\left(\int_{0}^{t_{0}}\sum_{\alpha=0}^{3}V^{\alpha}\left\langle\widehat{Q},Ad_{u^{-1}}\delta\grave{A}_{a}\right\rangle cdt\right)\varpi_{3}$

$=\int_{\Omega_{3}(0)}\left(\int_{0}^{t_{0}}\left\langle\widehat{Q},Ad_{u^{-1}}\left(\delta\grave{A}\left(V\right)\right)\right\rangle_{H}cdt\right)\varpi_{3}=\int_{\Omega_{3}(0)}\left(\int_{0}^{t_{0}}\left\langle Ad_{u}\widehat{Q},\delta\grave{A}\left(V\right)\right\rangle_{H}cdt\right)\varpi_{3}$

The quantity

$$J=Ad_{u}\widehat{Q}\otimes V\in\mathfrak{X}\left(T_{1}U\otimes TM\right)\tag{54}$$

is the current, common to all particles of the same type. It is built from the sections $\mathbf{u}\in\mathfrak{X}\left(P_{U}\right),\widehat{Q}\in\mathfrak{X}\left(P_{U}\left[T_{1}U,Ad\right]\right)$ and $V_{k}\in\mathfrak{X}\left(TM\right)$. This is a section $J\in\mathfrak{X}\left(T_{1}U\otimes TM\right)$ which is symmetric with respect to the vector field $\mathbf{V}_{k}$:

$\pounds_{\mathbf{V}_{k}}J=\pounds_{\mathbf{V}_{k}}\left(Ad_{u}\widehat{Q}\right)\otimes V_{k}+Ad_{u}\widehat{Q}\otimes\pounds_{V_{k}}V_{k}=0\Rightarrow J\left(\Phi_{V_{k}}\left(\tau,m\right)\right)=\Phi_{\mathbf{V}_{k}}\left(\tau,J\left(m\right)\right).$

It represents the variation of energy with respect to the potential.

The variational derivative with respect to $\grave{A}$ is the distribution, acting on sections $\delta\grave{A}\in\Lambda_{1}\left(TM;T_{1}U\right)$ and valued in $\mathbb{R}$:

$$\frac{\delta\ell_{p}}{\delta\grave{A}}\left(\delta\grave{A}\right)=\int_{\Omega}\sum_{\alpha=0}^{3}\left\langle J^{\alpha},\delta\grave{A}_{\alpha}\right\rangle_{H}\varpi_{4}\tag{55}$$

The distribution is equivariant in a global change of gauge, $\delta\grave{A}$ change with Ad and so does J.

8.1.3 Variational derivative with respect to the metric

The variational derivative with respect to $P=\sum_{i,\alpha=0}^{3}P_{i}^{\alpha}\varepsilon^{i}\otimes\partial\xi_{\alpha}$ is

$T=u^{-1}\cdot\frac{du}{dt}+Ad_{u^{-1}}\sum_{\beta=0}^{3}\grave{A}_{\beta}\left(\sum_{j=0}^{3}V^{j}P_{j}^{\beta}\right)$

$\frac{\partial T}{\partial P_{i}^{\alpha}}=Ad_{u^{-1}}\grave{A}_{\alpha}\left(V^{i}\right)$

$\frac{\partial T}{\partial P}=Ad_{u^{-1}}\sum_{a=1}^{16}\sum_{\alpha=0}^{3}\left(\grave{A}_{\alpha}^{a}\kappa_{a}\right)\left\{c\varepsilon_{0}+\sum_{i=1}^{3}\left(2\frac{\sinh\mu_{w}\cosh\mu_{w}}{1+2(\cosh\mu_{w})^{2}}\frac{W^{i}}{\mu_{w}}\right)\varepsilon_{i}\right\}\otimes\partial\xi_{\alpha}$

so that with $\delta P=\sum_{i,\alpha=0}^{3}\delta P_{i}^{\alpha}\varepsilon^{i}\otimes\partial\xi_{\alpha}$:

$\frac{\partial T}{\partial P}\left(\delta P\right)=Ad_{u^{-1}}\sum_{a=1}^{16}\sum_{\beta,i=0}^{3}\left(\grave{A}_{\beta}^{a}V^{i}\kappa_{a}\right)\delta P_{i}^{\beta}$

$\frac{\delta\ell_{p}}{\delta P}\left(\delta P\right)=\int_{\Omega}\left\langle Ad_{u}\widehat{Q},\sum_{a=1}^{16}\sum_{\beta,i=0}^{3}\left(\grave{A}_{\beta}^{a}V^{i}\kappa_{a}\right)\delta P_{i}^{\beta}\right\rangle_{H}\varpi_{4}$

$\left\langle Ad_{u}\widehat{Q},\sum_{a=1}^{16}\sum_{\beta,i=0}^{3}\left(\grave{A}_{\beta}^{a}V^{i}\kappa_{a}\right)\delta P_{i}^{\beta}\right\rangle=\sum_{a=1}^{16}\sum_{\beta,i=0}^{3}\eta_{aa}\left[Ad_{u}\widehat{Q}\right]^{a}V^{i}\grave{A}_{\beta}^{a}\delta P_{i}^{\beta}$

$=\sum_{a=1}^{16}\sum_{\beta,i=0}^{3}\eta_{aa}\left[Ad_{u}\widehat{Q}\right]^{a}\left(\sum_{\gamma=0}^{3}V^{\gamma}P_{\gamma}^{\prime i}\right)\grave{A}_{\beta}^{a}\delta P_{i}^{\beta}$

$=\sum_{a=1}^{16}\sum_{\beta,i,\gamma=0}^{3}\eta_{aa}\left[Ad_{u}\widehat{Q}\right]^{a}V^{\gamma}\grave{A}_{\beta}^{a}P_{\gamma}^{\prime i}\delta P_{i}^{\beta}$

$=\sum_{a=1}^{16}\sum_{\beta,\gamma=0}^{3}\eta_{aa}J^{a\gamma}\grave{A}_{\beta}^{a}\sum_{i=0}^{3}P_{\gamma}^{\prime i}\delta P_{i}^{\beta}$

$=\sum_{a=1}^{16}\sum_{\gamma=0}^{3}\eta_{aa}J^{a\gamma}\sum_{i,\beta=0}^{3}P_{\gamma}^{\prime i}\grave{A}_{\beta}^{a}\delta P_{i}^{\beta}=\left\langle J^{\gamma},\grave{A}_{\beta}\right\rangle_{H}\sum_{i,\beta=0}^{3}P_{\gamma}^{\prime i}\delta P_{i}^{\beta}$

The variational derivative with respect to P_i^α is the distribution which acts on sections $\delta P = \sum_{i,\alpha=0}^{3} \delta P_i^\alpha \varepsilon^i \otimes \partial\xi_\alpha$, valued in $\mathbb{R}$:

$$\frac{\delta \ell_p}{\delta P}(\delta P) = \int_\Omega \sum_{\alpha=0}^{3} \left\langle J^\alpha, \grave{A}_\beta \right\rangle_H \sum_{i,\beta=0}^{3} P_\alpha^{i\prime} \delta P_i^\beta \varpi_4 \tag{56}$$

8.2 Variational derivatives for the field

8.2.1 Variational derivative with respect to the potential

The variational derivative with respect to $\grave{A} = \sum_{\alpha=0}^{3} \sum_{a=1}^{16} \grave{A}_\alpha^a \kappa_a \otimes d\xi^\alpha \in \Lambda_1(TM; T_1U)$ is :

$\frac{\delta}{\delta \grave{A}} \left(\int_\Omega \langle \mathcal{F}, \mathcal{F} \rangle \varpi_4 \right) =$

$\int_\Omega \sum_{\alpha=0}^{3} \sum_{a=1}^{16} \left(\frac{\partial}{\partial \grave{A}_\alpha^a} \left(\langle \mathcal{F}, \mathcal{F} \rangle \sqrt{|\det g|} \right) - \sum_{\beta=0}^{3} \frac{d}{d\xi^\beta} \frac{\partial}{\partial \partial_\beta \grave{A}_\alpha^a} \left(\langle \mathcal{F}, \mathcal{F} \rangle \sqrt{|\det g|} \right) \right) d\xi$

with $d\xi = d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

It is more convenient to take $\langle \mathcal{F}, \mathcal{F} \rangle = \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \sum_{\lambda\mu} \mathcal{F}^{b\lambda\mu} \mathcal{F}_{\lambda\mu}^b$

We compute successively the 2 derivatives.

i) $\frac{\partial}{\partial \grave{A}_\alpha^a} \langle \mathcal{F}, \mathcal{F} \rangle = \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \sum_{\lambda\mu} \left(\left(\frac{\partial}{\partial \grave{A}_\alpha^a} \mathcal{F}^{b\lambda\mu} \right) \mathcal{F}_{\lambda\mu}^b + \mathcal{F}^{b\lambda\mu} \left(\frac{\partial}{\partial \grave{A}_\alpha^a} \mathcal{F}_{\lambda\mu}^b \right) \right)$

$\frac{\partial}{\partial \grave{A}_\alpha^a} \mathcal{F}^{b\lambda\mu} = \frac{\partial}{\partial \grave{A}_\alpha^a} \sum_{\eta\zeta} g^{\lambda\zeta} g^{\eta\mu} \mathcal{F}_{\zeta\eta}^b = \frac{\partial}{\partial \grave{A}_\alpha^a} \sum_{\eta\zeta} g^{\lambda\zeta} g^{\eta\mu} \left(\partial_\zeta \grave{A}_\eta - \partial_\eta \grave{A}_\zeta + \left[\grave{A}_\zeta, \grave{A}_\eta \right]^b \right)$

With the structure coefficients $C_{bc}^a = -C_{cb}^a : \left[\grave{A}_\zeta, \grave{A}_\eta \right]^b = \sum_{c,d=1}^{16} C_{cd}^b \grave{A}_\zeta^c \grave{A}_\eta^d$

$\frac{\partial}{\partial \grave{A}_\alpha^a} \mathcal{F}^{b\lambda\mu} = \sum_{\eta\zeta} g^{\lambda\zeta} g^{\eta\mu} \frac{\partial}{\partial \grave{A}_\alpha^a} \sum_{c,d=1}^{16} C_{cd}^b \grave{A}_\zeta^c \grave{A}_\eta^d$

$= \sum_{\eta\zeta} \sum_{c,d=1}^{16} C_{ad}^b g^{\lambda\alpha} g^{\eta\mu} \grave{A}_\eta^d + \sum_{\eta\zeta} \sum_{c,d=1}^{16} C_{ca}^b g^{\lambda\zeta} g^{\alpha\mu} \grave{A}_\zeta^c = \sum_{\eta=0}^{3} \sum_{c=1}^{16} C_{ac}^b g^{\lambda\alpha} g^{\eta\mu} \grave{A}_\eta^c + \sum_{\eta=0}^{3} \sum_{c=1}^{16} C_{ca}^b g^{\lambda\eta} g^{\alpha\mu} \grave{A}_\eta^c$

$\frac{\partial}{\partial \grave{A}_\alpha^a} \mathcal{F}^{b\lambda\mu} = \sum_{\eta=0}^{3} \sum_{c=1}^{16} C_{ac}^b \left(g^{\lambda\alpha} g^{\eta\mu} \grave{A}_\eta^c - g^{\lambda\eta} g^{\alpha\mu} \grave{A}_\eta^c \right)$

$\frac{\partial}{\partial \grave{A}_\alpha^a} \mathcal{F}_{\lambda\mu}^b = \frac{\partial}{\partial \grave{A}_\alpha^a} \left(\partial_\lambda \grave{A}_\mu - \partial_\mu \grave{A}_\lambda + \sum_{c,d=1}^{16} C_{cd}^b \grave{A}_\lambda^c \grave{A}_\mu^d \right)$

$= \sum_{c,d=1}^{16} C_{ad}^b \delta_{\lambda\alpha} \grave{A}_\mu^d + \sum_{c,d=1}^{16} C_{ca}^b \grave{A}_\lambda^c \delta_{\mu\alpha} = \sum_{c=1}^{16} C_{ac}^b \left(\delta_{\lambda\alpha} \grave{A}_\mu^c - \grave{A}_\lambda^c \delta_{\mu\alpha} \right)$

$\frac{\partial}{\partial \grave{A}_\alpha^a} \langle \mathcal{F}, \mathcal{F} \rangle = \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \sum_{\lambda\mu} \{ \left(\sum_{\eta=0}^{3} \sum_{c=1}^{16} C_{ac}^b \left(g^{\lambda\alpha} g^{\eta\mu} \grave{A}_\eta^c - g^{\lambda\eta} g^{\alpha\mu} \grave{A}_\eta^c \right) \right) \mathcal{F}_{\lambda\mu}^b$

$+ \mathcal{F}^{b\lambda\mu} \left(\sum_{c=1}^{16} C_{ac}^b \left(\delta_{\lambda\alpha} \grave{A}_\mu^c - \grave{A}_\lambda^c \delta_{\mu\alpha} \right) \right) \}$

$= \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \{ \left(\sum_{\eta=0}^{3} \sum_{c=1}^{16} C_{ac}^b \sum_{\lambda\mu} \left(\mathcal{F}_{\lambda\mu}^b g^{\lambda\alpha} g^{\eta\mu} \grave{A}_\eta^c - \mathcal{F}_{\lambda\mu}^b g^{\lambda\eta} g^{\alpha\mu} \grave{A}_\eta^c \right) \right)$

$+ \left(\sum_{c=1}^{16} C_{ac}^b \sum_{\lambda\mu} \left(\mathcal{F}^{b\lambda\mu} \delta_{\lambda\alpha} \grave{A}_\mu^c - \grave{A}_\lambda^c \mathcal{F}^{b\lambda\mu} \delta_{\mu\alpha} \right) \right) \}$

$= \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \left(\left(\sum_{\eta=0}^{3} \sum_{c=1}^{16} C_{ac}^b \left(\mathcal{F}^{b\alpha\eta} \grave{A}_\eta^c - \mathcal{F}^{b\eta\alpha} \grave{A}_\eta^c \right) \right) + \left(\sum_{c=1}^{16} C_{ac}^b \left(\mathcal{F}^{b\alpha\mu} \grave{A}_\mu^c - \grave{A}_\lambda^c \mathcal{F}^{b\lambda\alpha} \right) \right) \right)$

$= \sum_{b=1}^{16} \eta_{bb} \left(\left(\sum_{\lambda=0}^{3} \sum_{c=1}^{16} C_{ac}^b \mathcal{F}^{b\alpha\lambda} \grave{A}_\lambda^c \right) + \left(\sum_{\lambda=0}^{3} \sum_{c=1}^{16} C_{ac}^b \mathcal{F}^{b\alpha\lambda} \grave{A}_\lambda^c \right) \right)$

$= 2 \sum_{\lambda=0}^{3} \sum_{b=1}^{16} \eta_{bb} \mathcal{F}^{b\alpha\lambda} \left[\kappa_a, \grave{A}_\lambda \right]^b = 2 \sum_{\lambda=0}^{3} \left\langle \mathcal{F}^{\alpha\lambda}, \left[\kappa_a, \grave{A}_\lambda \right] \right\rangle_H$

$= -2 \sum_{\lambda=0}^{3} \left\langle \mathcal{F}^{\alpha\lambda}, \left[\grave{A}_\lambda, \kappa_a \right] \right\rangle_H = -2 \sum_{\lambda=0}^{3} \left\langle \left[\mathcal{F}^{\alpha\lambda}, \grave{A}_\lambda \right], \kappa_a \right\rangle_H$

with $\langle X, [Z,Y] \rangle_H = \langle [X,Z], Y \rangle_H$

$\frac{\partial}{\partial \grave{A}^a_\alpha} \langle \mathcal{F}, \mathcal{F} \rangle = 2\sum_{\lambda=0}^3 \left\langle \left[\grave{A}_\lambda, \mathcal{F}^{\alpha\lambda}\right], \kappa_a \right\rangle_H = 2\sum_{\lambda=0}^3 \eta_{aa} \left[\grave{A}_\lambda, \mathcal{F}^{\alpha\lambda}\right]^a$

The derivative of the function $\langle \mathcal{F}, \mathcal{F} \rangle$ with respect to the component of a form is a vector valued in the Lie algebra (Th.Physics 6.1.2).

The quantity

$$\phi(m) = 2\sum_{a=1}^{16} \sum_{\alpha,\beta=0}^{3} \left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}\right]^a \kappa_a \otimes \partial\xi_\alpha \in T_1U \otimes TM \qquad (57)$$

is the current associated to the field. It is defined everywhere on Ω.

The first part of the variational derivative is the distribution, valued in $\mathbb{R}$:

$\int_\Omega \sum_{\alpha=0}^3 \sum_{a=1}^{16} \left(\frac{\partial}{\partial \grave{A}^a_\alpha} \left(\langle \mathcal{F}, \mathcal{F} \rangle \sqrt{|\det g|}\right)\right) \left(\delta \grave{A}^a_\alpha\right) d\xi$

$= 2\int_\Omega \sum_{\alpha=0}^3 \left\langle \sum_{\beta=0}^3 \left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}\right], \delta \grave{A}_\alpha \right\rangle_H \varpi_4 = \int_\Omega \sum_{\alpha=0}^3 \left\langle \phi^\alpha, \delta \grave{A}_\alpha \right\rangle_H \varpi_4$

It is equivariant in a change of gauge, as well as the current ϕ which changes with Ad.

ii) Second part :

$\frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} (\langle \mathcal{F}, \mathcal{F} \rangle) = \frac{1}{2}\sum_{b=1}^{16} \eta_{bb} \sum_{\lambda\mu} \left(\left(\frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \mathcal{F}^{b\lambda\mu}\right) \mathcal{F}^b_{\lambda\mu} + \mathcal{F}^{b\lambda\mu} \left(\frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \mathcal{F}^b_{\lambda\mu}\right)\right)$

$\frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \mathcal{F}^{b\lambda\mu} = \frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \sum_{\eta\zeta} g^{\lambda\zeta} g^{\eta\mu} \left(\partial_\zeta \grave{A}^b_\eta - \partial_\eta \grave{A}^b_\zeta + \left[\grave{A}_\zeta, \grave{A}_\eta\right]^b\right) = \delta_{ab} \left(g^{\lambda\beta} g^{\alpha\mu} - g^{\lambda\alpha} g^{\beta\mu}\right)$

$\frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \mathcal{F}^b_{\lambda\mu} = \frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \left(\partial_\lambda \grave{A}^b_\mu - \partial_\mu \grave{A}^b_\lambda + \sum_{c,d=1}^{16} C^b_{cd} \grave{A}^c_\lambda \grave{A}^d_\mu\right) = \delta_{ab} \left(\delta_{\lambda\beta}\delta_{\mu\alpha} - \delta_{\mu\beta}\delta_{\lambda\alpha}\right)$

$\frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} (\langle \mathcal{F}, \mathcal{F} \rangle) = \frac{1}{2}\sum_{b=1}^{16} \eta_{bb} \sum_{\lambda\mu} \left(\delta_{ab} \left(g^{\lambda\beta} g^{\alpha\mu} - g^{\lambda\alpha} g^{\beta\mu}\right) \mathcal{F}^b_{\lambda\mu} + \mathcal{F}^{b\lambda\mu} \left(\delta_{ab} \left(\delta_{\lambda\beta}\delta_{\mu\alpha} - \delta_{\mu\beta}\delta_{\lambda\alpha}\right)\right)\right)$

$= \frac{1}{2}\eta_{aa} \sum_{\lambda\mu} \left(\left(g^{\lambda\beta} g^{\alpha\mu} \mathcal{F}^a_{\lambda\mu} - g^{\lambda\alpha} g^{\beta\mu} \mathcal{F}^a_{\lambda\mu}\right) + \left(\left(\delta_{\lambda\beta}\delta_{\mu\alpha} \mathcal{F}^{a\lambda\mu} - \delta_{\mu\beta}\delta_{\lambda\alpha} \mathcal{F}^{a\lambda\mu}\right)\right)\right)$

$= \frac{1}{2}\left(\eta_{aa}\left(\left(\mathcal{F}^{a\beta\alpha} - \mathcal{F}^{a\alpha\beta}\right) + \left(\mathcal{F}^{a\beta\alpha} - \mathcal{F}^{a\alpha\beta}\right)\right)\right) = 2\eta_{aa} \mathcal{F}^{a\beta\alpha} = 2\left\langle \mathcal{F}^{\beta\alpha}, \kappa_a \right\rangle_H$

$\sum_{\beta=0}^3 \frac{d}{d\xi^\beta} \frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \left(\langle \mathcal{F}, \mathcal{F} \rangle \sqrt{|\det g|}\right) = 2\sum_{\beta=0}^3 \frac{d}{d\xi^\beta} \left(\eta_{aa} \mathcal{F}^{a\beta\alpha} \sqrt{|\det g|}\right)$

The second part of the variational derivative is the distribution, acting on smooth sections $\delta \grave{A}$ and valued in $\mathbb{R}$:

$\delta \grave{A} \to \int_\Omega \sum_{\alpha=0}^3 \sum_{a=1}^{16} \sum_{\beta=0}^3 \left(\frac{d}{d\xi^\beta} \frac{\partial}{\partial \partial_\beta \grave{A}^a_\alpha} \left(\langle \mathcal{F}, \mathcal{F} \rangle \sqrt{|\det g|}\right)\right) \left(\delta \grave{A}^a_\alpha\right) d\xi$

$= 2\int_\Omega \sum_{\alpha=0}^3 \sum_{a=1}^{16} \eta_{aa} \sum_{\beta=0}^3 \left(\delta \grave{A}^a_\alpha\right) \partial_\beta \left(\eta_{aa} \mathcal{F}^{a\beta\alpha} \sqrt{|\det g|}\right) d\xi$

The differential of $*\mathcal{F}^a$ with respect to the coordinates in M is the 3 form :

$d(*\mathcal{F}^a) = \sum_{\alpha=0}^3 (-1)^\alpha \sum_{\beta=0}^3 \partial_\beta \left(\mathcal{F}^{a\alpha\beta} \sqrt{|\det g|}\right) d\xi^0 \wedge ... \wedge \widehat{d\xi^\alpha} \wedge ... d\xi^3$

$\left(\sum_{\beta=0}^3 \delta \grave{A}^a_\beta d\xi^\beta\right) \wedge d(*\mathcal{F}^a) = \sum_{\alpha,\beta=0}^3 \delta \grave{A}^a_\alpha \partial_\beta \left(\mathcal{F}^{a\alpha\beta} \sqrt{|\det g|}\right) d\xi$

$= -\sum_{\alpha,\beta=0}^3 \delta \grave{A}^a_\alpha \partial_\beta \left(\mathcal{F}^{a\beta\alpha} \sqrt{|\det g|}\right) d\xi$

The codifferential D is the operator acting on $r+1$ form on M :

$D : \Lambda_{r+1}(TM; \mathbb{R}) \to \Lambda_r(TM; \mathbb{R}) :: D\lambda_{r+1} = *d * \lambda_r$

$D\mathcal{F}^a = *d * \mathcal{F}^a$ is a 1 form

$*D\mathcal{F}^a = **d * \mathcal{F}^a = -d * \mathcal{F}^a$ is a 3 form

$\delta \grave{A}^a \wedge d(*\mathcal{F}^a) = -\delta \grave{A}^a \wedge *D\mathcal{F}^a = -(-1)^{1\times 3} *D\mathcal{F}^a \wedge \delta \grave{A}^a = G_1\left(D\mathcal{F}^a, \delta \grave{A}^a\right) \varpi_4$

$\sum_{\alpha,\beta=0}^{3} \delta \grave{A}_{\alpha}^{a} \partial_{\beta} \left(\mathcal{F}^{a\alpha\beta} \sqrt{|\det g|} \right) d\xi = -G_1 \left(D\mathcal{F}^a, \delta \grave{A}^a \right) \varpi_4$
So
$\int_{\Omega} \sum_{\alpha=0}^{3} \sum_{a=1}^{16} \sum_{\beta=0}^{3} \left(\frac{d}{d\xi^\beta} \frac{\partial}{\partial \partial_\beta \grave{A}_\alpha^a} \left(\langle \mathcal{F}, \mathcal{F} \rangle \sqrt{|\det g|} \right) \right) \left(\delta \grave{A}_\alpha^a \right) d\xi$
$= -2 \int_\Omega \sum_{a=1}^{16} \eta_{aa} G_1 \left(D\mathcal{F}^a, \delta \grave{A}^a \right) \varpi_4 = -2 \int_\Omega \left\langle D\mathcal{F}, \delta \grave{A} \right\rangle \varpi_4$
Whenever $\delta \grave{A}$ has an exterior differential $d\delta \grave{A}$ we have :
$G_1 \left(D\mathcal{F}^a, \delta \grave{A}^a \right) = G_2 \left(\mathcal{F}^a, d \left(\delta \grave{A}^a \right) \right)$
$\int_\Omega \sum_{a=1}^{16} \eta_{aa} G_1 \left(D\mathcal{F}^a, \delta \grave{A}^a \right) \varpi_4 = \int_\Omega \sum_{a=1}^{16} \eta_{aa} G_2 \left(\mathcal{F}^a, d \left(\delta \grave{A}^a \right) \right) \varpi_4 = \int_\Omega \left\langle \mathcal{F}, d\delta \grave{A} \right\rangle \varpi_4$
because the codifferential is the adjoint of the exterior differential (Maths.2400).

The variational derivative with respect to $\widehat{\grave{A}}$ is then the distribution, acting on smooth sections $\delta \grave{A} \in \Lambda_{1c\infty} (TM; T_1U)$ defined over a compact support and valued in $\mathbb{R}$:

$$\frac{\delta}{\delta \grave{A}} \left(\int_\Omega \langle \mathcal{F}, \mathcal{F} \rangle \varpi_4 \right) \left(\delta \grave{A} \right) = \int_\Omega \sum_{\alpha=0}^{3} \left(\left\langle \phi^\alpha, \delta \grave{A}_\alpha \right\rangle_H - 2 \left\langle \mathcal{F}, d\delta \grave{A} \right\rangle \right) \varpi_4 \qquad (58)$$

In the computations above :
- the expression of the first distribution holds when $\mathcal{F}$ is the result of a 1st jet extension $\mathcal{F} \left(j^1 \grave{A} \right)$.
- in the second distribution we have used the external differential of $\mathcal{F}$, and the expression assumes that $\mathcal{F}$ is computed from the partial derivatives $\partial_\alpha \grave{A}_\beta$. We will come back on this point below.

8.2.2 Variational derivative with respect to the metric

The metric is involved through $\sqrt{|\det g|}$ with $g_{\alpha\beta} = \sum_{j=0}^{3} \eta_{jj} P_\alpha'^j P_\beta'^j \Leftrightarrow g^{\alpha\beta} = \sum_{j=0}^{3} \eta_{jj} P_j^\alpha P_j^\beta$ and in $\langle \mathcal{F}, \mathcal{F} \rangle$ through $*\mathcal{F}$.
The variational derivative is :
$\frac{\delta}{\delta P} \left(\int_\Omega \langle \mathcal{F}, \mathcal{F} \rangle \varpi_4 \right) = \sum_{i,\alpha=0}^{3} \frac{\delta}{\delta P_i^\alpha} \left(\int_\Omega \langle \mathcal{F}, \mathcal{F} \rangle \varpi_4 \right) \varepsilon_i \otimes d\xi^\alpha$
$\frac{\delta}{\delta P_i^\alpha} \left(\int_\Omega \langle \mathcal{F}, \mathcal{F} \rangle \varpi_4 \right) = \int_\Omega \sum_{\lambda,\mu=0}^{3} \left(\frac{\partial}{\partial g^{\lambda\mu}} \left(\langle \mathcal{F}, \mathcal{F} \rangle \frac{1}{\sqrt{|\det g^{-1}|}} \right) \frac{\partial g^{\lambda\mu}}{\partial P_i^\alpha} \varepsilon_i \otimes d\xi^\alpha \right) d\xi$
$\frac{\partial}{\partial g^{\lambda\mu}} \left(\langle \mathcal{F}, \mathcal{F} \rangle \frac{1}{\sqrt{|\det g^{-1}|}} \right) = \left(\frac{\partial}{\partial g^{\lambda\mu}} \langle \mathcal{F}, \mathcal{F} \rangle \right) \sqrt{|\det g|} + \langle \mathcal{F}, \mathcal{F} \rangle \frac{\partial}{\partial g^{\lambda\mu}} \left(\frac{1}{\sqrt{|\det g^{-1}|}} \right)$
$\frac{\partial}{\partial g^{\lambda\mu}} \langle \mathcal{F}, \mathcal{F} \rangle = \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \frac{\partial}{\partial g^{\lambda\mu}} \left(\sum_{\xi\eta} \mathcal{F}^{b\xi\eta} \mathcal{F}_{\xi\eta}^b \right) = \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \left(\sum_{\xi\eta} \left(\frac{\partial}{\partial g^{\lambda\mu}} \mathcal{F}^{b\xi\eta} \right) \mathcal{F}_{\xi\eta}^b \right)$
$= \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \left(\sum_{\xi\eta} \left(\frac{\partial}{\partial g^{\lambda\mu}} \sum_{cd} g^{c\xi} g^{d\eta} \mathcal{F}_{cd}^b \right) \mathcal{F}_{\xi\eta}^b \right)$
$= \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \left(\sum_{c,d} g^{d\eta} \mathcal{F}_{\lambda d}^b \mathcal{F}_{\mu\eta}^b + g^{c\xi} \mathcal{F}_{c\lambda}^b \mathcal{F}_{\xi\mu}^b \right)$
$= \frac{1}{2} \sum_{b=1}^{16} \eta_{bb} \left(\sum_{c,d} g^{c\eta} \mathcal{F}_{\lambda c}^b \mathcal{F}_{\mu\eta}^b + g^{c\eta} \mathcal{F}_{c\lambda}^b \mathcal{F}_{\eta\mu}^b \right) = \sum_{b=1}^{16} \eta_{bb} \sum_{c\eta} g^{c\eta} \mathcal{F}_{\lambda c}^b \mathcal{F}_{\mu\eta}^b$
$= \sum_{c\eta} g^{c\eta} \langle \mathcal{F}_{\lambda c} \mathcal{F}_{\mu\eta} \rangle_H$
$\frac{\partial}{\partial g^{\lambda\mu}} \left(\frac{1}{\sqrt{|\det g^{-1}|}} \right) = \left(-\frac{1}{2} \left(|\det g^{-1}| \right)^{-3/2} \left(-\frac{\partial}{\partial g^{\lambda\mu}} \det g^{-1} \right) \right)$

$$\frac{\partial}{\partial g^{\lambda\mu}}\det g^{-1} = [g]_{\mu}^{\lambda}\det g^{-1} = g_{\lambda\mu}\det g^{-1}$$
$$\frac{\partial}{\partial g^{\lambda\mu}}\left(\frac{1}{\sqrt{|\det g^{-1}|}}\right) = \frac{1}{2}\left(g_{\lambda\mu}\right)\left(\left|\det g^{-1}\right|\right)^{-1/2} = \frac{1}{2}\left(g_{\lambda\mu}\right)\sqrt{|\det g|}$$
$$\frac{\partial}{\partial g^{\lambda\mu}}\left(\langle\mathcal{F},\mathcal{F}\rangle\frac{1}{\sqrt{|\det g^{-1}|}}\right) = \left(\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\lambda c}\mathcal{F}_{\mu\eta}\right\rangle_{H} + \langle\mathcal{F},\mathcal{F}\rangle\frac{1}{2}\left(g_{\lambda\mu}\right)\right)\sqrt{|\det g|}$$
$$\frac{\partial g^{\lambda\mu}}{\partial P_{i}^{\alpha}} = \frac{\partial}{\partial P_{i}^{\alpha}}\sum_{j=0}^{3}\eta_{jj}P_{j}^{\lambda}P_{j}^{\mu} = \sum_{j=0}^{3}\eta_{jj}\delta_{ij}\delta_{\lambda\alpha}P_{j}^{\mu}+\eta_{jj}\delta_{ij}P_{j}^{\lambda}\delta_{\mu\alpha} = \eta_{ii}\left(\delta_{\lambda\alpha}P_{i}^{\mu}+P_{i}^{\lambda}\delta_{\mu\alpha}\right)$$
$$\sum_{\lambda,\mu=0}^{3}\left(\frac{\partial}{\partial g^{\lambda\mu}}\left(\langle\mathcal{F},\mathcal{F}\rangle\frac{1}{\sqrt{|\det g^{-1}|}}\right)\frac{\partial g^{\lambda\mu}}{\partial P_{\alpha}^{i}}\right)$$
$$= \sum_{\lambda,\mu=0}^{3}\left\{\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\lambda c}^{b}\mathcal{F}_{\mu\eta}^{b}\right\rangle_{H} + \langle\mathcal{F},\mathcal{F}\rangle\frac{1}{2}\left(g_{\lambda\mu}\right)\right\}\left(\eta_{ii}\left(\delta_{\lambda\alpha}P_{i}^{\mu}+P_{i}^{\lambda}\delta_{\mu\alpha}\right)\right)\sqrt{|\det g|}$$
$$= \sum_{\lambda,\mu,c,j\eta=0}^{3}\{\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\lambda c},\mathcal{F}_{\mu\eta}\right\rangle_{H}\left(\eta_{ii}\left(\delta_{\lambda\alpha}P_{i}^{\mu}+P_{i}^{\lambda}\delta_{\mu\alpha}\right)\right)$$
$$+\langle\mathcal{F},\mathcal{F}\rangle\frac{1}{2}\left(g_{\lambda\mu}\right)\left(\eta_{ii}\left(\delta_{\lambda\alpha}P_{i}^{\mu}+P_{i}^{\lambda}\delta_{\mu\alpha}\right)\right)\}\sqrt{|\det g|}$$
$$= \sum_{\lambda,\mu,c,j\eta=0}^{3}\{\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\lambda c},\mathcal{F}_{\mu\eta}\right\rangle_{H}\delta_{\lambda\alpha}P_{i}^{\mu} + g^{c\eta}g^{c\eta}\left\langle\mathcal{F}_{\lambda c},\mathcal{F}_{\mu\eta}\right\rangle_{H}P_{i}^{\lambda}\delta_{\mu\alpha}$$
$$+\langle\mathcal{F},\mathcal{F}\rangle\left(\sum_{j=0}^{3}\frac{1}{2}\left(g_{\lambda\mu}\right)\delta_{\lambda\alpha}P_{i}^{\mu}+\frac{1}{2}\left(g_{\lambda\mu}\right)P_{i}^{\lambda}\delta_{\mu\alpha}\right)\}\eta_{ii}\sqrt{|\det g|}$$
$$= \sum_{\lambda,\mu,c,j\eta=0}^{3}\{\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\alpha c}\mathcal{F}_{\mu\eta}\right\rangle_{H}P_{i}^{\mu} + g^{c\eta}\left\langle\mathcal{F}_{\lambda c},\mathcal{F}_{\alpha\eta}\right\rangle_{H}P_{i}^{\lambda}$$
$$+\langle\mathcal{F},\mathcal{F}\rangle\left(\frac{1}{2}\left(g_{\alpha\mu}\right)P_{i}^{\mu}+\frac{1}{2}\left(g_{\lambda\alpha}\right)P_{i}^{\lambda}\right)\}\eta_{ii}\sqrt{|\det g|}$$
$$= \sum_{\lambda,\mu,c,j\eta=0}^{3}\{\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\alpha c}\mathcal{F}_{\lambda\eta}\right\rangle_{H}P_{i}^{\lambda} + g^{c\eta}\left\langle\mathcal{F}_{\lambda c},\mathcal{F}_{\alpha\eta}\right\rangle_{H}P_{i}^{\lambda}$$
$$+\langle\mathcal{F},\mathcal{F}\rangle\left(\frac{1}{2}\left(g_{\alpha\lambda}\right)P_{i}^{\lambda}+\frac{1}{2}\left(g_{\lambda\alpha}\right)P_{i}^{\lambda}\right)\}\eta_{ii}\sqrt{|\det g|}$$
$$= \sum_{\lambda,\mu,c,j\eta=0}^{3}\left\{\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\alpha c}\mathcal{F}_{\lambda\eta}\right\rangle_{H} + g^{\eta c}\left\langle\mathcal{F}_{\lambda\eta},\mathcal{F}_{\alpha c}\right\rangle_{H} + \langle\mathcal{F},\mathcal{F}\rangle g_{\alpha\lambda}\right\}\eta_{ii}P_{i}^{\lambda}\sqrt{|\det g|}$$
$$= \sum_{\lambda,\mu,c,j\eta=0}^{3}\left\{2\sum_{c\eta} g^{c\eta}\left\langle\mathcal{F}_{\alpha c}\mathcal{F}_{\lambda\eta}\right\rangle_{H} + \langle\mathcal{F},\mathcal{F}\rangle g_{\alpha\lambda}\right\}\eta_{ii}P_{i}^{\lambda}\sqrt{|\det g|}$$
$$= \left\{2\sum_{\lambda\mu} g^{\lambda\mu}\left\langle\mathcal{F}_{\alpha\lambda}\mathcal{F}_{\beta\mu}\right\rangle_{H} + \langle\mathcal{F},\mathcal{F}\rangle g_{\alpha\beta}\right\}\eta_{ii}P_{i}^{\beta}\sqrt{|\det g|}$$
$$\frac{\delta}{\delta P}\left(\int_{\Omega}\langle\mathcal{F},\mathcal{F}\rangle\varpi_{4}\right) = \int_{\Omega}\sum_{i\alpha\beta=0}^{3}\left\{\langle\mathcal{F},\mathcal{F}\rangle g_{\alpha\beta} + 2\sum_{\lambda\mu} g^{\lambda\mu}\left\langle\mathcal{F}_{\alpha\lambda}\mathcal{F}_{\beta\mu}\right\rangle_{H}\right\}\left(\eta_{ii}P_{i}^{\beta}\varepsilon_{i}\otimes d\xi^{\alpha}\right)\varpi_{4}$$
$$\sum_{\beta=0}^{3} g_{\alpha\beta}\eta_{ii}P_{i}^{\beta} = \sum_{\beta=0}^{3}\sum_{k=0}^{3}\eta_{kk}\left[P'\right]_{\alpha}^{k}\left[P'\right]_{\beta}^{k}\eta_{ii}P_{i}^{\beta} = \sum_{k=0}^{3}\left[P'\right]_{\alpha}^{k}\delta_{ik} = \left[P'\right]_{\alpha}^{i}$$
$$\sum_{\beta\mu i=0}^{3} g^{\lambda\mu}\mathcal{F}_{\beta\mu}\eta_{ii}P_{i}^{\beta}\delta P_{i}^{\alpha} = \sum_{\beta\gamma\mu\zeta i=0}^{3} g^{\lambda\mu}\mathcal{F}^{\gamma\zeta}g_{\gamma\beta}g_{\zeta\mu}\eta_{ii}P_{i}^{\beta}\delta P_{i}^{\alpha} = \sum_{\beta\gamma i=0}^{3}\mathcal{F}^{\gamma\lambda}g_{\gamma\beta}\eta_{ii}P_{i}^{\beta}\delta P_{i}^{\alpha}$$
$$= \sum_{\beta\gamma ik=0}^{3}\mathcal{F}^{\gamma\lambda}\eta_{kk}\left[P'\right]_{\gamma}^{k}\left[P'\right]_{\beta}^{k}\eta_{ii}P_{i}^{\beta}\delta P_{i}^{\alpha} = \sum_{\gamma i=0}^{3}\mathcal{F}^{\gamma\lambda}\eta_{ii}\left[P'\right]_{\gamma}^{i}\eta_{ii}\delta P_{i}^{\alpha} = \sum_{\gamma i=0}^{3}\mathcal{F}^{\gamma\lambda}\left[P'\right]_{\gamma}^{i}\delta P_{i}^{\alpha}$$
$$\sum_{i\alpha\beta=0}^{3}\left\{\langle\mathcal{F},\mathcal{F}\rangle g_{\alpha\beta} + \sum_{\lambda\mu=0}^{3}2g^{\lambda\mu}\left\langle\mathcal{F}_{\alpha\lambda},\mathcal{F}_{\beta\mu}\right\rangle_{H}\right\}\left(\eta_{ii}P_{i}^{\beta}\delta P_{i}^{\alpha}\right)$$
$$= \sum_{i\alpha=0}^{3}\langle\mathcal{F},\mathcal{F}\rangle\left[P'\right]_{\alpha}^{i}\delta P_{i}^{\alpha} + 2\sum_{\alpha\lambda\gamma i=0}^{3}\left\langle\mathcal{F}_{\alpha\lambda},\mathcal{F}^{\gamma\lambda}\right\rangle_{H}\left[P'\right]_{\gamma}^{i}\delta P_{i}^{\alpha}$$
$$= \sum_{i\alpha\gamma=0}^{3}\langle\mathcal{F},\mathcal{F}\rangle\delta_{\alpha\gamma}\left[P'\right]_{\gamma}^{i}\delta P_{i}^{\alpha} + 2\sum_{\alpha\lambda\gamma i=0}^{3}\left\langle\mathcal{F}_{\alpha\lambda},\mathcal{F}^{\gamma\lambda}\right\rangle_{H}\left[P'\right]_{\gamma}^{i}\delta P_{i}^{\alpha}$$
$$= \sum_{\alpha\gamma=0}^{3}\left(\langle\mathcal{F},\mathcal{F}\rangle\delta_{\alpha\gamma} + 2\sum_{\lambda=0}^{3}\left\langle\mathcal{F}_{\alpha\lambda},\mathcal{F}^{\gamma\lambda}\right\rangle_{H}\right)\left(\sum_{i=0}^{3}\left[P'\right]_{\gamma}^{i}\delta P_{i}^{\alpha}\right)$$
$$\int_{\Omega}\sum_{i\alpha\beta=0}^{3}\left\{\langle\mathcal{F},\mathcal{F}\rangle g_{\alpha\beta} + \sum_{\lambda\mu=0}^{3}2g^{\lambda\mu}\left\langle\mathcal{F}_{\alpha\lambda},\mathcal{F}_{\beta\mu}\right\rangle_{H}\right\}\left(\eta_{ii}P_{i}^{\beta}\delta P_{i}^{\alpha}\right)\varpi_{4} =$$
$$\int_{\Omega}\sum_{\alpha\gamma=0}^{3}\left(\langle\mathcal{F},\mathcal{F}\rangle\delta_{\alpha\gamma} + 2\sum_{\lambda=0}^{3}\left\langle\mathcal{F}_{\alpha\lambda},\mathcal{F}^{\gamma\lambda}\right\rangle_{H}\right)\left(\sum_{i=0}^{3}\left[P'\right]_{\gamma}^{i}\delta P_{i}^{\alpha}\right)\varpi_{4}$$

The variational derivative with respect to P_{i}^{α} is the distribution which acts on sections $\delta P = \sum_{i,\alpha=0}^{3}\delta P_{i}^{\alpha}\varepsilon^{i}\otimes\partial\xi_{\alpha}$, valued in $\mathbb{R}$:

$$\frac{\delta}{\delta P}\left(\int_{\Omega}\langle\mathcal{F},\mathcal{F}\rangle\varpi_4\right)(\delta P)=\int_{\Omega}\sum_{\alpha\beta=0}^{3}\left(\langle\mathcal{F},\mathcal{F}\rangle\delta_{\alpha\beta}+2\sum_{\lambda=0}^{3}\left\langle\mathcal{F}_{\alpha\lambda},\mathcal{F}^{\beta\lambda}\right\rangle_H\right)\left(\sum_{i=0}^{3}[P']^i_{\beta}\,\delta P^{\alpha}_i\right)\varpi_4 \quad (59)$$

In the vacuum the field does not interact, and thus does not exchange energy with the particles, but, as this variational derivative shows, it "interacts" with the metric, whose deformation carries the energy of the field.

9 Equilibrium in a continuous process

The conditions for an equilibrium are then given by the equations :

$\frac{\delta}{\delta\grave{A}}\left(\ell_F\left(\grave{A},P\right)+\sum_{j,\psi_0}\ell_P\left(\mathbf{u},\grave{A},P,q_j(0)\right)\right)=0$

$\frac{\delta}{\delta P}\left(\ell_F\left(\grave{A},P\right)+\sum_{j,\psi_0}\ell_P\left(\mathbf{u},\grave{A},P,q_j(0)\right)\right)=0$

$\frac{\delta}{\delta u}\left(\ell_F\left(\grave{A},P\right)+\sum_{j,\psi_0}\ell_P\left(\mathbf{u},\grave{A},P,q_j(0)\right)\right)=0$

9.1 Conditions for the state of the particles

The variable $\mathbf{u}_k$ appears only on the variational derivative

$\frac{\delta\ell_p}{\delta u}(\delta u)=\int_0^{t_0}\left\langle Ad_uQ,\left[\frac{du}{dt}\cdot u^{-1}+\grave{A}(V),\delta u\cdot u^{-1}\right]+\grave{A}(\delta V)\right\rangle_H|_{q(t)}cdt$

The conditions for an equilibrium, with a given value of the field, are then, for a section representing a family of particles :

$$a=1...16:\left\langle Ad_u\widehat{Q}_k,\left\{\left[\frac{du_k}{dt}\cdot u_k^{-1}+\grave{A}(V_k),\frac{\partial u_k}{\partial z_a}\cdot u_k^{-1}\right]+\grave{A}\left(\frac{dV_k}{dz_a}\right)\right\}\right\rangle_H|_m=0 \quad (60)$$

$\Leftrightarrow a=1...16:\sum_{a=1}^{16}\left\langle\widehat{Q}_k,\left[\nabla_{V_k}u_k,\frac{\partial u_k}{\partial z_a}\cdot u_k^{-1}\right]+Ad_{u_k^{-1}}\sum_{\alpha=0}^{3}\grave{A}_\alpha\frac{\partial V_k^\alpha}{\partial z^a}\right\rangle=0$

Which implies in a continuous motion :

$\left\langle Q,\left[u^{-1}\cdot\frac{du}{dt},u^{-1}\cdot\frac{du}{dt}\right]\right\rangle=0=\left\langle Q,\sum_{a=1}^{16}\frac{dz_a}{dt}\left[u^{-1}\cdot\frac{du}{dt},u^{-1}\cdot\frac{\partial u}{\partial z_a}\right]\right\rangle$

$=-\left\langle Q,\sum_{a=1}^{16}\left(\left[Ad_{u^{-1}}\grave{A}(V),\frac{dz_a}{dt}u^{-1}\cdot\frac{\partial u}{\partial z_a}\right]+Ad_{u^{-1}}\sum_{\alpha=0}^{3}\grave{A}_\alpha\frac{dV^\alpha}{dz_a}\frac{dz_a}{dt}\right)\right\rangle$

$$\left\langle Ad_{u_k}\widehat{Q}_k,\left[\grave{A}(V_k),\frac{du_k}{dt}\cdot u_k^{-1}\right]+\grave{A}\left(\frac{dV_k}{dt}\right)\right\rangle_H=0 \quad (61)$$

where $\frac{dV^\alpha}{dt}=\sum_{j=1}^{3}\frac{d}{dt}\left(2\frac{\sinh\mu_w\cosh\mu_w}{1+2(\cosh\mu_w)^2}\sum_{j=1}^{3}\frac{W^j}{\mu_w}\right)P_j^\alpha$ (the tetrad of the observer is constant), which can be seen as the equation of motion.

For the antiparticles $Q\to-Q$, $t\to-t$: the equation is the same, but $V_k=c\varepsilon_0+\overrightarrow{v}_k\to V_k^c=c\varepsilon_0-\overrightarrow{v}_k$. The spatial trajectory is opposite.

For a bonded particle $W=0$ the condition reads : $\left\langle\widehat{Q}_k,\left[\nabla_{V_k}u_k,\frac{\partial u_k}{\partial z_a}\cdot u_k^{-1}\right]\right\rangle_H|_m=0$. More generally, if the spatial speed $\overrightarrow{v}$ is such that $c\gg\|\overrightarrow{v}\|$:

$\left\langle \widehat{Q}_k, \left[\nabla_{V_k} u_k, \frac{\partial u_k}{\partial z_a} \cdot u_k^{-1}\right] + Ad_{u_k^{-1}} \sum_{\alpha=0}^3 \grave{A}_\alpha \frac{\partial V_k^\alpha}{\partial z^a} \right\rangle \simeq \left\langle \widehat{Q}_k, \left[\nabla_{V_k} u_k, \frac{\partial u_k}{\partial z_a} \cdot u_k^{-1}\right] \right\rangle$

then the conditions are met for geodesics : $\nabla_{V_k} \mathbf{u}_k = 0 = \pounds_{\chi H(V_k)} \mathbf{u}_k$. We retrieve the assumption of Einstein's Theory of gravitation (but here the connection involves all the fields).

9.2 Conditions for the field

9.2.1 Codifferential

Whenever $\mathcal{F}$ is differentiable we have the identity (equilibrium or not) :

$\forall \delta \grave{A} \in \Lambda_{1c,\infty}(TM; T_1U) : \left\langle D\mathcal{F}, \delta \grave{A} \right\rangle = \left\langle \mathcal{F}, d\delta \grave{A} \right\rangle$

so, if $\delta \grave{A}$ is any closed form : $\left\langle D\mathcal{F}, \delta \grave{A} \right\rangle = 0$ and we can state that :

$$D\mathcal{F} = 0 \tag{62}$$

$D\mathcal{F} = 0 = *d * \mathcal{F} \Leftrightarrow d * \mathcal{F} = 0$

Which gives the equations :

$$a = 1...16, \alpha = 0...3 : \sum_{\beta=0}^3 \partial_\beta \left(\mathcal{F}^{a\alpha\beta} \sqrt{|\det g|} \right) = 0 \tag{63}$$

The Hodge dual of $\mathcal{F}^a$ is a closed form for all components $a = 1...16$. By the Poincaré's lemna there is a, non unique, one form $K \in \Lambda_1(TM; T_1U)$ such that :

$$*\mathcal{F} = dK \Leftrightarrow \mathcal{F} = *dK \tag{64}$$

9.2.2 Potential

Accounting for the previous result, the condition reads :

$$\forall \delta \grave{A} \in \Lambda_{1c,\infty}(TM; T_1U) : \int_\Omega \sum_{\alpha=0}^3 \left\langle \phi^\alpha + \sum_{k=1}^p J_k^\alpha, \delta \grave{A}_\alpha \right\rangle_H \varpi_4 = 0 \tag{65}$$

9.3 Condition for the metric

The condition is :

$\forall \delta P = \sum_{i,\beta=0}^3 \delta P_i^\beta \partial\xi_\beta \otimes \varepsilon^i \in \mathfrak{X}_{c\infty}\left(\mathbb{R}^4 \otimes TM\right)$:

$\frac{\delta \ell_p}{\delta P}(\delta P) = \int_\Omega \sum_{\alpha=0}^3 \left\langle J^\alpha, \grave{A}_\beta \right\rangle_H \sum_{i,\beta=0}^3 P_\alpha^{\prime i} \delta P_i^\beta \varpi_4$

$\frac{\delta}{\delta P}\left(\int_\Omega \langle \mathcal{F}, \mathcal{F}\rangle \varpi_4\right)(\delta P) = \int_\Omega \sum_{\alpha\beta=0}^3 \left(\langle \mathcal{F}, \mathcal{F}\rangle \delta_{\alpha\beta} + 2\sum_{\lambda=0}^3 \left\langle \mathcal{F}_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \right\rangle_H \right) \left(\sum_{i=0}^3 [P']_\beta^i \delta P_i^\alpha \right) \varpi_4$

$\int_\Omega \sum_{\beta=0}^3 \left\langle J^\beta, \grave{A}_\alpha \right\rangle_H \left(\sum_{i,\alpha=0}^3 P_\beta^{\prime i} \delta P_i^\alpha \right) \varpi_4$

$+ \int_\Omega \sum_{\alpha\beta=0}^3 \left(\langle \mathcal{F}, \mathcal{F}\rangle \delta_{\alpha\beta} + 2\sum_{\lambda=0}^3 \left\langle \mathcal{F}_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \right\rangle_H \right) \left(\sum_{i=0}^3 [P']_\beta^i \delta P_i^\alpha \right) \varpi_4 = 0$

Let us define : $\sum_{i,\alpha=0}^{3} P_{\beta}^{i\prime} \grave{A}_{\alpha}^{a} \delta P_{i}^{\alpha} = \delta \grave{A}_{\beta}^{a}$

$\forall \beta : \int_{\Omega} \sum_{\beta=0}^{3} \left\langle J^{\beta}, \grave{A}_{\alpha} \right\rangle_{H} \left(\sum_{i,\beta=0}^{3} P_{\beta}^{i\prime} \delta P_{i}^{\alpha} \right) \varpi_4 = \int_{\Omega} \sum_{\beta=0}^{3} \left\langle J^{\beta}, \delta \grave{A}_{\beta} \right\rangle_{H} \varpi_4$

With the previous equation :

$\int_{\Omega} \sum_{\beta=0}^{3} \left\langle J^{\beta}, \delta \grave{A}_{\beta} \right\rangle_{H} \varpi_4 = -\int_{\Omega} \sum_{\gamma=0}^{3} \left\langle \phi^{\beta}, \delta \grave{A}_{\beta} \right\rangle_{H} \varpi_4 = -\int_{\Omega} \left(\sum_{\beta=0}^{3} \sum_{a=1}^{16} \eta_{aa} \phi^{a\beta} \delta \grave{A}_{\beta}^{a} \right) \varpi_4$

$= -\int_{\Omega} \left(\sum_{\beta=0}^{3} \sum_{a=1}^{16} \eta_{aa} \phi^{a\beta} \sum_{i,\alpha=0}^{3} P_{\beta}^{i\prime} \grave{A}_{\alpha}^{a} \delta P_{i}^{\alpha} \right) \varpi_4$

$= -\int_{\Omega} \left(\sum_{\alpha\beta=0}^{3} \sum_{a=1}^{16} \eta_{aa} \phi^{a\beta} \grave{A}_{\alpha}^{a} \left(\sum_{i=0}^{3} P_{\beta}^{i\prime} \delta P_{i}^{\alpha} \right) \right) \varpi_4$

$= -\int_{\Omega} \left(\sum_{\alpha\beta=0}^{3} \left\langle \phi^{\beta}, \grave{A}_{\alpha} \right\rangle \left(\sum_{i=0}^{3} [P']_{\beta}^{i} \delta P_{i}^{\alpha} \right) \right) \varpi_4$

The condition reads :

$\forall \delta P \in \mathfrak{X}_{c\infty} \left(TM^* \otimes \mathbb{R}^4 \right)$:

$\int_{\Omega} \sum_{\alpha\gamma=0}^{3} \left(\langle \mathcal{F}, \mathcal{F} \rangle \delta_{\alpha\beta} + 2 \sum_{\lambda=0}^{3} \langle \mathcal{F}_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \rangle_{H} - \left\langle \phi^{\beta}, \grave{A}_{\alpha} \right\rangle_{H} \right) \left(\sum_{i=0}^{3} [P']_{\beta}^{i} \delta P_{i}^{\alpha} \right) \varpi_4 = 0$

or equivalently that the distribution, acting on sections $\delta X = \sum_{\alpha,\beta=0}^{3} \delta X_{\beta}^{\alpha} d\xi^{\beta} \otimes \partial \xi_{\alpha} \in \mathfrak{X}_{c\infty} \left(\otimes_1^1 TM \right)$:

$\int_{\Omega} \sum_{\alpha\beta=0}^{3} \left(\langle \mathcal{F}, \mathcal{F} \rangle \delta_{\alpha\beta} + 2 \sum_{\lambda=0}^{3} \langle \mathcal{F}_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \rangle_{H} - \left\langle \phi^{\beta}, \grave{A}_{\alpha} \right\rangle_{H} \right) \left(\delta X_{\beta}^{\alpha} \right) \varpi_4 = 0$

$\phi(m) = 2 \sum_{a=1}^{16} \sum_{\beta=0}^{3} \left[\grave{A}_{\beta}, \mathcal{F}^{\alpha\beta} \right]^{a} \kappa_a \otimes \partial \xi_{\alpha} \in T_1 U \otimes TM.$

$\phi^{a\beta} = 2 \sum_{\lambda=0}^{3} \left\langle \left[\grave{A}_{\lambda}, \mathcal{F}^{\beta\lambda} \right], \kappa_a \right\rangle_{H}$

$\left\langle \phi^{\beta}, \grave{A}_{\alpha} \right\rangle_{H} = 2 \sum_{\lambda=0}^{3} \left\langle \left[\grave{A}_{\lambda}, \mathcal{F}^{\beta\lambda} \right], \grave{A}_{\alpha} \right\rangle_{H} = -2 \sum_{\lambda=0}^{3} \left\langle \left[\mathcal{F}^{\beta\lambda}, \grave{A}_{\lambda} \right], \grave{A}_{\alpha} \right\rangle_{H} = -2 \sum_{\lambda=0}^{3} \left\langle \mathcal{F}^{\beta\lambda}, \left[\grave{A}_{\lambda}, \grave{A}_{\alpha} \right] \right\rangle_{H}$

$= 2 \sum_{\lambda=0}^{3} \left\langle \mathcal{F}^{\beta\lambda}, \left[\grave{A}_{\alpha}, \grave{A}_{\lambda} \right] \right\rangle_{H} = 2 \sum_{\lambda=0}^{3} \left\langle \left[\grave{A}_{\alpha}, \grave{A}_{\lambda} \right], \mathcal{F}^{\beta\lambda} \right\rangle_{H}$

$2 \sum_{\lambda=0}^{3} \langle \mathcal{F}_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \rangle_{H} - \left\langle \phi^{\beta}, \grave{A}_{\alpha} \right\rangle_{H} = 2 \sum_{\lambda=0}^{3} \langle \mathcal{F}_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \rangle_{H} - \left\langle \left[\grave{A}_{\alpha}, \grave{A}_{\lambda} \right], \mathcal{F}^{\beta\lambda} \right\rangle_{H}$

$= 2 \sum_{\lambda=0}^{3} \left\langle \left[d\grave{A} \right]_{\alpha\lambda} + \left[\grave{A}_{\alpha}, \grave{A}_{\lambda} \right], \mathcal{F}^{\beta\lambda} \right\rangle_{H} - \left\langle \left[\grave{A}_{\alpha}, \grave{A}_{\lambda} \right], \mathcal{F}^{\beta\lambda} \right\rangle_{H} = 2 \sum_{\lambda=0}^{3} \left\langle \left[d\grave{A} \right]_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \right\rangle_{H}$

The distribution reads :

$$\forall \delta X \in \mathfrak{X}_{c\infty} \left(\otimes_1^1 TM \right) : \int_{\Omega} \sum_{\alpha\beta=0}^{3} \left(\langle \mathcal{F}, \mathcal{F} \rangle \delta_{\alpha\beta} + 2 \sum_{\lambda=0}^{3} \left\langle \left[d\grave{A} \right]_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \right\rangle_{H} \right) \left(\delta X_{\beta}^{\alpha} \right) \varpi_4 = 0 \tag{66}$$

and the condition at equilibrium is satisfied if :

$$\forall \alpha, \beta = 0...3 : \sum_{\alpha\beta=0}^{3} \left(\langle \mathcal{F}, \mathcal{F} \rangle \delta_{\alpha\beta} + 2 \sum_{\lambda=0}^{3} \left\langle \left[d\grave{A} \right]_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \right\rangle_{H} \right) = 0 \tag{67}$$

Computing, for all values of α, β, the quantity : $Z_{\alpha}^{\beta} = \sum_{\lambda=0}^{3} \left\langle \left[d\grave{A} \right]_{\alpha\lambda}, \mathcal{F}^{\beta\lambda} \right\rangle_{H}$ with the notation :

$\left[d\grave{A}_r\right] = \left[\ \left[d\grave{A}\right]_{32} \quad \left[d\grave{A}\right]_{13} \quad \left[d\grave{A}\right]_{21}\ \right]_{16\times 3} ; \left[d\grave{A}_w\right] = \left[\ \left[d\grave{A}\right]_{01} \quad \left[d\grave{A}\right]_{02} \quad \left[d\grave{A}\right]_{03}\ \right]_{16\times 3}$
$[\mathcal{F}^r] = \left[\ \mathcal{F}^{32} \quad \mathcal{F}^{13} \quad \mathcal{F}^{21}\ \right]_{16\times 3} ; [\mathcal{F}^w] = \left[\ \mathcal{F}^{01} \quad \mathcal{F}^{02} \quad \mathcal{F}^{03}\ \right]_{16\times 3}$
$\langle \mathcal{F}, \mathcal{F} \rangle = \sum_{\{\alpha\beta\}} \mathcal{F}^{\alpha\beta} \mathcal{F}_{\alpha\beta} = \sum_{p=1}^{3} \left\langle [\mathcal{F}_r]_p , [\mathcal{F}^r]_p \right\rangle + \left\langle [\mathcal{F}_w]_p , [\mathcal{F}^w]_p \right\rangle$
One gets :

$$\forall p, q = 1, 2, 3 :$$
$$\left\langle \left[d\grave{A}_r\right]_p , [\mathcal{F}^w]_q \right\rangle_H = \left\langle \left[d\grave{A}_r\right]_q , [\mathcal{F}^w]_p \right\rangle_H$$
$$\left\langle \left[d\grave{A}_r\right]_p , [\mathcal{F}^r]_q \right\rangle_H = \left\langle \left[d\grave{A}_w\right]_q , [\mathcal{F}^w]_p \right\rangle_H$$
$$\langle \mathcal{F}, \mathcal{F} \rangle = -2 \sum_{q=1}^{3} \left\langle \left[d\grave{A}_w\right]_q , [\mathcal{F}^w]_q \right\rangle_H = -2 \sum_{q=1}^{3} \left\langle \left[d\grave{A}_r\right]_q , [\mathcal{F}^r]_q \right\rangle_H$$

$\langle \mathcal{F}, \mathcal{F} \rangle = \frac{1}{2} \sum_{\alpha\beta} \left\langle \mathcal{F}_{\alpha\beta}, \mathcal{F}^{\alpha\beta} \right\rangle = \frac{1}{2} \sum_{\alpha\beta} \left\langle \left[d\grave{A}\right]_{\alpha\beta} , \mathcal{F}^{\alpha\beta} \right\rangle + \frac{1}{2} \sum_{\alpha\beta} \left\langle \left[\grave{A}_\alpha, \grave{A}_\beta\right] , \mathcal{F}^{\alpha\beta} \right\rangle$
$= \frac{1}{2} \left(2 \sum_{q=1}^{3} \left(\left\langle \left[d\grave{A}_w\right]_q , [\mathcal{F}^w]_q \right\rangle_H + \left\langle \left[d\grave{A}_r\right]_q , [\mathcal{F}^r]_q \right\rangle_H \right) \right) + \frac{1}{2} \sum_{\alpha\beta} \left\langle \grave{A}_\alpha, \left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}\right] \right\rangle$
$= \left\langle \left[d\grave{A}_w\right]_q , [\mathcal{F}^w]_q \right\rangle_H + \left\langle \left[d\grave{A}_r\right]_q , [\mathcal{F}^r]_q \right\rangle_H + \frac{1}{4} \sum_\alpha \left\langle \grave{A}_\alpha, \phi^\alpha \right\rangle$
$\langle \mathcal{F}, \mathcal{F} \rangle = -2 \sum_{q=1}^{3} \left\langle \left[d\grave{A}_w\right]_q , [\mathcal{F}^w]_q \right\rangle_H = -2 \sum_{q=1}^{3} \left\langle \left[d\grave{A}_r\right]_q , [\mathcal{F}^r]_q \right\rangle_H$
$\langle \mathcal{F}, \mathcal{F} \rangle = -\frac{1}{2} \langle \mathcal{F}, \mathcal{F} \rangle - \frac{1}{2} \langle \mathcal{F}, \mathcal{F} \rangle + \frac{1}{4} \sum_\alpha \left\langle \grave{A}_\alpha, \phi^\alpha \right\rangle$

$$\langle \mathcal{F}, \mathcal{F} \rangle = \frac{1}{8} \sum_{\beta=0}^{3} \left\langle \grave{A}_\beta, \phi^\beta \right\rangle_H$$

We can express these equations by involving the metric. Using :
$\mathcal{F}^{\alpha\beta} = \sum_{\lambda\mu=0}^{3} g^{\alpha\lambda} g^{\beta\mu} \mathcal{F}_{\lambda\mu}$
$*\mathcal{F}_r = -\left(\mathcal{F}^{01} d\xi^3 \wedge d\xi^2 + \mathcal{F}^{02} d\xi^1 \wedge d\xi^3 + \mathcal{F}^{03} d\xi^2 \wedge d\xi^1 \right) \sqrt{|\det g|}$
$*\mathcal{F}_w = -\left(\mathcal{F}^{32} d\xi^0 \wedge d\xi^1 + \mathcal{F}^{13} d\xi^0 \wedge d\xi^2 + \mathcal{F}^{21} d\xi^0 \wedge d\xi^3 \right) \sqrt{|\det g|}$
$[*\mathcal{F}_r] = [\mathcal{F}_w] [g_3]^{-1} \sqrt{\det g_3} = -[\mathcal{F}^w] \sqrt{\det g_3}$
$[*\mathcal{F}_w] = -[\mathcal{F}_r] [g_3] / \sqrt{\det g_3} = -[\mathcal{F}^r] \sqrt{\det g_3}$
$[\mathcal{F}^w] = -[\mathcal{F}_w] [g_3]^{-1}$
$[\mathcal{F}^r] = [\mathcal{F}_r] [g_3] / \det g_3$
$\left\langle \left[d\grave{A}_r\right]_p , [\mathcal{F}^w]_q \right\rangle_H = \left\langle \left[d\grave{A}_r\right]_q , [\mathcal{F}^w]_p \right\rangle_H$
$\left\langle \left[d\grave{A}_r\right]_p , -\left([\mathcal{F}_w] [g_3]^{-1} \right)_q \right\rangle_H = \left\langle \left[d\grave{A}_r\right]_q , -\left([\mathcal{F}_w] [g_3]^{-1} \right)_p \right\rangle_H$
$\left(\left[d\grave{A}_r\right]_p \right)^t [\eta] \left([\mathcal{F}_w] [g_3]^{-1} \right)_q = \left(\left[d\grave{A}_r\right]_q \right)^t [\eta] \left([\mathcal{F}_w] [g_3]^{-1} \right)_p$

$$\left(\left(\left[d\grave{A}_r\right]\right)^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right)_q^p = \left(\left(\left[d\grave{A}_r\right]\right)^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right)_p^q$$
$$\left(\left[d\grave{A}_r\right]\right)^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1} = \left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t [\eta]\left[d\grave{A}_r\right]$$
$$\left\langle \left[d\grave{A}_r\right]_p, \left[\mathcal{F}^r\right]_q \right\rangle_H = \left\langle \left[d\grave{A}_w\right]_q, \left[\mathcal{F}^w\right]_p \right\rangle_H$$
$$\left\langle \left[d\grave{A}_r\right]_p, \left(\left[\mathcal{F}_r\right]\left[g_3\right] / \det g_3\right)_q \right\rangle_H = \left\langle \left[d\grave{A}_w\right]_q, \left(-\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right)_p \right\rangle_H$$
$$\left(\left[d\grave{A}_r\right]_p\right)^t \left([\eta]\left[\mathcal{F}_r\right]\left[g_3\right]\right)_q / \det g_3 = -\left(\left[d\grave{A}_w\right]_q\right)^t [\eta]\left(\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right)_p$$
$$\left(\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_r\right]\left[g_3\right]\right)_q^p = -\left(\left[d\grave{A}_w\right]^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right)_p^q \det g_3$$
$$\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_r\right]\left[g_3\right] = -\left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t [\eta]\left[d\grave{A}_w\right] \det g_3$$
$$\langle \mathcal{F}, \mathcal{F} \rangle = -2\sum_{q=1}^3 \left\langle \left[d\grave{A}_w\right]_q, -\left(\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right)_q \right\rangle_H = -2\sum_{q=1}^3 \left\langle \left[d\grave{A}_r\right]_q, \left(\left[\mathcal{F}_r\right]\left[g_3\right] / \det g_3\right)_q \right\rangle_H$$
$$\langle \mathcal{F}, \mathcal{F} \rangle = 2\sum_{q=1}^3 \left(\left[d\grave{A}_w\right]_q\right)^t [\eta]\left(\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right)_q = -2\sum_{q=1}^3 \left(\left[d\grave{A}_r\right]_q\right)^t [\eta]\left(\left[\mathcal{F}_r\right]\left[g_3\right]\right)_q / \det g_3$$
$$\langle \mathcal{F}, \mathcal{F} \rangle = 2Tr\left(\left[d\grave{A}_w\right]^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right) = -2Tr\left(\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_r\right]\left[g_3\right]\right) / \det g_3$$

we get :

$$\begin{gathered}
\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1} = \left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t [\eta]\left[d\grave{A}_r\right] \\
\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_r\right]\left[g_3\right] = -\left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t [\eta]\left[d\grave{A}_w\right] \det g_3 \\
\langle \mathcal{F}, \mathcal{F} \rangle = 2Tr\left(\left[d\grave{A}_w\right]^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right) = 2Tr\left(\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_r\right]\left[g_3\right]\right) / \det g_3
\end{gathered} \tag{68}$$

10 Solution of the equations

10.1 Equations

i) The state of the particles is defined by the potential :

$a = 1...16 : \left\langle Ad_u \widehat{Q}_k, \left\{\left[\frac{du_k}{dt} \cdot u_k^{-1} + \grave{A}\left(V_k\right), \frac{\partial u_k}{\partial z_a} \cdot u_k^{-1}\right] + \grave{A}\left(\frac{dV_k}{dz_a}\right)\right\}\right\rangle_H |_m = 0$

$\Rightarrow \left\langle Ad_{u_k} \widehat{Q}_k, \left[\grave{A}\left(V_k\right), \frac{du_k}{dt} \cdot u_k^{-1}\right] + \grave{A}\left(\frac{dV_k}{dt}\right)\right\rangle_H = 0$

ii) The metric is defined by the field :

$\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1} = \left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t [\eta]\left[d\grave{A}_r\right]$

$\left[d\grave{A}_r\right]^t [\eta]\left[\mathcal{F}_r\right]\left[g_3\right] = \left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t [\eta]\left[d\grave{A}_w\right] \det g_3$

$\langle \mathcal{F}, \mathcal{F} \rangle = 2Tr\left(\left[d\grave{A}_r\right]^t [\eta] [\mathcal{F}_r] [g_3]\right) = 2Tr\left(\left[d\grave{A}_w\right]^t [\eta] [\mathcal{F}_w] [g_3]^{-1}\right) \det g_3$

iii) For the field we have 2 identities :

the Codifferential equation : $D\mathcal{F} = 0 \Rightarrow \mathcal{F} = *dK$

the Chern-Weil theorem : $Tr\left([\mathcal{F}_r]^t [\eta] [\mathcal{F}_w]\right)$ does not depend on the connection

and the Currents equations :

$\forall \delta\grave{A} \in \Lambda_{1c,\infty}(TM; T_1U) : \int_\Omega \sum_{\beta=0}^3 \left\langle \phi^\beta + \sum_{k=1}^p J_k^\beta, \delta\grave{A}_\beta \right\rangle_H \varpi_4 = 0$

with

$J_k = Ad_{u_k}\widehat{Q}_k \otimes V_k \in \mathfrak{X}(T_1U \otimes TM)$

$\phi = 2\sum_{a=1}^{16}\sum_{\alpha,\beta=0}^3 \left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}\right]^a \kappa_a \otimes \partial\xi_\alpha \in \mathfrak{X}(T_1U \otimes TM)$

Except for the Codifferential equation and the Chern-Weil theorem, all the equations hold when $\grave{A}$ is not continuously differentiable and $\mathcal{F}$ is expressed with the first jet extension $j^1\grave{A}$.

10.2 The codifferential equation

10.2.1 The codifferential as a differential operator

The codifferential equation is an identity : it is verified whenever $\mathcal{F}$ is computed from the partial derivatives of $\grave{A}$. There can be discontinuities. To study the problem we need to express the equation with the potential itself, and for this to use the jet formalism.

An element of $J^1\Lambda_1(TM; T_1U)$ reads $j^1\grave{A} = \left(m, \grave{A}_\alpha^a, \grave{A}_{\beta\alpha}^a, a = 1..16, \alpha, \beta = 0..3\right)$ where $\sum_{\alpha=0}^3\sum_{a=1}^{16} \grave{A}_{\beta\alpha}^a d\xi^\alpha \otimes \kappa_a$ are 4 independent one form (Maths.6.2). If $j^1\grave{A}$ is the prolongation of a section then $\grave{A}_{\beta\alpha}^a = \partial_\beta \grave{A}_\alpha^a$.

The strength can then be seen as a differential operator :

$\mathcal{F} : J^1\Lambda_1(TM; T_1U) \to \Lambda_2(TM; T_1U) ::$

$\mathcal{F}\left(j^1\grave{A}\right) = \sum_{\{\alpha,\beta\}=0..3}\sum_{a=1}^{16}\left(\grave{A}_{\alpha\beta}^a - \grave{A}_{\beta\alpha}^a + \left[\grave{A}_\alpha, \grave{A}_\beta\right]^a\right) d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a$

The codifferential is, in a general chart :

$D\mathcal{F} =$

$\sum_{\alpha=0}^3\sum_{a=1}^{16}(-1)^\alpha g_{\alpha\alpha}\left(\sum_{\beta=0}^3 \partial_\beta\left(\mathcal{F}^{\alpha\beta}\sqrt{\det g_3}\right)\right)/\sqrt{|\det g|} d\xi^0 \wedge ..\widehat{d\xi^\alpha} \wedge ..d\xi^3$

and as a differential operator :

$D\mathcal{F} : J^1\Lambda_1(TM; T_1U) \to \Lambda_3(TM; T_1U) ::$

$D\mathcal{F}\left(j^1\grave{A}\right) = \sum_{\alpha=0}^3\sum_{a=1}^{16}(-1)^\alpha \times$

$g_{\alpha\alpha}\left(\sum_{\beta=0}^3 \partial_\beta\left(\sum_{\lambda,\mu=0}^3 g^{\alpha\lambda}g^{\beta\mu}\left(\grave{A}_{\lambda\mu}^a - \grave{A}_{\mu\lambda}^a + \left[\grave{A}_\lambda, \grave{A}_\mu\right]^a\right)\sqrt{|\det g|}\right)\right)/\sqrt{|\det g|}\kappa_a \otimes d\xi^0 \wedge ..\widehat{d\xi^\alpha} \wedge ..d\xi^3$

$\sum_{\beta=0}^3 \partial_\beta\left(\sum_{\lambda,\mu=0}^3 g^{\alpha\lambda}g^{\beta\mu}\left(\left(\grave{A}_{\lambda\mu}^a - \grave{A}_{\mu\lambda}^a + \left[\grave{A}_\lambda, \grave{A}_\mu\right]^a\right)\right)\sqrt{|\det g|}\right)$

$= 2\sum_{\beta=0}^3 \partial_\beta\left(\sum_{\mu=1}^3\left(g^{\alpha 0}g^{\beta\mu} - g^{\alpha\mu}g^{\beta 0}\right)\left(\left(\grave{A}_{0\mu}^a - \grave{A}_{\mu 0}^a + \left[\grave{A}_0, \grave{A}_\mu\right]^a\right)\right)\sqrt{|\det g|}\right)$

$+2\sum_{\beta=0}^3 \partial_\beta\left(\left(g^{\alpha 1}g^{\beta 2} - g^{\alpha 2}g^{\beta 1}\right)\left(\left(\grave{A}_{12}^a - \grave{A}_{21}^a + \left[\grave{A}_1, \grave{A}_2\right]^a\right)\right)\sqrt{|\det g|}\right)$

$+2\sum_{\beta=0}^{3}\partial_\beta\left(\left(g^{\alpha1}g^{\beta3}-g^{\alpha3}g^{\beta1}\right)\left(\left(\grave{A}^a_{13}-\grave{A}^a_{31}+\left[\grave{A}_1,\grave{A}_3\right]^a\right)\right)\sqrt{|\det g|}\right)$
$+2\sum_{\beta=0}^{3}\partial_\beta\left(\left(g^{\alpha2}g^{\beta3}-g^{\alpha3}g^{\beta2}\right)\left(\left(\grave{A}^a_{23}-\grave{A}^a_{32}+\left[\grave{A}_2,\grave{A}_3\right]^a\right)\right)\sqrt{|\det g|}\right)$
Let us denote as usual
$[\mathcal{F}^a_r]=\left[\begin{array}{ccc}\mathcal{F}^a_{32} & \mathcal{F}^a_{13} & \mathcal{F}^a_{21}\end{array}\right];[\mathcal{F}^a_w]=\left[\begin{array}{ccc}\mathcal{F}^a_{01} & \mathcal{F}^a_{02} & \mathcal{F}^a_{03}\end{array}\right]$
and denote :
$\left[D^{\alpha\beta}_w\right]=\left[\begin{array}{c}g^{\alpha0}g^{\beta1}-g^{\alpha1}g^{\beta0}\\ g^{\alpha0}g^{\beta2}-g^{\alpha2}g^{\beta0}\\ g^{\alpha0}g^{\beta3}-g^{\alpha3}g^{\beta0}\end{array}\right]=g^{\alpha0}\left[g_3^{-1}\right]_\beta-g^{\beta0}\left[g_3^{-1}\right]_\alpha;$
$\left[D^{\alpha\beta}_r\right]=\left[\begin{array}{c}g^{\alpha3}g^{\beta2}-g^{\alpha2}g^{\beta3}\\ g^{\alpha1}g^{\beta3}-g^{\alpha3}g^{\beta1}\\ g^{\alpha2}g^{\beta1}-g^{\alpha1}g^{\beta2}\end{array}\right]=j\left(\left[g_3^{-1}\right]_\beta\right)\left[g_3^{-1}\right]_\alpha$
$\sum_{\beta=0}^{3}\partial_\beta\left(\sum_{\lambda,\mu=0}^{3}g^{\alpha\lambda}g^{\beta\mu}\left(\left(\grave{A}^a_{\lambda\mu}-\grave{A}^a_{\mu\lambda}+\left[\grave{A}_\lambda,\grave{A}_\mu\right]^a\right)\right)\sqrt{|\det g|}\right)$
$=\sum_{\beta=0}^{3}\partial_\beta\left(\left([\mathcal{F}^a_w]\left[D^{\alpha\beta}_w\right]+[\mathcal{F}^a_r]\left[D^{\alpha\beta}_r\right]\right)\sqrt{|\det g|}\right)$

$$D\mathcal{F}\left(j^1\grave{A}\right)=\sum_{\alpha,\beta=0}^{3}g_{\alpha\alpha}\partial_\beta\left(\left([\mathcal{F}^a_w]\left[D^{\alpha\beta}_w\right]+[\mathcal{F}^a_r]\left[D^{\alpha\beta}_r\right]\right)\sqrt{|\det g|}\right)\frac{1}{\sqrt{|\det g|}}\kappa_a\otimes(-1)^\alpha d\xi^0\wedge..\widehat{d\xi^\alpha}\wedge..d\xi^3 \tag{69}$$

In the standard chart :

$$\begin{gathered}D\mathcal{F}\left(j^1\grave{A}\right)=\\ \sum_{a=1}^{16}\sum_{\beta=1}^{3}\partial_\beta\left([\mathcal{F}^a_w]\left[g_3^{-1}\right]_\beta\sqrt{|\det g|}\right)\frac{1}{\sqrt{|\det g|}}\kappa_a\otimes d\xi^1\wedge d\xi^2\wedge d\xi^3\\ +\sum_{\alpha=1}^{3}(-1)^\alpha g_{\alpha\alpha}\left(\partial_0\left([\mathcal{F}^a_w]\left[g_3^{-1}\right]_\alpha\sqrt{|\det g|}\right)+\sum_{\beta=1}^{3}\partial_\beta\left([\mathcal{F}^a_r]\left[D^{\alpha\beta}_r\right]\sqrt{|\det g|}\right)\right)\\ \times\frac{1}{\sqrt{|\det g|}}\kappa_a\otimes d\xi^0\wedge..\widehat{d\xi^\alpha}\wedge..d\xi^3\end{gathered} \tag{70}$$

It is not a linear differential operator, because of the brackets. But it defines a distribution.
$D\mathcal{F}\left(j^1\grave{A}\right)$ acts linearly on $\delta Z\in\Lambda_1(TM;T_1U)$ by :
$D\mathcal{F}\left(j^1\grave{A}\right)(\delta Z)=\sum_{a=1}^{16}\eta_{aa}\delta Z^a\wedge D\mathcal{F}\left(j^1\grave{A}^a\right)|\det g|$
$=2\sum_{\alpha,\beta=0}^{3}\left\langle\delta Z_\alpha,g_{\alpha\alpha}\partial_\beta\left(\left([\mathcal{F}^a_w]\left[D^{\alpha\beta}_w\right]+[\mathcal{F}^a_r]\left[D^{\alpha\beta}_r\right]\right)\sqrt{|\det g|}\right)\right\rangle_H\varpi_4\in\Lambda_4(TM;T_1U)$
Let us define :

$$\widehat{D\mathcal{F}}\left(j^1\grave{A}\right)(\delta Z)=\int_\Omega\sum_{a=1}^{16}\eta_{aa}\delta Z^a\wedge D\mathcal{F}\left(j^1\grave{A}^a\right)|\det g| \tag{71}$$

$\widehat{D\mathcal{F}}\left(j^1\grave{A}\right)\in\mathfrak{X}_{\infty c}\left(\Lambda_1(TM;T_1U)\right)'$ the space of distributions on smooth compactly supported sections $\Lambda_1(TM;T_1U)$

We have a map : $\widehat{D\mathcal{F}}:J^1\Lambda_1(TM;T_1U)\to\mathfrak{X}_{\infty c}\left(\Lambda_1(TM;T_1U)\right)'$

We will say that $j^1\grave{A}$ meets the codifferential equation if

$$\forall \delta Z \in \mathfrak{X}_{\infty c}\left(\Lambda_1\left(TM;T_1U\right)\right) : \widehat{D\mathcal{F}}\left(j^1\grave{A}\right)\left(\delta Z\right) = 0 \quad (72)$$

In the standard chart if there is $j^1\grave{A}$ such that $\mathcal{F}\left(j^1\grave{A}\right)$:
$\sum_{\beta=1}^{3}\partial_\beta\left(\left[\mathcal{F}_w^a\right]\left[g_3^{-1}\right]_\beta\sqrt{\det g_3}\right) = 0$
$\alpha = 1,2,3 : \partial_0\left(\left[\mathcal{F}_w^a\right]\left[g_3^{-1}\right]_\alpha\sqrt{\det g_3}\right) + \sum_{\beta=1}^{3}\partial_\beta\left(\left[\mathcal{F}_r^a\right]\left[D_r^{\alpha\beta}\right]\sqrt{\det g_3}\right) = 0$
the condition is met.
If there is $K \in \Lambda_1\left(TM;T_1U\right)$:
$\left[\mathcal{F}_r^a\right] = \left[dK_w^a\right]\left[g_3\right]^{-1}\sqrt{\det g_3}; \left[\mathcal{F}_w^a\right] = -\left[dK_r^a\right]\left[g_3\right]/\sqrt{\det g_3}$
The first equation is always met :
$\sum_{\beta=1}^{3}\partial_\beta\left(\left[dK_r^a\right]_\beta\right) = \partial_1\left(\partial_3K_2 - \partial_2K_3\right) + \partial_2\left(\partial_1K_3 - \partial_3K_1\right) + \partial_3\left(\partial_2K_1 - \partial_1K_2\right)$
$= \partial_1\partial_3K_2 - \partial_1\partial_2K_3 + \partial_2\partial_1K_3 - \partial_2\partial_3K_1 + \partial_3\partial_2K_1 - \partial_3\partial_1K_2 = 0$
For the second equation :
$\left[\mathcal{F}_w^a\right]\left[g_3^{-1}\right]_\alpha\sqrt{\det g_3} = -\left[dK_r^a\right]\left[g_3\right]\left[g_3^{-1}\right]_\alpha = -\left[dK_r^a\right]_\alpha$
$\alpha = 1,2,3 : -\partial_0\left(\left[dK_r^a\right]_\alpha\right) + \sum_{\beta=1}^{3}\partial_\beta\left(\left[\mathcal{F}_r^a\right]\left[D_r^{\alpha\beta}\right]\sqrt{\det g_3}\right) = 0$
using the identity :
$\left[M\right]^t j\left(r\right)\left[M\right] = j\left(\left[M\right]^{-1}\left[r\right]\right)\det M$
$\left[g_3\right]^{-1}\left[D_r^{\alpha\beta}\right] = \left[g_3\right]^{-1} j\left(\left[g_3^{-1}\right]_\beta\right)\left[g_3^{-1}\right]_\alpha = \left(\left[g_3\right]^{-1} j\left(\left[g_3^{-1}\right]_\beta\right)\left[g_3^{-1}\right]\right)_\alpha$
$= \left[j\left(\left[g_3\right]\left[g_3^{-1}\right]_\beta\right)\right]_\alpha \det\left(\left[g_3\right]^{-1}\right) = \left(j\left(\varepsilon_\beta\right)\right)_\alpha \det\left(\left[g_3\right]^{-1}\right)$
$\left[\mathcal{F}_r^a\right]\left[D_r^{\alpha\beta}\right]\sqrt{\det g_3} = -\left[dK_w^a\right]\left(j\left(\varepsilon_\alpha\right)\right)_\beta$
the second equation reads :
$\alpha = 1,2,3 : -\partial_0\left(\left[dK_r^a\right]_\alpha\right) - \sum_{\beta=1}^{3}\partial_\beta\left(\left[dK_w^a\right]\left(j\left(\varepsilon_\alpha\right)\right)_\beta\right) = 0$
and one can check that it is always met.

So we can safely conclude that the solutions of the problem at equilibrium are sections $j^1\grave{A}$ such that :

$$\exists K \in \Lambda_1\left(TM;T_1U\right) : \left[\mathcal{F}_r\left(j^1\grave{A}\right)\right] = \left[dK_w\right]\left[g_3\right]^{-1}\sqrt{\det g_3}; \left[\mathcal{F}_w\left(j^1\grave{A}\right)\right] = -\left[dK_r\right]\left[g_3\right]/\sqrt{\det g_3} \quad (73)$$

$\left[\mathcal{F}_w\right] = \left[\begin{array}{ccc} \grave{A}_{01} - \grave{A}_{10} + \left[\grave{A}_0, \grave{A}_1\right] & \grave{A}_{02} - \grave{A}_{20} + \left[\grave{A}_0, \grave{A}_2\right] & \grave{A}_{03} - \grave{A}_{30} + \left[\grave{A}_0, \grave{A}_3\right] \end{array}\right] =$
$-\left[\begin{array}{ccc} \partial_3K_2 - \partial_2K_3 & \partial_1K_3 - \partial_3K_1 & \partial_2K_1 - \partial_1K_2 \end{array}\right]\left[g_3\right]/\sqrt{\det g_3}$
$\left[\mathcal{F}_r\right] = \left[\begin{array}{ccc} \grave{A}_{32} - \grave{A}_{23} + \left[\grave{A}_3, \grave{A}_2\right] & \grave{A}_{13} - \grave{A}_{31} + \left[\grave{A}_1, \grave{A}_3\right] & \grave{A}_{21} - \grave{A}_{12} + \left[\grave{A}_2, \grave{A}_1\right] \end{array}\right] =$
$\left[\begin{array}{ccc} \left(\partial_0K_1 - \partial_1K_0\right) & \left(\partial_0K_2 - \partial_2K_0\right) & \left(\partial_0K_3 - \partial_3K_0\right) \end{array}\right]\left[g_3\right]^{-1}\sqrt{\det g_3}$

10.2.2 The Chern-Weil Theorem

The Chern-Weil theorem must be met whenever the strength is defined through partial derivatives.

Let $K \in \Lambda_1(TM; T_1U)$ such that $\mathcal{F} = *dK$

With the matrix notation for 2 forms :

$[*dK_r^a] = [dK_w^a][g_3]^{-1}\sqrt{\det g_3}; [*dK_w^a] = -[dK_r^a][g_3]/\sqrt{\det g_3}$

$[\mathcal{F}_r^a] = [dK_w^a][g_3]^{-1}\sqrt{\det g_3}; [\mathcal{F}_w^a] = -[dK_r^a][g_3]/\sqrt{\det g_3}$

Then :

$Tr[\mathcal{F}_r]^t[\eta][\mathcal{F}_w] = -Tr[g_3]^{-1}[dK_w]^t[\eta][dK_r][g_3] = -Tr[dK_w]^t[\eta][dK_r]$

does not depend on the connection (that is $\grave{A}, \mathcal{F}$).

10.2.3 The codifferential equation for the EM field

The present model holds when the field is restricted to one of its components, be the EM field or the gravitational field, and of course in the approximation of SR geometry. For the EM field ($a = 1$) the current is null, and the codifferential equation reads :

$\alpha = 0...3 : \sum_{\beta=0}^3 \partial_\beta\left(\mathcal{F}^{1\alpha\beta}\sqrt{|\det g|}\right) = 0$

In the Special Relativity Geometry it is :

$\alpha = 0...3 : \sum_{\beta\lambda\mu=0}^3 \partial_\beta\left(\eta_{\alpha\lambda}\eta_{\beta\mu}\mathcal{F}_{\lambda\mu}^1\right) = \sum_{\beta=0}^3 \partial_\beta\left(\eta_{\alpha\alpha}\eta_{\beta\beta}\mathcal{F}_{\alpha\beta}^1\right)$

$= \eta_{\alpha\alpha}\sum_{\beta=0}^3 \eta_{\beta\beta}\partial_\beta\left(\left(\partial_\alpha\grave{A}_\beta^1 - \partial_\beta\grave{A}_\alpha^1\right)\right) = 0$

$-\partial_{0\alpha}^2\grave{A}_0^1 + \partial_{00}^2\grave{A}_\alpha^1 + \sum_{\beta=1}^3 \partial_{\beta\alpha}^2\grave{A}_\beta^1 - \partial_{\beta\beta}^2\grave{A}_\alpha^1 = 0$

$\partial_{0\alpha}\grave{A}_0^1 - \sum_{\beta=1}^3 \partial_{\beta\alpha}^2\grave{A}_\beta^1 = \partial_\alpha\left(\partial_0\grave{A}_0^1 - \sum_{\beta=1}^3 \partial_\beta\grave{A}_\beta^1\right)$

With the Coulomb gauge it sums up to : $\square\grave{A}^1 = 0$. That is the wave equation, which is the most typical feature of the force field.

The solutions are found using the Fourier transform (Maths.7.5). The fundamental solution (or Green functions) is the family of distributions $U(t)$ acting on functions defined on $\mathbb{R}^3$, depending on t :

$U(t) : C_\infty\left(\mathbb{R}^3; \mathbb{C}\right) \to \mathbb{C} :: U(t)(\psi) = \frac{t}{4\pi c^2}\int_{S^2}\psi(s)\,d\sigma_2(s)$ where $S^2(0,c)$ is the sphere of radius c and center 0 in $\mathbb{R}^3$.

Then the solution of $\square U = \delta_0$, with the distribution U acting on functions $f \in C_\infty\left(\mathbb{R}; C_\infty\left(\mathbb{R}^3; \mathbb{C}\right)\right)$ is the distribution :

$U(f) = \int_0^\infty (U(t)(f(t,tx)))\,dt = \int_0^\infty \left(H(t)\frac{t}{4\pi c^2}\int_{S^2} f(t,tx)\,d\sigma_2(x)\right)dt$ with the Heavyside function $H(0) = 0, t > 0 : H(t) = 1$

Its support is the hypercone $t \geqq 0, \|x\| = ct$.

The solutions of the problem in SR :

$\grave{A} \in C(\Omega; \mathbb{R})$ with $\Omega = [0, t_0] \times \Omega_3(0)$

$\square\grave{A} = 0$

$m \in \Omega_3(0) : \grave{A} = f_0; \frac{\partial}{\partial t}\grave{A}|_{t=0} = f_1$

are then $\grave{A}(t,x) = U(t) * f_0(x) + \frac{dU}{dt} * f_1(x)$ with the convolution of functions and the derivative of distributions using :

$\square U(t) = \delta_0 \to \square(U(t) * f_0) = f_0$

$\frac{d}{dt}\square U(t) = \delta_0 \to \square\left(\frac{d}{dt}U(t) * f_1\right) = f_1$

On a manifold with a Riemann metric on $\Omega_3(t)$ there is still a unique solution, which is an integrable function with compact support.

In both cases one goes from the fundamental solution to the specific solution by the usual association of a function to a distribution through an integral :

$U \to F(t,x) = \frac{1}{4\pi t}\int_{S^2(0,ct)} f_0(x-s)\, d\sigma_2(s) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S^2(0,ct)} f_1(x-s)\, d\sigma_2(s)\right)$

which is how Green's functions are usually seen.

The computation of the field created by a single particle is the "point particle problem", about which there is a vast literature : a field, usually scalar, defined by a map $F \in C(M;\mathbb{R})$ is such that $\Box F = 0$ in the vacuum, and $\Box F = f$ on the trajectory of the particle. One basic idea is that F is given by f on the trajectory and in the vacuum should be some image of f. This is consistent with a general fact in Partial Differential Equations : the solutions are image of the initial conditions and, here, the particles, on their trajectories, add initial conditions for the field.

In all cases the physical meaning is that the field originates from interactions with particles, which are the sources, propagate by spherical waves, and the value of the field at a given point is the sum of the field originating from the sources.

10.2.4 The issue

It is quite clear that the equations listed previously and resulting from the implementation of the Principle of Least Action, are not sufficient to provide a solution of the problem, even in the simple case of a continuous model. We need to call for additional properties of the field, and of course for the assumption we have made that the field propagates along Killing curves.

The Fourier transform uses the abelian group of translations on $\mathbb{R}^4$, which is not accessible in General Relativity. But, with the assumption that the field propagates along Killing curves, we have access to another group of transformations : the isometries. And because the field is no longer scalar, we must extend the theory of distributions to vector bundles.

In a continuous model the variables related to the particles show a symmetry, given by the vector fields V_k. As a consequence there should be also some symmetry for the field. Because the field is defined on Ω, and not on trajectories, this symmetry is expressed in a different way. At any point m in Ω there should be an incoming Killing curve along which some copy of the value of the field is transported. The field is measured through its action on particles by the potential. So the symmetry should apply to the potential.

The usual distributions S are continuous linear functional acting on scalar functions φ. They are associated to functions f acting on the functions φ through an integral : $S \to f$ such that $S(\varphi) = \int f\varphi dx$. Variational derivatives are linear functional acting on sections Y of vector bundles.(Maths.7.2.3), then distributions are associated to forms X over the vector bundle through the integral such as $S(Y) = \int_\Omega X(Y)\,\varpi_0$. Here the variables are either sections of vector bundles belonging to $\mathfrak{X}(P_C \otimes TM)$ such as the currents ϕ, J, or forms valued in T_1U, such as $\grave{A}, \mathcal{F}$. To a linear scalar functional ℓ acting on sections $Y \in \mathfrak{X}(P_C \otimes TM)$ is associated a section $X \in \Lambda_1(TM; T_1U)$ through

$\ell \to X :: \ell(Y) = \int_\Omega \left\langle X_\beta, Y^\beta \right\rangle_H \varpi_0$

10.3 Morphisms induced by isometries

(see Annex for the mathematical details)

A diffeomorphism on Ω is a bijective map $f : \Omega \to \Omega$ such that its derivative $f' : T_m\Omega \to T_{f(m)}\Omega$ is itself invertible. By pullback or push forward it defines a morphism on the space of tensors on the tangent bundle $\otimes TM$. They preserve the type of the tensors, the product of tensors and the exterior product of forms, commute with the exterior differential, and can be composed.

An isometry is a diffeomorphism which preserves the metric : $f_* g(m) = g(f(m))$. Then it preserves the scalar product $G_r(\lambda, \mu)$ of forms as well as the volume form ϖ_4, and commutes with Hodge duality : $*(f^*\lambda) = f^*(*\lambda)$. The derivative $f'(m)$ can be extended to a Clifford morphism, and further to a Clifford morphism F_C on P_C which is real and preserves the Hermitian product. It can be expressed as $Ad_{C(S)}$ where S is the product of at most 4 vectors : it is either the scalar multiple of a fixed vector, or an element of $Spin(3,1)$.

A one parameter group of diffeomorphisms is defined by the flow of a vector field V :

$f_t : M \to M :: f_\tau(m) = \Phi_V(\tau, m)$

$f'_\tau : T_mM \to T_{\Phi_V(\tau,m)} :: f_\tau(m) = \Phi'_{Vm}(\tau, m)$

which is then transported along an integral curve : $V(\Phi_V(\tau, m)) = \Phi'_{Vm}(\tau, m)(V(m))$

A one parameter group of isometries is defined by the flow of a Killing vector field V, which then transports the metric, and this is equivalent to $\pounds_V g = 0 \Leftrightarrow \Phi_V(\tau, m)_* g(m) = g(\Phi_V(\tau, m))$.

A one parameter group of isometries can be extended to a one parameter group of morphisms on P_C :

$f_t : M \to M :: f_\tau(m) = \Phi_V(\tau, m)$

$F_\tau : P_C \to P_C : F_\tau(m)\left(\sum_{a=1}^{16} Z^a F_a(m)\right) = \sum_{a,b=1}^{16} \left[Ad_{C(S(\Phi_V(\tau,m)))}\right]_b^a Z^b F_a(\Phi_V(\tau, m))$

where $C(S(\Phi_V(\tau, m))) = \exp \tau C(V) \cdot C(S(m))$ and $C(S(m))$ is given by $\Phi'_{Vm}(0, m)$.

A Killing vector field defines uniquely a one parameter group, and then both V, S at each point.

We have assumed that the field propagates along null, future oriented, Killing vector fields $V \in \mathfrak{X}(K)$ so we will restrict ourselves to this case. Because $\langle V, V\rangle = 0$, $\exp \tau C(V) = 1 : C(S(\Phi_V(\tau, m))) = C(S(m))$ is constant along a propagation curve and the morphism $S(m)$ on $Cl(3,1)$ is necessarily the product of a spatial rotation and a translation, defined by $V = c\varepsilon_0 + \widetilde{v}$ with the components $\widetilde{v}$ of v in the tetrad, and a parameter r which can be seen as a polarisation :

$\widetilde{v}, r : \Omega \to \mathbb{R}^3$

$C(S(m)) \in C(Spin(3,1))$

$C(S(m)) = \left(\epsilon\sqrt{1-\frac{1}{c^2}\left(\widetilde{v}^t r\right)^2}, 0, 0, -\frac{1}{c} ij(r)\widetilde{v}, r, 0, 0, 0\right), \epsilon = \pm 1$

V is necessarily an eigen vector of Ad_S with eigen value 1 (usually it is not the only one).

Moreover the push forward by V preserves the time vector : $f_{V*}\varepsilon_0 = \varepsilon_0$. The only elements of $C(Spin(3,1))$ which have this property are the pure rotations : $-\frac{1}{c} ij(r)\widetilde{v} = 0 \Rightarrow r = \lambda\widetilde{v}$ and :

$$C(S(m)) = \left(\epsilon\sqrt{1-\lambda^2}, 0, 0, 0, \lambda\widetilde{v}, 0, 0, 0\right), \epsilon = \pm 1 \tag{74}$$

The parameters are fixed along the curve, so $\lambda, \widetilde{v}, \epsilon$ are fixed at m.

$C(S(m))^{-1} = CC(C(S(m)))^t = \left(\epsilon\sqrt{1-\lambda^2}, 0, 0, 0, -\lambda\widetilde{v}, 0, 0, 0\right)$

$\Leftrightarrow \lambda \rightarrow -\lambda$

The matrix of the adjoint map is then :

$$\left[Ad_{C(S)}\right]_{16\times 16} = \begin{bmatrix} & a & v_0 & v & w & r & x_0 & x & b \\ \\ a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v & 0 & 0 & J & 0 & 0 & 0 & 0 & 0 \\ w & 0 & 0 & 0 & J & 0 & 0 & 0 & 0 \\ r & 0 & 0 & 0 & 0 & J & 0 & 0 & 0 \\ x_0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & J & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $[J]_{3\times 3} = 1 + 2\lambda\left(\sqrt{1-\lambda^2} j(\widetilde{v}) + \lambda j(\widetilde{v}) j(\widetilde{v})\right)$ is the matrix of a spatial rotation of the tetrad. Thus $[J]^t[J] = 1$.

The restriction of $Ad_{C(S)}$ to the space of $Cl(3,1)$ spanned by $(\varepsilon_j)_{j=0..3}$:

$\left[\widehat{N}\right]_{4\times 4} = \begin{bmatrix} 1 & 0 \\ 0 & [J] \end{bmatrix}$ and $\left[\widetilde{V}\right] = \left[\widehat{N}\right]\left[\widetilde{V}\right]$

This matrix is orthogonal : $\left[\widehat{N}\right]^t [\eta] \left[\widehat{N}\right] = [\eta]$

The maps $f_{\tau*}$ are expressed, in the basis of the chart, through the jacobian $[L] = [\Phi'_{Vm}(\tau, m)]$:

$[L] = \begin{bmatrix} L_0^0 & [L^0] \\ [L_0] & [l]_{3\times 3} \end{bmatrix}$ and because ε_0 is preserved $[L] = \begin{bmatrix} 1 & 0 \\ 0 & [l]_{3\times 3} \end{bmatrix}$

There is a relation between $[L]$ and the restriction $\widehat{N}$ of Ad_S to the space spanned by $(\varepsilon_j)_{j=0..3}$:

$[L(\Phi_V(\tau, m))] = [P(\Phi_V(\tau, m))]\left[\widehat{N}\right][P'(m)]$

so in the standard chart :

$$[L(\tau)] = [\Phi'_{Vm}(\tau, m)] = \begin{bmatrix} 1 & 0 \\ 0 & [l]_{3\times 3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & [Q(\Phi_V(\tau, m))][J][Q'(m)] \end{bmatrix} \tag{75}$$

The same one parameter group can be used to transport tensors on TM, as well as elements of P_C, and so of elements of the tensorial bundles $TM \otimes T_1U, TM^* \otimes T_1U$. The currents, the potential and the one form K are equivariant in a global change of gauge. We can transport them by the same one parameter group : we transport the tensor (vector or form) by the push forward with $\Phi_V(\tau, m)$ on one hand, and we transport the vector of $Cl(\mathbb{C}, 4)$ by $Ad_{C(S(\Phi_V(\tau,m)))}$ on the other hand. So we combine two operations based on the same Killing vector.

Push forward of vectors $X = Y \otimes Z \in \mathfrak{X}(TM \otimes T_1U)$ from m to $\Phi_V(\tau, m)$:

$F_{V\tau*}((Y \otimes Z)(m))(\Phi_V(\tau, m)) = f_{V\tau*}Y \otimes Ad_{C(S(m))}Z(m)$

$F_{V\tau*}(X(m))(\Phi_V(\tau, m)) = \sum_{a,n=1}^{16} \left[Ad_{C(m)}\right]_b^a \sum_{\beta,\gamma=0}^{3} \left[L\left(\Phi'_{Vm}(\tau, m)\right)\right]_\gamma^\beta X^{b\gamma}(m)\, \partial\xi_\beta \otimes \kappa_a(\Phi_V(\tau, m))$

with the matrix $\left[L\left(\Phi'_{Vm}(\tau, m)\right)\right] = \begin{bmatrix} 1 & 0 \\ 0 & [l]_{3\times3} \end{bmatrix} = \left[\Phi'_{Vm}(\tau, m)\right]$

Push forward of one forms : $\lambda \in \Lambda_1(TM; T_1U)$ from m to $\Phi_V(\tau, m)$:

$F_{V\tau*}(\lambda(m))(\Phi_V(\tau, m)) = Ad_{C(S(m))}(f_{V\tau*}\lambda)$

$F_{V\tau*}(\lambda(m))(\Phi_V(\tau, m))$

$= \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]_b^a \sum_{\gamma,\beta=0}^{3} \left[L\left(\Phi'_{Vm}(-\tau, \Phi_V(\tau, m))\right)\right]_\beta^\gamma \lambda_\gamma^b(m)\, d\xi^\beta \otimes \kappa_a(\Phi_V(\tau, m))$

$\Phi'_{Vm}(-\tau, \Phi_V(\tau, m)) = \left(\Phi_{V'm}(\tau, m)\right)^{-1}$

with the matrix $\left[L\left(\Phi'_{Vm}(-\tau, \Phi_V(\tau, m))\right)\right] = \left[L\left(\Phi'_{Vm}(\tau, m)\right)\right]^{-1} = \left[L'\left(\Phi'_{Vm}(\tau, m)\right)\right] = \begin{bmatrix} 1 & 0 \\ 0 & [l]_{3\times3}^{-1} \end{bmatrix}$

$F_{V\tau*}(\lambda(m))(\Phi_V(\tau, m)) = \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]_b^a \sum_{\gamma,\beta=0}^{3} \left[L'\left(\Phi_{V'm}(\tau, m)\right)\right]_\beta^\gamma \lambda_\gamma^b(m)\, d\xi^\beta \otimes \kappa_a(\Phi_V(\tau, m))$

So that for a vector $Y \in \mathfrak{X}(TM)$

$F_{V\tau*}(\lambda(f_{V\tau*}Y)(m))(\Phi_V(\tau, m)) = \left[Ad_{C(S(m))}\right]\lambda(Y)(m)$

$F_{V\tau*}(\lambda(f_{V\tau*}Y)(m))(\Phi_V(\tau, m))$

$= \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]_b^a \sum_{\gamma,\beta=0}^{3} \left[L\left(\Phi'_{Vm}(-\tau, \Phi_V(\tau, m))\right)\right]_\beta^\gamma \lambda_\gamma^b(m) \left[L\left(\Phi'_{Vm}(\tau, m)\right)\right]_\gamma^\beta X^\gamma(m)\, \kappa_a(\Phi_V(\tau, m))$

$= \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]_b^a \sum_{\gamma,\beta=0}^{3} \lambda_\gamma^b(m)\, X^\gamma(m)\, \kappa_a(\Phi_V(\tau, m))$

and for $Y \in \mathfrak{X}(TM \otimes T_1U)$

$\sum_{\beta=0}^{3} \left\langle F_{V\tau*}(\lambda(m))(\Phi_V(\tau, m))_\beta, F_{V\tau*}(Y(m))(\Phi_V(\tau, m))^\beta \right\rangle_H$

$= \sum_{\beta=0}^{3} \left\langle \left[Ad_{C(S(m))}\right]\lambda(m)_\beta, \left[Ad_{C(S(m))}\right] Y^\beta(m) \right\rangle_H = \sum_{\beta=0}^{3} \left\langle \lambda(m)_\beta, Y^\beta(m) \right\rangle_H$

Push forward of 2 forms $\lambda \in \Lambda_2(TM; T_1U)$ from m to $\Phi_V(\tau, m)$:

$F_{V\tau*}(\lambda(m))(\Phi_V(\tau, m))$

$= \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]_b^a \sum_{\gamma}^{3} \left[L'\left(\Phi'_{Vm}(\tau, m)\right)\right]_\alpha^\lambda \left[L'\left(\Phi'_{Vm}(\tau, m)\right)\right]_\beta^\mu \lambda_{\lambda\mu}^b(m)\, d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a(\Phi_V(\tau, m))$

which can be written

$F_{V\tau*}(\lambda(m))(\Phi_V(\tau, m))$

$= \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]_b^a \sum_{\lambda,\beta=0}^{3} \widetilde{\Lambda}_{\alpha\beta}^b(\Phi_V(\tau, m))\, d\xi^\alpha \wedge d\xi^\beta(\Phi_V(\tau, m))$

where, using the usual notation

$\left[\left[\widetilde{\Lambda}_r\right], \left[\widetilde{\Lambda}_w\right]\right]|_{\Phi_V(\tau,m)} = \left[\left[\Lambda_r\right], \left[\Lambda_w\right]\right]|_{\Phi_V(\tau,m)} \left[L_{L'}\left(\Phi'_{Vm}(\tau, m)\right)\right]$

with the matrix $[L_{L'}(\Phi_V(\tau,m))]_{6\times 6}=\begin{bmatrix}\left([l']^{-1}\right)^t\det l' & 0\\ 0 & l'\end{bmatrix}=\begin{bmatrix}[l]^t(\det l)^{-1} & 0\\ 0 & [l]^{-1}\end{bmatrix}$

$[L_{L'}]=[L_L]^{-1}=[L_{L^{-1}}]$

$\left[\widetilde{\Lambda}_r\right]=[\Lambda_r][l]^t(\det l)^{-1}$

$\left[\widetilde{\Lambda}_w\right]=[\Lambda_w][l]^{-1}$

Push forward of the metric from m to $\Phi_V(\tau,m)$:

$F_{V\tau*}(g(m))(\Phi_V(\tau,m))$

$=\sum_{\gamma,\beta=0}^{3}[L'(\Phi_{V'm}(\tau,m))]_\alpha^\lambda[L'(\Phi_{V'm}(\tau,m))]_\beta^\mu g_{\lambda\mu}(m)\,d\xi^\alpha\otimes d\xi^\beta(\Phi_V(\tau,m))$

$[F_{V\tau*}g]=[L'(\Phi_{V'm}(\tau,m))]^t[g(m)][L'(\Phi_{V'm}(\tau,m))]$

$[F_{V\tau*}g_3]=[l']^t[g_3(m)][l']=\left([l]^{-1}\right)^t[g_3(m)][l]^{-1}$

which gives :

$[\widetilde{g}(\Phi_V(\tau,m))]=\begin{bmatrix}-1 & 0\\ 0 & [l']^t[g_3][l']\end{bmatrix}$

and we can check that $[F_{V\tau*}(g(m))(\Phi_V(\tau,m))]=[g(\Phi_V(\tau,m))]$:

10.4 The equations for the field

10.4.1 The current equation

i) Because the field propagates on $\Omega_3(t)$ the current equation

$\forall\delta\grave{A}\in\Lambda_{1c,\infty}(TM;T_1U):\int_\Omega\sum_{\alpha=0}^{3}\left\langle\phi^\alpha+\sum_{k=1}^{p}J_k^\alpha,\delta\grave{A}_\alpha\right\rangle_H\varpi_4=0$

reads :

$\forall t:\int_0^t\left(\int_{\Omega_3(t)}\sum_{\alpha=0}^{3}\left\langle\phi^\alpha,\delta\grave{A}_\alpha\right\rangle_H\varpi_3\right)cdt=-\sum_{k=1}^{p}\int_0^t\left(\int_{\Omega_3(t)}\sum_{\alpha=0}^{3}\left\langle J_k^\alpha,\delta\grave{A}_\alpha\right\rangle_H\varpi_3\right)cdt=-\int_0^t\sum_{k=1}^{p}\left(\sum_{\alpha=0}^{3}\left\langle J_k^\alpha,\delta\grave{A}_\alpha\right\rangle_H|_{q_k(t)}\right)cdt$

that is

$\int_{\Omega_3(t)}\sum_{\alpha=0}^{3}\left\langle\phi^\alpha,\delta\grave{A}_\alpha\right\rangle_H\varpi_3=-\sum_{k=1}^{p}\sum_{\alpha=0}^{3}\left\langle J_k^\alpha,\delta\grave{A}_\alpha\right\rangle_H|_{q_k(t)}$

$\mu(\phi,t):\Lambda_{1c,\infty}(TM;T_1U)\to\mathbb{R}::\mu(\phi,t)\left(\delta\grave{A}\right)=\int_{\Omega_3(t)}\sum_{\alpha=0}^{3}\left\langle\phi^\alpha,\delta\grave{A}_\alpha\right\rangle_H\varpi_3$

is a distribution acting in $\Omega_3(t)$ and depending on t

$\mu(J,t)::\Lambda_{1c,\infty}(TM;T_1U)\to\mathbb{R}::\mu(J,t)\left(\delta\grave{A}\right)=\sum_{k=1}^{p}\sum_{\alpha=0}^{3}\left\langle J_k^\alpha,\delta\grave{A}_\alpha\right\rangle_H|_{q_k(t)}$

is a finite sum of Dirac's distribution acting in $\Omega_3(t)$ and depending on t.

$u_k,\widehat{Q}_k,V_k$ are symmetric with respect to $\mathbf{V}_k$, so is $J_k:J_k(\Phi_{V_k}(\tau,m))=\Phi_{\mathbf{V}_k,\tau*}J_k(m)$.

$\mu(J_k,t)\left(\delta\grave{A}\right)=\sum_{\alpha=0}^{3}\left\langle J_k^\alpha,\delta\grave{A}_\alpha\right\rangle_H|_{\Phi_{\mathbf{V}_k}(t,x_k)}$ with the location x_k of each particle of the family k at $t=0$, so that $\mu(J_k,t)$ is symmetric with respect to $V_k:\mu(J_k,t)=\Phi_{\mathbf{V}_k,t*}\mu$

Assuming that J is known, we look for a continuous map ϕ which provides an equivalent distribution ϕ acting on $\delta\grave{A}$.

The current equation is the key relation between the field and the particles. The distribution $\mu(J_k,t)$ is symmetric with respect to V_k, we can guess that there is some symmetry for ϕ so we will focus on maps symmetric with respect to all isometries. To be consistent we need to assume that $\delta \grave{A}$ is also symmetric.

ii) Let $\theta \geq 0$, $O(\tau) = \varphi_o(c\tau, x_0)$, the propagation curves with origin $O(t-\theta) = \varphi_o(c(t-\theta), x_0)$ tangent $Y = c\varepsilon_0 + y$ intersect $\Omega_3(t)$ at points $q = \varphi_o(ct, x)$ which belong to a sphere $S(O(t), \theta) \subset \Omega_3(t)$ with center $O(t) = \varphi_o(ct, x_0)$ and radius $c\theta$. The spatial vector y is orthogonal to the sphere. The components of y in the standard chart do not depend on t.

$S(O(t), \theta) = \{q : \exists Y = c\varepsilon_0 + y \in \mathfrak{X}(K), q = \Phi_Y(\theta, O(t-\theta))\}$

Conversely, the propagation curves originating from $\Omega_3(t)$ and reaching $O(t+\theta)$ come from points belonging to $S(O(t), \theta)$

$S(O(t), \theta) = \{q : \exists Y = c\varepsilon_0 + y \in \mathfrak{X}(K), O(t+\theta) = \Phi_Y(\theta, q)\}$

So that the system of spheres $S(O(t), \theta)$ has a dual meaning : as the destination points of propagation curves originating from $O(t-\theta)$, and as points of origin of propagation curves reaching $O(t+\theta)$.

$$\begin{array}{ccccccccc}
\Omega_3(t-\theta) & & & & \Omega_3(t) & & & & \Omega_3(t+\theta) \\
 & & & & S(O(t), \theta) & & & & \\
 & & & \nearrow & \uparrow & \searrow & & & \\
 & & Y_1 & & c\theta & & Y_2 & & \\
 & \nearrow & & & \downarrow & & & \searrow & \\
O(t-\theta) & \rightarrow & c\varepsilon_0 & \rightarrow & O(t) & \rightarrow & c\varepsilon_0 & \rightarrow & O(t+\theta)
\end{array}$$

The volume form on $S(O(t), \theta)$ is $d\sigma_2(q) = \frac{1}{\langle y, y \rangle_3} \varpi_3(t)(y) = \frac{1}{c} \sqrt{\det g_3} d\xi^1 \wedge d\xi^2 \wedge d\xi^3(y)$

The surface $\sigma_2(O(t), \theta) = \int_{S(O,\theta)} d\sigma_2(q)$ of $S(O(t), \theta)$ is the scalar computed with the image of the form in any chart. $S(O(t), \theta)$ is isomorphic to $S(O(t), 1), \sigma_2(O(t), \theta) = (c\theta)^2 \rho(O(t))$ where $\rho(O(t)) = \sigma_2(O(t), 1)$ depends on the metric on $\Omega_3(t)$. In Special Relativity : $\sigma_2(O(t), \theta) = \frac{4}{3}\pi(c\theta)^2$.

iii) Let $X \in \mathfrak{X}(TM \otimes T_1U), Z \in \Lambda_{1c,\infty}(TM; T_1U)$ both symmetric with respect to isometries and Z smooth, compactly supported with support in Ω: $\forall Y \in \mathfrak{X}(K) : X(\Phi_Y(\theta, m)) = F_{Y\theta*}(X)(\Phi_Y(\theta, m)), Z(\Phi_Y(\theta, m)) = F_{Y\theta*}(Z)(\Phi_Y(\theta, m)) \Rightarrow$

$\sum_{\beta=0}^{3} \left\langle F_{Y\theta*}(X(m))(\Phi_Y(\theta, m))^\beta, F_{Y\theta*}(Z(m))(\Phi_Y(\theta, m))_\beta \right\rangle_H = \sum_{\beta=0}^{3} \left\langle X^\beta(m), Z_\beta(m) \right\rangle$

Let be $O(\tau) = \varphi_o(c\tau, x_0), x_0 = Ct$

$\forall q \in S(O(t), \theta) : \exists Y \in \mathfrak{X}(K) : O(t+\theta) = \Phi_Y(\theta, q) \Leftrightarrow q = \Phi_Y(-\theta, O(t+\theta))$

$\sum_{\beta=0}^{3} \langle X^\beta, Z_\beta \rangle_H |_q =$

$\sum_{\beta=0}^{3} \left\langle F_{Y(-\theta)*}(X(q))(\Phi_Y(-\theta, O(t+\theta)))^\beta, F_{Y(-\theta)*}(Z(q))(\Phi_Y(-\theta, O(t+\theta)))_\beta \right\rangle_H$

$= \sum_{\beta=0}^{3} \langle X^\beta, Z_\beta \rangle_H |_{O(t+\theta)}$

$\int_{S(O(t),\theta)} \sum_{\beta=0}^{3} \left\langle X(q)^\beta, Z(q)_\beta \right\rangle_H d\sigma_2(q) = \sigma_2(O(t), \theta) \sum_{\beta=0}^{3} \langle X^\beta, Z_\beta \rangle_H |_{O(t+\theta)}$

Similarly :

$\sum_{\alpha=0}^{3} \langle J_k^\alpha, Z_\alpha \rangle_H |_{q_k(t) \in S(O(t),\theta)} =$

$\sum_{\alpha=0}^{3}\langle F_{Y\theta*}(J_k(q_k))^{\alpha}(\Phi_Y(\theta,q_k(t))),F_{Y\theta*}(Z(q_k))_{\alpha}(\Phi_Y(\theta,q_k(t)))\rangle_H$
$=\sum_{k=1}^{p}\sum_{\alpha=0}^{3}\langle F_{Y\theta*}(J_k(q_k))^{\alpha},Z_{\alpha}\rangle_H|_{O(t+\theta)}$
$\int_{S(O(t),\theta)}\sum_{\beta=0}^{3}\left\langle X(q)^{\beta},Z(q)_{\beta}\right\rangle_H d\sigma_2(q)=\sigma_2(O(t),\theta)\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t+\theta)}$

Because Ω is relatively compact, for a given $O(t)$ there is $\theta_M(O(t))$ such that $\forall j,\exists\theta\leq\theta_M(O(t)):q_j(t)\in S(O(t),\theta)$. The choice of $O(t)$ is arbitrary, as well of Ω. We assume that there is θ_M such that :

$\Omega_3(t)=\cup_{\theta=0...\theta_M}S(O(t),\theta),\forall j,\exists\theta\leq\theta_M:q_j(t)\in S(O(t),\theta)$

$\Omega_3(t)$ is a ball of center $O(t)$ and radius $c\theta_M$ which includes all the particles of the system.

$\int_{\Omega_3(t)}\sum_{\beta=0}^{3}\left\langle X(q)^{\beta},Z(q)_{\beta}\right\rangle_H\varpi_3(q)=\int_0^{\theta_M}\left(\int_{S(O(t),\theta)}\sum_{\beta=0}^{3}\left\langle X(q)^{\beta},Z(q)_{\beta}\right\rangle_H d\sigma_2(q)\right)d\theta$
$=\int_0^{\theta_M}\sigma_2(O(t),\theta)\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t+\theta)}\right)d\theta$
$\int_0^{t_0}\int_{\Omega_3(t)}\sum_{\beta=0}^{3}\left\langle X(q)^{\beta},Z(q)_{\beta}\right\rangle_H\varpi_3(q)=\int_0^{t_0}\left(\int_0^{\theta_M}\sigma_2(O(t),\theta)\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t+\theta)}\right)d\theta\right)cdt$

Proceed to the change of variables (Maths.3.2.3) : $(t,\theta)\to(\xi=t+\theta,\eta=\theta)$, then the jacobian is $\begin{bmatrix}1&1\\1&0\end{bmatrix}$, $\det\begin{bmatrix}1&1\\1&0\end{bmatrix}=-1$, the domain becomes $(0\times(t_0+\theta_M))$:

$\int_0^{t_0}\left(\int_0^{\theta_M}\sigma_2(O(t),\theta)\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t+\theta)}\right)d\theta\right)cdt$
$=\int_{\xi=0}^{t_0+\theta_M}\int_0^{\theta_M}\sigma_2(O(\xi-\eta),\eta)\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(\xi)}\right)|-1|\,cd\xi d\eta$
$=\int_{\xi=0}^{t_0+\theta_M}\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(\xi)}\right)\left(\int_0^{\theta_M}\sigma_2(O(\xi-\eta),\eta)\,d\eta\right)cd\xi$

The quantity $\int_0^{\theta_M}\sigma_2(O(\xi-\eta),\eta)\,cd\eta=c^2\int_0^{\theta_M}\rho(O(\xi-\eta))\eta^2d\eta=c^3\int_0^{\theta_M}\rho(O(t))\theta^2d\theta$ is the volume of the domain $\Omega_3(t)$, it is finite and we can assume that it is constant equal to Ω_0.

$\int_0^{t_0}\left(\int_0^{\theta_M}\sigma_2(O(t),\theta)\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t+\theta)}\right)d\theta\right)cdt=\Omega_0\int_{\xi=0}^{t_0+\theta_M}\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(\xi)}\right)d\xi$

or equivalently $\Omega_0\int_{t=0}^{t_0}\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t)}\right)dt$

$\int_{t=0}^{t_0}\int_{\Omega_3(t)}\sum_{\beta=0}^{3}\left\langle X(q)^{\beta},Z(q)_{\beta}\right\rangle_H\varpi_3(q)=\Omega_0\int_{t=0}^{t_0}\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t)}\right)dt$

And the current equation reads :

$\Omega_0\int_{t=0}^{t_0}\left(\sum_{\beta=0}^{3}\left\langle X^{\beta},Z_{\beta}\right\rangle_H|_{O(t)}\right)dt$
$=-\int_{t=0}^{t_0}\left(\int_0^{\theta_M}\sum_{k=1}^{p}\left(\sum_{\alpha=0}^{3}\langle F_{Y_k\theta*}(J_k(q_k))^{\alpha},Z_{\alpha}(q_k)\rangle_H|_{O(t+\theta)}\right)|_{q_k(t)\in S(O(t),\theta)}d\theta\right)cdt$

The relation holds for any O thus this is a definition for $m\in\Omega$ and we say that ϕ meets the current equation if

$$\forall O(t):\phi(O(t))=-\frac{1}{\Omega_0}\int_0^{\theta_M}\sum_{k=1}^{p}F_{Y_k\theta*}(J_k(q_k(t)))(O(t))|_{q_k(t-\theta)\in S(O(t-\theta),\theta)}d\theta \tag{76}$$

that is the integral over θ of the push forward by Y_k of $J_k(q_k(t-\theta))$ from $q_k(t-\theta)\in S(O(t-\theta),\theta)$ to $O(t)$.

So if the field is symmetric for any isometry, then the current equation implies that the right hand side is also symmetric.

iv) The equation above shows that particles are sources of the field, and the current related to the field propagates from an interaction with a particle in spherical waves, 2 dimensional spheres located in $\Omega_3(t+\tau)$ with radius $c\tau$. We consider the interactions with all particles, "coming from the past", and located at increasing spatial distance from $O(t)$ .By the choice of θ_M all particles of the system impact the value of ϕ at $O(t)$. However :

The particles which are involved in the right hand side are located on propagation curves reaching $O(t)$. A particle never follows a propagation curve (its spatial speed would be c) so a propagation curve meets at most once a given particle. The particles which are included in the increasing spheres $S(O(t-\theta),\theta)$ are located, on $\Omega_3(t)$, at increasing spatial distance from $O(t)=\varphi_0(ct,x_0)$. For a particle j if $q_j(t-\theta)\in S(O(t-\theta),\theta)$ then $q_j(t-\theta')\notin S(O(t-\theta'),\theta')$ because the spheres $S(O(t-\tau),\tau)$ have for center $O(t-\tau)=\varphi_0(c(t-\tau),x_0)$ and the spatial distance from $q_j(t-\theta)$ to $q_j(t-\theta')$ is certainly smaller than $c(\theta-\theta')$ (if the particle were immobile then $q_j(t)=\varphi_o(ct,x_j)$). As a consequence :

A particle is "seen" only once by $O(t)$.

By taking $O(t)=q_j(t)$ a particle j does not interact with a copy of itself in the past.

v) The section ϕ is defined as $\phi^{a\beta}=\frac{\partial}{\partial\grave{A}^a_\beta}\langle\mathcal{F},\mathcal{F}\rangle=2\sum_{\lambda=0}^3\eta_{aa}\left[\grave{A}_\lambda,\mathcal{F}^{\beta\lambda}\right]^a$. It is continuous. The solution ϕ that we have found for the current equation is actually a distribution, based on a map

$\phi\in\mathfrak{X}(T_1U\otimes TM)$ such that

$\frac{\delta}{\delta\grave{A}}\left(\int_\Omega\langle\mathcal{F},\mathcal{F}\rangle\varpi_4\right)\left(\delta\grave{A}\right)=\int_\Omega\sum_{\alpha=0}^3\left\langle\phi^\alpha,\delta\grave{A}_\alpha\right\rangle_H\varpi_4=-\int_\Omega\sum_{k=1}^p\left(\sum_{\alpha=0}^3\left\langle J_k^\alpha,\delta\grave{A}_\alpha\right\rangle_H\right)\varpi_4$

Which is necessary in order to compute an integral. The formula above for $\phi(O(t))$ is a bit abusive because J is defined locally.

vi) The current equation comes from the computation of the partial derivative for the energy of the field :

$\frac{\delta}{\delta\grave{A}}\left(\int_\Omega\langle\mathcal{F},\mathcal{F}\rangle\varpi_4\right)\left(\delta\grave{A}\right)=\int_\Omega\sum_{\alpha=0}^3\left(\left\langle\phi^\alpha,\delta\grave{A}_\alpha\right\rangle_H-2\left\langle\mathcal{F},d\delta\grave{A}\right\rangle\right)\varpi_4$

in which the second item is null. It represents the variation of energy carried by the field in its propagation, *with respect to the potential* $\grave{A}$. So it is not a density of energy with respect to the geometric volume, or a flow of energy. There is obviously a variation of potential when the field interacts with a particle, which is represented by $-J_k(q_k(t))$, and the meaning of the current equation is that this variation is spread continuously in a spherical propagation.

10.4.2 Morphism for the potential and the strength

We assume that ϕ is transported along a propagation curve with tangent Y, that is :

$$\phi\left(\Phi_Y(\tau,m)\right)=\sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\sum_{\beta,\gamma=0}^{3}\left[L\left(\Phi'_{Ym}(\tau,m)\right)\right]_\gamma^\beta\phi^{b\gamma}(m)\,\partial\xi_\beta\otimes\kappa_a\left(\Phi_V(\tau,m)\right) \tag{77}$$

Let us assume that $\grave{A},\mathcal{F}$ are also transported similarly along the same curve.
That is :

$\grave{A}\left(\Phi_Y(\tau,m)\right)=F_{Y\tau*}\grave{A}\left(\Phi_Y(\tau,m)\right)=\sum_{a=1}^{16}\sum_{\beta=0}^{3}\widetilde{\grave{A}_\beta^a}\left(\Phi_Y(\tau,m)\right)\kappa_a\otimes d\xi^\beta\left(\Phi_Y(\tau,m)\right)$

$\widetilde{\grave{A}_\beta^a}=\sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\sum_{\gamma,\beta=0}^{3}\left[L\left(\Phi'_{Ym}\left(-\tau,\Phi_Y(\tau,m)\right)\right)\right]_\beta^\gamma\grave{A}_\gamma^b(m)$

$\mathcal{F}\left(\Phi_Y(\tau,m)\right)=F_{Y\tau*}\mathcal{F}\left(\Phi_Y(\tau,m)\right)$

$=\sum_{a=1}^{16}\sum_{\{\alpha,\beta\}=0..3}\widetilde{\mathcal{F}_{\alpha\beta}^a}\left(\Phi_Y(\tau,m)\right)d\xi^\alpha\wedge d\xi^\beta\otimes\kappa_a\left(\Phi_Y(\tau,m)\right)$

$\widetilde{\mathcal{F}_{\alpha\beta}^a}$

$=\sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\sum_{\gamma,\beta=0}^{3}\left[L\left(\Phi'_{Ym}\left(-\tau,\Phi_Y(\tau,m)\right)\right)\right]_\alpha^\lambda\left[L\left(\Phi'_{Ym}\left(-\tau,\Phi_Y(\tau,m)\right)\right)\right]_\beta^\mu\mathcal{F}_{\lambda\mu}^b(m)$

$\left[L\left(\Phi'_{Ym}\left(-\tau,\Phi_Y(\tau,m)\right)\right)\right]=\left[L\left(\Phi'_{Ym}(\tau,m)\right)\right]^{-1}$

Let us denote $\left[L\left(\Phi_{Y'm}(\tau,m)\right)\right]=[L],\left[L\left(\Phi'_{Ym}\left(-\tau,\Phi_Y(\tau,m)\right)\right)\right]=[L']=[L]^{-1}$

$\widetilde{\grave{A}}\left(\Phi_Y(\tau,m)\right)_\beta^a=\sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\sum_{\gamma,\beta=0}^{3}[L']_\beta^\gamma\grave{A}_\gamma^b(m)$

$\widetilde{\mathcal{F}}\left(\Phi_Y(\tau,m)\right)_{\alpha\beta}^a=\sum_{a=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\left([L']^t\left[\mathcal{F}^b\right][L']\right)_\beta^\alpha$

The computed value of the current at $\Phi_Y(\tau,m)$ is then :

$\widetilde{\phi}$

$=2\sum_{a=1}^{16}\sum_{\alpha,\beta=0}^{3}\left[\widetilde{\grave{A}}_\beta,\sum_{\lambda\mu=0}^{3}g^{\alpha\lambda}\left(\Phi_Y(\tau,m)\right)g^{\beta\mu}\left(\Phi_Y(\tau,m)\right)\widetilde{\mathcal{F}}_{\lambda\mu}\right]^a\kappa_a\otimes\partial\xi_\alpha$

$=2\sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\sum_{\alpha,\beta=0}^{3}\left[\sum_{\gamma=0}^{3}[L']_\beta^\gamma\grave{A}_\gamma,\sum_{\lambda\mu=0}^{3}g^{\alpha\lambda}g^{\beta\mu}\left([L']^t[\mathcal{F}][L']\right)_\mu^\lambda\right]^b\kappa_a\otimes\partial\xi_\alpha$

$=2\sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\sum_{\alpha,\beta,\gamma,\lambda,\mu=0}^{3}[L']_\beta^\gamma\left[\grave{A}_\gamma,\left[g^{-1}\right]_\lambda^\alpha\left([L']^t[\mathcal{F}][L']\right)_\mu^\lambda\left[g^{-1}\right]_\beta^\mu\right]^b\kappa_a\otimes\partial\xi_\alpha$

$=2\sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a\sum_{\alpha,\beta,\gamma=0}^{3}[L']_\beta^\gamma\left[\grave{A}_\gamma,\left(\left[g^{-1}\right][L']^t[\mathcal{F}][L']\left[g^{-1}\right]\right)_\beta^\alpha\right]^b\kappa_a\otimes\partial\xi_\alpha$

The metric is transported along the Killing curve and :

$\left[g\left(\Phi_Y(\tau,m)\right)\right]=[L']^t[g(m)][L']$

$\left[g^{-1}\left(\Phi_Y(\tau,m)\right)\right]=[L]\left[g^{-1}(m)\right][L]^t$

$\left(\left[g^{-1}\left(\Phi_Y(\tau,m)\right)\right][L']^t[\mathcal{F}][L']\left[g^{-1}\left(\Phi_Y(\tau,m)\right)\right]\right)_\beta^\alpha$

$=\left([L]\left[g^{-1}(m)\right][L]^t[L']^t[\mathcal{F}][L'][L]\left[g^{-1}(m)\right][L]^t\right)_\beta^\alpha$

$=\left([L]\left[g^{-1}(m)\right][\mathcal{F}(m)]\left[g^{-1}(m)\right][L]^t\right)_\beta^\alpha=\sum_{\lambda\mu=0}^{3}[L]_\lambda^\alpha\left(\left[g^{-1}(m)\right][\mathcal{F}(m)]\left[g^{-1}(m)\right]\right)_\mu^\lambda\left([L]^t\right)_\beta^\mu$

$=\sum_{\lambda\mu=0}^{3}[L]_\lambda^\alpha\left[\mathcal{F}^{\lambda\mu}(m)\right][L]_\mu^\beta$

$\widetilde{\phi} = 2\sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]^a_b \sum_{\alpha,\beta,\gamma=0}^{3} [L']^\gamma_\beta \left[\grave{A}_\gamma, \sum_{\lambda\mu=0}^{3} [L]^\alpha_\lambda \left[\mathcal{F}^{\lambda\mu}(m)\right] [L]^\beta_\mu\right]^b \kappa_a \otimes \partial\xi_\alpha$

$= 2\sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]^a_b \sum_{\alpha,\beta,\gamma,\lambda,\mu=0}^{3} [L']^\gamma_\beta [L]^\beta_\mu [L]^\alpha_\lambda \left[\grave{A}_\gamma, \left[\mathcal{F}^{\lambda\mu}(m)\right]\right]^b \kappa_a \otimes \partial\xi_\alpha$

$= 2\sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]^a_b \sum_{\alpha,\gamma,\lambda=0}^{3} [L]^\alpha_\lambda \left[\grave{A}_\gamma, \left[\mathcal{F}^{\lambda\gamma}(m)\right]\right]^b \kappa_a \otimes \partial\xi_\alpha$

$= 2\sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]^a_b \sum_{\alpha,\gamma,\lambda=0}^{3} [L]^\alpha_\lambda \phi^{b\lambda}(m) \kappa_a \otimes \partial\xi_\alpha$

that is $\widetilde{\phi} = F_{Y\tau*}\phi(\Phi_Y(\tau,m))$

The assumption that ϕ is transported on a propagation curve is consistent with the assumption that $\grave{A}, \mathcal{F}$ are also transported similarly. This is consistent with the metric equation which tells that the metric is defined by the field. So we state :

$$\grave{A}(\Phi_V(\tau,m)) = \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]^a_b \sum_{\gamma,\beta=0}^{3} [L'((\Phi'_{Vm}(\tau,m)))]^\gamma_\beta \grave{A}^b_\gamma(m)\, d\xi^\beta \otimes \kappa_a(\Phi_V(\tau,m)) \quad (78)$$

$$\mathcal{F}(\Phi_V(\tau,m)) = \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]^a_b \sum_{\gamma,\beta=0}^{3} [L'((\Phi'_{Vm}(\tau,m)))]^\lambda_\alpha [L'((\Phi'_{Vm}(\tau,m)))]^\mu_\beta \mathcal{F}^b_{\lambda\mu}(m)\, d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a(\Phi_V(\tau,m)) \quad (79)$$

$$[g(\Phi_V(\tau,m))] = [L'((\Phi'_{Vm}(\tau,m)))]^t [g(m)] [L'((\Phi'_{Vm}(\tau,m)))] \quad (80)$$

$\mathcal{F} = *dK$ and $F_{Y\tau*}$ commutes with the Hodge duality and exterior differentiation, so dK transforms as a 2 form as $\mathcal{F}$ and K as $\grave{A}$.

$dK(\Phi_V(\tau,m)) = \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]^a_b \sum_{\gamma,\beta=0}^{3} [L'((\Phi'_{Vm}(\tau,m)))]^\lambda_\alpha [L'((\Phi'_{Vm}(\tau,m)))]^\mu_\beta dK^b_{\lambda\mu}(m)\, d\xi^\alpha \wedge d\xi^\beta \otimes \kappa_a(\Phi_V(\tau,m))$

The decomposition of the field along the components in the basis $\kappa_a, a = 1...16$ of T_1U is, in some way, similar to the wave lengths of the EM field.

$T_1U = \left\{(iA, V_0, iV, iW, R, X_0, iX, B), A, V_0, X_0, B \in \mathbb{R}, V, W, R, X \in \mathbb{R}^3\right\}$

The EM field corresponds to $(iA, 0, 0, 0, 0, 0, 0, 0)$, the gravitational field to $(0, 0, 0, iW, R, 0, 0, 0)$, the weak field to $(0, 0, 0, 0, R, 0, 0, 0)$, the strong field to $(iA, 0, iV, 0, R, X_0, 0, 0)$. The matrix of the adjoint map is :

$$\left[Ad_{C(S)}\right]_{16\times 16} = \begin{bmatrix} & A & V_0 & V & W & R & X_0 & X & X \\ \\ A & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ V_0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ V & 0 & 0 & J & 0 & 0 & 0 & 0 & 0 \\ W & 0 & 0 & 0 & J & 0 & 0 & 0 & 0 \\ R & 0 & 0 & 0 & 0 & J & 0 & 0 & 0 \\ X_0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & J & 0 \\ B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The structure of the field is preserved in propagation.

10.4.3 PDE for the field

The relations above hold for any propagation curve. There is a unique propagation curve starting from a point with a given spatial direction. The formula above tells how the potential is split along all the directions. However this is also given by the strength $\mathcal{F}$. Coming back to its definition : $\mathcal{F}(y) = -\mathbf{p}_u^*\left(\pounds\widehat{A}\right)(y)$

we take any vector field y on TM, that we lift y on P_U as a vector field $Y(\mathbf{p}_u(m)) = \mathbf{p}_u'(m)\, y(m)$ using the standard gauge $\mathbf{p}_u$

we transport the connection form $\widehat{A}$ along Y : $(\Phi_Y(\tau,.))^*\,\widehat{A}(\mathbf{p}_u), (\Phi_Y(-\tau,.))^*\,\widehat{A}(\mathbf{p}_u)$

we compute the variations

$\Delta_R(\tau)\,\widehat{A}(p_u) = \frac{1}{\tau}\left((\Phi_Y(\tau,.))^*\,\widehat{A}(\mathbf{p}_u) - \widehat{A}(\mathbf{p}_u)\right)$

$\Delta_L(\tau)\,\widehat{A}(p_u) = \frac{1}{\tau}\left(\widehat{A}(p_u) - (\Phi_Y(-\tau,.))^*\,\widehat{A}(p_u)\right)$

the Lie derivative of the connection form $\widehat{A}$ along Y is defined as

$\pounds_Y\widehat{A}(p_u) = \frac{d}{d\tau}(\Phi_Y(\tau,.))^*\,\widehat{A}(p_u)\,|_{\tau=0} = \lim_{\tau\to 0}\Delta_R(\tau)\,\widehat{A}(p_u) = -\lim_{\tau\to 0}\Delta_L(\tau)\,\widehat{A}(p_u)$

According to the formulas above, along a Killing vector field :

$\grave{A}(\Phi_Y(\tau,m)) = Ad_{C(S(m))}\left(\Phi_{Y\tau*}\left(\grave{A}(m)\right)\right)$

$C(S)$ is constant on the curve, but depends on the curve, then, for any propagation curve at m :

$\pounds_Y\widehat{A}(p_u) = \mathcal{F}(m)(Y(m)) = \frac{d}{d\tau}\left(Ad_{C(Y)^{-1}}\grave{A}(\Phi_Y(\tau,m))\right)|_{\tau=0} = Ad_{C(Y)^{-1}}\grave{A}'(m)\,Y(m)$

$\frac{1}{2}\sum_{\alpha,\beta=0}^{3}\mathcal{F}_{\alpha\beta}^{a}Y^{\beta}d\xi^{\alpha} = \sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\sum_{\beta=0}^{3}\left(\partial_\beta\grave{A}_\alpha^b\right)Y^\beta d\xi^\alpha$

$$\alpha = 0...3 : \frac{1}{2}\sum_{\alpha,\beta=0}^{3}\mathcal{F}_{\alpha\beta}Y^{\beta} = \left[Ad_{C(Y)^{-1}}\right]\sum_{\beta=0}^{3}\left(\partial_\beta\grave{A}_\alpha^b\right)Y^\beta \quad (81)$$

The left hand side reads, for $a = 1...16$

$\frac{1}{2}\sum_{\beta=1}^{3}\mathcal{F}_{0\beta}^{a}y^{\beta} = \sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left(\partial_0\grave{A}_0^b\right)c + \sum_{\beta=1}^{3}\left(\partial_\beta\grave{A}_0^b\right)y^\beta\right)$

$\frac{1}{2}\left(\mathcal{F}_{10}^{a}c + \mathcal{F}_{12}^{a}y^2 + \mathcal{F}_{13}^{a}y^3\right) = \sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left(\partial_0\grave{A}_1^b\right)c + \sum_{\beta=1}^{3}\left(\partial_\beta\grave{A}_1^b\right)y^\beta\right)$

$\frac{1}{2}\left(\mathcal{F}_{20}c + \mathcal{F}_{21}y^1 + \mathcal{F}_{23}y^3\right) = \sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left(\partial_0\grave{A}_2^b\right)c + \sum_{\beta=1}^{3}\left(\partial_\beta\grave{A}_2^b\right)y^\beta\right)$

$\frac{1}{2}\left(\mathcal{F}_{30}c + \mathcal{F}_{31}y^1 + \mathcal{F}_{32}y^2\right) = \sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left(\partial_0\grave{A}_3^b\right)c + \sum_{\beta=1}^{3}\left(\partial_\beta\grave{A}_3^b\right)y^\beta\right)$

In matrix form :

$$\begin{bmatrix} 0 & \mathcal{F}_{01}^a & \mathcal{F}_{02}^a & \mathcal{F}_{03}^a \\ -\mathcal{F}_{01}^a & 0 & -\mathcal{F}_{21}^a & \mathcal{F}_{13}^a \\ -\mathcal{F}_{02}^a & \mathcal{F}_{21}^a & 0 & -\mathcal{F}_{32}^a \\ -\mathcal{F}_{03}^a & -\mathcal{F}_{13}^a & \mathcal{F}_{32}^a & 0 \end{bmatrix}\begin{bmatrix} c \\ y^1 \\ y^2 \\ y^3 \end{bmatrix} = [\mathcal{F}^a][Y] = 2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left[\partial\grave{A}^b\right][Y]$$

With $\left[\partial\grave{A}^b\right] = \begin{bmatrix} \partial_0\grave{A}_0^b & \partial_1\grave{A}_0^b & \partial_2\grave{A}_0^b & \partial_3\grave{A}_0^b \\ \partial_0\grave{A}_1^b & \partial_1\grave{A}_1^b & \partial_2\grave{A}_1^b & \partial_3\grave{A}_1^b \\ \partial_0\grave{A}_2^b & \partial_1\grave{A}_2^b & \partial_2\grave{A}_2^b & \partial_0\grave{A}_2^b \\ \partial_0\grave{A}_3^b & \partial_1\grave{A}_3^b & \partial_2\grave{A}_3^b & \partial_3\grave{A}_3^b \end{bmatrix}$

$$a=1...16:[\mathcal{F}^a][Y]=2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left[\partial\grave{A}^b\right][Y] \tag{82}$$

We can choose any null, future oriented, Killing vector $Y(m)$ at m. Let us take successively the vectors $Y_j(m)=c(\varepsilon_0(m)+\varepsilon_j(m)),j=1,2,3$ with the vectors $\varepsilon_j(m)$ of the tetrad at m : $j,\beta=0,1,2,3:Y_j^\beta=c\left(P_0^\beta(m)+P_j^\beta(m)\right)$

The matrices $Ad_{C(Y_j)}$ are built with the matrices

$[J_j]_{3\times3}=1+2\lambda_j\left(\sqrt{1-\lambda_j^2}j(\widetilde{y}_j)+\lambda_j j(\widetilde{y}_j)j(\widetilde{y}_j)\right)$

with $\widetilde{y}_j$ the vector y_j expressed in the tetrad at m. Ad_{C_j} depends on a single scalar fixed parameter λ_j. And $Ad_{C(Y_j)^1}$ is computed by taking $\lambda_j\to-\lambda_j$.

$j=1,2,3:[\mathcal{F}^a][Y_j]=2\sum_{b=1}^{16}\left[Ad_{C(Y_j)^{-1}}\right]_b^a\left[\partial\grave{A}^b\right][Y_j]$

and we get the equations :

$[\mathcal{F}_w^a][Q_j]=2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\partial_0\grave{A}_0^b+\left[M_1^b\right][Q_j]\right)$

$-[\mathcal{F}_w^a]^t+j(\mathcal{F}_r^a)[Q_j]=2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left[M_2^b\right]+\left[M_3^b\right][Q_j]\right)$

with

$[Q_j]=\begin{bmatrix}Q_j^1\\Q_j^2\\Q_j^3\end{bmatrix};\left[M_1^b\right]=\begin{bmatrix}\partial_1\grave{A}_0^b & \partial_2\grave{A}_0^b & \partial_3\grave{A}_0^b\end{bmatrix};\left[M_2^b\right]=\begin{bmatrix}\partial_0\grave{A}_1^b\\\partial_0\grave{A}_2^b\\\partial_0\grave{A}_3^b\end{bmatrix};\left[M_3^b\right]=$
$\begin{bmatrix}\partial_1\grave{A}_1^b & \partial_2\grave{A}_1^b & \partial_3\grave{A}_1^b\\\partial_1\grave{A}_2^b & \partial_2\grave{A}_2^b & \partial_3\grave{A}_2^b\\\partial_1\grave{A}_3^b & \partial_2\grave{A}_3^b & \partial_3\grave{A}_3^b\end{bmatrix}$

$[\mathcal{F}_w^a]=\left(j(\mathcal{F}_r^a)[Q_j]-2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left[M_2^b\right]+\left[M_3^b\right][Q_j]\right)\right)^t$

$=\left(-j(Q_j)\left([\mathcal{F}_r^a]^t\right)-2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left[M_2^b\right]+\left[M_3^b\right][Q_j]\right)\right)^t$

$=[\mathcal{F}_r^a]j(Q_j)-2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left[M_2^b\right]+\left[M_3^b\right][Q_j]\right)^t$

$[\mathcal{F}_w^a][Q_j]=-2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\left[M_2^b\right]+\left[M_3^b\right][Q_j]\right)^t[Q_j]$

$=2\sum_{b=1}^{16}\left[Ad_{C(Y)^{-1}}\right]_b^a\left(\partial_0\grave{A}_0^b+\left[M_1^b\right][Q_j]\right)$

$\Rightarrow-([M_2]+[M_3][Q_j])^t[Q_j]=\left(\partial_0\grave{A}_0+[M_1][Q_j]\right)$

$\Rightarrow\partial_0\grave{A}_0+[M_1][Q_j]+([M_2]+[M_3][Q_j])^t[Q_j]=0$

$\partial_0\grave{A}_0+\left([M_1]+[M_2]^t\right)[Q_j]+[Q_j]^t[M_3]^t[Q_j]=0$

We get the set of PDE :

$j=1,2,3$

$\partial_0\grave{A}_0^b+\left([M_1]+[M_2]^t\right)[Q_j]+[Q_j]^t[M_3]^t[Q_j]=0$

$[\mathcal{F}_w]=[\mathcal{F}_r]j(Q_j)-2\left[Ad_{C(Y)^{-1}}\right]\left([M_2]^t+[Q_j]^t[M_3]^t\right)$

which read :

$$j=1,2,3$$
$$\partial_0 \grave{A}_0+\sum_{\gamma=1}^3\left(\partial_\gamma \grave{A}_0+\partial_0 \grave{A}_\gamma+\sum_{\beta=1}^3 Q_j^\beta \partial_\gamma \grave{A}_\beta\right) Q_j^\gamma=0$$
$$\mathcal{F}_{01}=-\mathcal{F}_{21} Q_j^2+\mathcal{F}_{13} Q_j^3-2\left[A d_{C\left(Y_j\right)^{-1}}\right]\left(\partial_0 \grave{A}_1+\sum_{\beta=1}^3 Q_j^\beta \partial_1 \grave{A}_\beta\right)$$
$$\mathcal{F}_{02}=\mathcal{F}_{21} Q_j^1-\mathcal{F}_{32} Q_j^3-2\left[A d_{C\left(Y_j\right)^{-1}}\right]\left(\partial_0 \grave{A}_2+\sum_{\beta=1}^3 Q_j^\beta \partial_2 \grave{A}_\beta\right)$$
$$\mathcal{F}_{03}=-\mathcal{F}_{13} Q_j^1+\mathcal{F}_{32} Q_j^2-2\left[A d_{C\left(Y_j\right)^{-1}}\right]\left(\partial_0 \grave{A}_3+\sum_{\beta=1}^3 Q_j^\beta \partial_3 \grave{A}_\beta\right) \tag{83}$$

The components of the tetrad are fixed parameters in these equations. We have a large freedom of gauge.

For the EM field the equations read

$\partial_0 \grave{A}_0+\sum_{\gamma=1}^3\left(\partial_\gamma \grave{A}_0+\partial_0 \grave{A}_\gamma+\sum_{\beta=1}^3 Q_j^\beta \partial_\gamma \grave{A}_\beta\right) Q_j^\gamma=0$

$\partial_1 \grave{A}_0-3 \partial_0 \grave{A}_1-2 Q_j^1 \partial_1 \grave{A}_1-Q_j^2 \partial_2 \grave{A}_1-Q_j^3 \partial_3 \grave{A}_1-Q_j^2 \partial_1 \grave{A}_2-Q_j^3 \partial_1 \grave{A}_3=0$

$\partial_2 \grave{A}_0-Q_j^1 \partial_2 \grave{A}_1-3 \partial_0 \grave{A}_2-Q_j^1 \partial_1 \grave{A}_2-2 Q_j^2 \partial_2 \grave{A}_2-Q_j^3 \partial_3 \grave{A}_2-Q_j^3 \partial_2 \grave{A}_3=0$

$\partial_3 \grave{A}_0-Q_j^1 \partial_3 \grave{A}_1-Q_j^2 \partial_3 \grave{A}_2-3 \partial_0 \grave{A}_3-Q_j^1 \partial_1 \grave{A}_3-Q_j^2 \partial_2 \grave{A}_3-2 Q_j^3 \partial_3 \grave{A}_3=0$

10.5 The metric equation

Denoting

$\left[d \grave{A}_r\right]=\left[\begin{array}{lll}\grave{A}_{32}-\grave{A}_{23} & \grave{A}_{13}-\grave{A}_{31} & \grave{A}_{21}-\grave{A}_{12}\end{array}\right]$

$\left[d \grave{A}_w\right]=\left[\begin{array}{lll}\grave{A}_{01}-\grave{A}_{10} & \grave{A}_{02}-\grave{A}_{20} & \grave{A}_{03}-\grave{A}_{30}\end{array}\right]$

$\left[d K_r\right]=\left[\begin{array}{lll}\partial_3 K_2-\partial_2 K_3 & \partial_1 K_3-\partial_3 K_1 & \partial_2 K_1-\partial_1 K_2\end{array}\right]$

$\left[d K_w\right]=\left[\begin{array}{lll}\partial_0 K_1-\partial_1 K_0 & \partial_0 K_2-\partial_2 K_0 & \partial_0 K_3-\partial_3 K_0\end{array}\right]$

and similarly :

$\left[X_w\right]_p=\left[\grave{A}_0, \grave{A}_p\right],\left[X_r\right]_1=\left[\grave{A}_3, \grave{A}_2\right],\left[X_r\right]_2=\left[\grave{A}_1, \grave{A}_3\right],\left[X_r\right]_3=\left[\grave{A}_2, \grave{A}_1\right]$

The codifferential equation is met if there is $K \in \Lambda_1\left(T M ; T_1 U\right)$ such that :

$\left[\mathcal{F}_w\right]=\left[d \grave{A}_w\right]+\left[X_w\right]=-\left[d K_r\right]\left[g_3\right] / \sqrt{\det g_3}$

$\left[\mathcal{F}_r\right]=\left[d \grave{A}_r\right]+\left[X_r\right]=\left[d K_w\right]\left[g_3\right]^{-1} \sqrt{\det g_3}$

Then the metric equation

$\left[d \grave{A}_r\right]^t[\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}=\left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t[\eta]\left[d \grave{A}_r\right]$

$\left[d \grave{A}_r\right]^t[\eta]\left[\mathcal{F}_r\right]\left[g_3\right]=\left[g_3\right]^{-1}\left[\mathcal{F}_w\right]^t[\eta]\left[d \grave{A}_w\right] \det g_3$

$\langle\mathcal{F}, \mathcal{F}\rangle=2 Tr\left(\left[d \grave{A}_r\right]^t[\eta]\left[\mathcal{F}_r\right]\left[g_3\right]\right)=2 Tr\left(\left[d \grave{A}_w\right]^t[\eta]\left[\mathcal{F}_w\right]\left[g_3\right]^{-1}\right) \det g_3$

reads with dK :

$$\left[d\grave{A}_r\right]^t[\eta][dK_r] = [dK_r]^t[\eta]\left[d\grave{A}_r\right]$$
$$\left[d\grave{A}_r\right]^t[\eta][dK_w] = [dK_r]^t[\eta]\left[d\grave{A}_w\right]$$
$$\langle\mathcal{F},\mathcal{F}\rangle = 2Tr\left(\left[d\grave{A}_r\right]^t[\eta][dK_w]\right)\sqrt{\det g_3} = 2Tr\left(\left[d\grave{A}_w\right]^t[\eta][dK_r]\right)\sqrt{\det g_3} \quad (84)$$

$\left[d\grave{A}\right]$ changes along a Killing curve as $[\mathcal{F}]$. The equations should hold along a Killing curve, that is :

$\left[d\grave{A}_r(\Phi_V(\tau,m))\right]^t[\eta][dK_r(\Phi_V(\tau,m))] = [dK_r(\Phi_V(\tau,m))]^t[\eta]\left[d\grave{A}_r(\Phi_V(\tau,m))\right]$

$\left[d\grave{A}_r(\Phi_V(\tau,m))\right]^t[\eta][dK_w(\Phi_V(\tau,m))] = [dK_r(\Phi_V(\tau,m))]^t[\eta]\left[d\grave{A}_w(\Phi_V(\tau,m))\right]$

With the previous formula :

$\left[\left[\widetilde{\Lambda}_r\right],\left[\widetilde{\Lambda}_w\right]\right]|_{\Phi_V(\tau,m)} = [[\Lambda_r],[\Lambda_w]]|_{\Phi_V(\tau,m)}[L_{L'}(\Phi'_{Vm}(\tau,m))]$

with the matrices

$[L_{L'}(\Phi_V(\tau,m))]_{6\times 6} = \begin{bmatrix} [l]^t(\det l)^{-1} & 0 \\ 0 & [l]^{-1} \end{bmatrix}$

$[L(\Phi'_{Vm}(\tau,m))]_{3\times 3} = \begin{bmatrix} 1 & 0 \\ 0 & [l] \end{bmatrix}$

$\left[d\grave{A}_r(\Phi_V(\tau,m))\right] = \left[Ad_{C(S(m))^{-1}}\right]\left[d\grave{A}_r(m)\right][l]^t(\det l)^{-1}$

$\left[d\grave{A}_w(\Phi_V(\tau,m))\right] = \left[Ad_{C(S(m))^{-1}}\right]\left[d\grave{A}_w(m)\right][l]^{-1}$

$[dK_r(\Phi_V(\tau,m))] = \left[Ad_{C(S(m))^{-1}}\right][dK_r(m)][l]^t(\det l)^{-1}$

$[dK_w(\Phi_V(\tau,m))] = \left[Ad_{C(S(m))^{-1}}\right][dK_w(m)][l]^{-1}$

and accounting for :

$S\in Spin(3,1)\Rightarrow$

$\left[Ad_{C(S(m))}\right]^t = [\eta]\left[Ad_{C(S(m))^t}\right][\eta]$

$\left[Ad_{(C(S(m)))^t}\right][\eta][\eta]\left[Ad_{C(S(m))}\right] = \left[Ad_{(C(S(m)))^t\cdot C(S(m))}\right] = 1$

The first equation reads :

$\left[d\grave{A}_r(\Phi_V(\tau,m))\right]^t[\eta][dK_r(\Phi_V(\tau,m))] = [l](\det l)^{-1}\left[d\grave{A}_r(m)\right]^t[\eta][dK_r(m)][l]^t(\det l)^{-1}$

$[dK_r(\Phi_V(\tau,m))]^t[\eta]\left[d\grave{A}_r(\Phi_V(\tau,m))\right] = [l](\det l)^{-1}[dK_r(m)]^t[\eta]\left[d\grave{A}_r(m)\right][l]^t(\det l)^{-1}$

$\left[d\grave{A}_r(m)\right]^t[\eta][dK_r(m)] = [dK_r(m)]^t[\eta]\left[d\grave{A}_r(m)\right]$

The second equation reads :

$\left[d\grave{A}_r(\Phi_V(\tau,m))\right]^t[\eta][dK_w(\Phi_V(\tau,m))] = [l](\det l)^{-1}\left[d\grave{A}_r(m)\right]^t[\eta][dK_w(m)][l]^{-1}$

$[dK_r(\Phi_V(\tau,m))]^t[\eta]\left[d\grave{A}_w(\Phi_V(\tau,m))\right] = [l](\det l)^{-1}[dK_r(m)]^t[\eta]\left[d\grave{A}_w(m)\right][l]^{-1}$

$\left[d\grave{A}_r(m)\right]^t[\eta][dK_w(m)] = [dK_r(m)]^t[\eta]\left[d\grave{A}_w(m)\right]$

So the metric equation is preserved by propagation.

These matricial equations are quite strong, and should be met at each point and whatever the spatial chart. The one form K is not unique. With these equations it is logical to assume a linear relation between $\left[d\grave{A}_r\right], [dK_r]$ and $\left[d\grave{A}_w\right], [dK_w]$.

Let $[M], [N]$ 16×16 matrices acting on T_1U which is a real vector space with the real basis κ_a. It suffices that the matrices are real.

$[dK_w] = [N]\left[d\grave{A}_w\right], [dK_r] = [M]\left[d\grave{A}_r\right]$

$\left[d\grave{A}_r\right]^t [\eta][M]\left[d\grave{A}_r\right] = \left[d\grave{A}_r\right]^t [M]^t [\eta]\left[d\grave{A}_r\right]$

$\left[d\grave{A}_r\right]^t [\eta][N]\left[d\grave{A}_w\right] = \left[d\grave{A}_r\right]^t [M]^t [\eta]\left[d\grave{A}_w\right]$

$[M], [M]$ must be such that : $[M]^t = [\eta][M][\eta] = [\eta][N][\eta], [N]^t = [\eta][N][\eta], [M] = [N]$

that is :

$$[dK] = [N]\left[d\grave{A}\right]; [N]^t = [\eta][N][\eta] \tag{85}$$

Then the metric equation is always met.

Conversely if $[dK(m)] = [N(m)]\left[d\grave{A}(m)\right]$ then the transformations rules along a Killing curve are met if

$\left[d\grave{A}_r(\Phi_V(\tau, m))\right] = \left[Ad_{C(S(m))^{-1}}\right]\left[d\grave{A}_r(m)\right][l]^t(\det l)^{-1}$

$\left[d\grave{A}_w(\Phi_V(\tau, m))\right] = \left[Ad_{C(S(m))^{-1}}\right]\left[d\grave{A}_w(m)\right][l]^{-1}$

$[dK_r(\Phi_V(\tau, m))] = \left[Ad_{C(S(m))^{-1}}\right][N(m)]\left[d\grave{A}_r(m)\right][l]^t(\det l)^{-1}$

$= [N(\Phi_V(\tau, m))]\left[d\grave{A}_r(\Phi_V(\tau, m))\right]$

$[dK_w(\Phi_V(\tau, m))] = \left[Ad_{C(S(m))^{-1}}\right][N(m)]\left[d\grave{A}_w(m)\right][l]^{-1}$

$= [N(\Phi_V(\tau, m))]\left[d\grave{A}_w(\Phi_V(\tau, m))\right]$

$\left[Ad_{C(S(m))^{-1}}\right][N(m)]\left[d\grave{A}_r(m)\right][l]^t(\det l)^{-1}$

$= [N(\Phi_V(\tau, m))]\left[Ad_{C(S(m))^{-1}}\right]\left[d\grave{A}_r(m)\right][l]^t(\det l)^{-1}$

$\left[Ad_{C(S(m))^{-1}}\right][N]\left[d\grave{A}_w(m)\right][l]^{-1} = [N(\Phi_V(\tau, m))]\left[Ad_{C(S(m))^{-1}}\right]\left[d\grave{A}_w(m)\right][l]^{-1}$

$\left[Ad_{C(S(m))^{-1}}\right][N(m)] = [N(\Phi_V(\tau, m))]\left[Ad_{C(S(m))^{-1}}\right]$

$[N(\Phi_V(\tau, m))] = \left[Ad_{C(S(m))^{-1}}\right][N(m)]\left[Ad_{C(S(m))^{-1}}\right]^t = \left[Ad_{C(S(m))^{-1}}\right][N(m)][\eta]\left[Ad_{C(S(m))}\right][\eta]$

$C(S(m))$ is constant along a propagation curve, so $[N(m)][\eta]$ is constant and commute with $\left[Ad_{C(S(m))^{-1}}\right]$

$\left[Ad_{C(S(m))}\right][N(m)][\eta] = [N(m)][\eta]\left[Ad_{C(S(m))}\right]$

The only elements which commute with all $C(S(m))$ are the scalars, and then $[Ad_g] = I$

The solution for K is not unique, the choice of N meeting these conditions is somewhat arbitrary and the sensible choice is $[N] = I$

$$[dK] = \left[d\grave{A}\right] \tag{86}$$

Replacing dK by its value from $\grave{A}$:
$[\mathcal{F}_w] = \left[d\grave{A}_w\right] + [X_w] = -[N]\left[d\grave{A}_r\right][g_3]/\sqrt{\det g_3}$
$[\mathcal{F}_r] = \left[d\grave{A}_r\right] + [X_r] = [N]\left[d\grave{A}_w\right][g_3]^{-1}\sqrt{\det g_3}$
$\left[d\grave{A}_w\right] + [N]\left[d\grave{A}_r\right][g_3]/\sqrt{\det g_3} = -[X_w]$
$[N]\left[d\grave{A}_w\right][g_3]^{-1}\sqrt{\det g_3} - \left[d\grave{A}_r\right] = [X_r]$

We have 6×16 first order PDE, linear in the derivative, for $\grave{A}$ with the parameter $[N]^t = [\eta][N][\eta]$:
$\left[d\grave{A}_w\right] = \left(1 + [N]^2\right)^{-1}\left(-[X_w] + [N][X_r][g_3]/\sqrt{\det g_3}\right)$
$\left[d\grave{A}_r\right] = -[N]^{-1}\left(1 + [N]^2\right)^{-1}[N][X_r] + [N]^{-1}\left(\left(1 + [N]^2\right)^{-1} - 1\right)[X_w][g_3]^{-1}\sqrt{\det g_3}$
With $N = I$

$$\begin{aligned}\left[d\grave{A}_w\right] &= \tfrac{1}{2}\left(-[X_w] + [X_r][g_3]/\sqrt{\det g_3}\right)\\ \left[d\grave{A}_r\right] &= -\tfrac{1}{2}\left([X_r] + [X_w][g_3]^{-1}\sqrt{\det g_3}\right)\end{aligned} \tag{87}$$

The metric is defined by the value of the field, as expected. For the EM field one retrieves $d\grave{A}^1 = 0$.

We have the identity $\sum_{p=1}^3 \left[d\grave{A}_r\right]_p = 0 \Rightarrow$
$\sum_{p=1}^3 \left(-[N]^{-1}\left(1 + [N]^2\right)^{-1}[N][X_r]_p + \sum_{q=1}^3 [N]^{-1}\left(\left(1 + [N]^2\right)^{-1} - 1\right)[X_w]_q g^{pq}\sqrt{\det g_3}\right)$
$= 0$
$\sum_{p=1}^3 \left(-[N][X_r]_p + \sum_{q=1}^3 \left(1 - \left(1 + [N]^2\right)\right)[X_w]_q g^{pq}\sqrt{\det g_3}\right) = 0$

$$\sum_{p=1}^{3}\left([X_r]_p + \sum_{q=1}^{3}[X_w]_q\, g^{pq}\sqrt{\det g_3}\right) = 0 \tag{88}$$

We have a set of 16 linear equations in g^{pq}.

The results still hold in the Geometry of Special Relativity : Killing curves are straight lines, $[g] = [\eta]$ and $S = 1, [N] = I$. The differential equations read :
$\left[d\grave{A}_w\right] = -\frac{1}{2}[X_w] + \frac{1}{2}[X_r]$
$\left[d\grave{A}_r\right] = -\frac{1}{2}[X_r] - \frac{1}{2}[X_w]$
$[dK] = \left[d\grave{A}\right]$

10.5.1 Chern-Weil theorem

The Chern-Weil theorem reads :

$Tr\left[\mathcal{F}_r\right]^t[\eta]\left[\mathcal{F}_w\right] = -Tr\left[dK_w\right]^t[\eta]\left[dK_r\right] = -Tr\left[d\grave{A}_w\right]^t[N]^t[\eta][N]\left[d\grave{A}_r\right] = -Tr\left[d\grave{A}_w\right]^t[\eta][N]^2\left[d\grave{A}_r\right]$

If $[N]=I : Tr\left[d\grave{A}_w\right]^t[\eta]\left[d\grave{A}_r\right]$ does not depend on the field.

$Tr\left[d\grave{A}_w\right]^t[\eta]\left[d\grave{A}_r\right] = -\frac{1}{4}\left(-\left[X_w\right]+\left[X_r\right]\left[g_3\right]/\sqrt{\det g_3}\right)^t[\eta]\left(\left(\left[X_r\right]+\left[X_w\right]\left[g_3\right]^{-1}\sqrt{\det g_3}\right)\right)$
$= -\frac{1}{4}Tr\{-\left[X_w\right]^t[\eta]\left[X_r\right]-\left[X_w\right]^t[\eta]\left[X_w\right]\left[g_3\right]^{-1}\sqrt{\det g_3}$
$+\left[g_3\right]\left[X_r\right]^t[\eta]\left[X_r\right]/\sqrt{\det g_3}+\left[g_3\right]\left[X_r\right]^t[\eta]\left[X_w\right]\left[g_3\right]^{-1}\}$
$= -\frac{1}{4}\{-Tr\left[X_w\right]^t[\eta]\left[X_r\right]+Tr\left[g_3\right]\left[X_r\right]^t[\eta]\left[X_w\right]\left[g_3\right]^{-1}$
$-Tr\left(\left[X_w\right]^t[\eta]\left[X_w\right]\left[g_3\right]^{-1}\right)\sqrt{\det g_3}+Tr\left(\left[g_3\right]\left[X_r\right]^t[\eta]\left[X_r\right]\right)/\sqrt{\det g_3}\}$
$= -\frac{1}{4}\left\{-Tr\left(\left[X_w\right]^t[\eta]\left[X_w\right]\left[g_3\right]^{-1}\right)\sqrt{\det g_3}+Tr\left(\left[g_3\right]\left[X_r\right]^t[\eta]\left[X_r\right]\right)/\sqrt{\det g_3}\right\}$

11 One and two particles systems

It gives the opportunity to show how the previous material can be used to get a good understanding of the evolution of the system.

11.1 One particle

We have 2 cases.

i) The hypothetical model of "one free particle in its own field". We can choose the particle as the observer : then by definition the state of the particle fixes the gauge, $u=1$ and $\psi=\psi_0$, which means that the field is null.

ii) If :

- at $t=0$ the particle has another state, then the initial field is not null with respect to an observer, it has a fixed value over $\Omega_3(0)$ and propagates
- the particle is subject to an external field

then part of the variables are known with respect to a given observer. The state of the particle or the value of the field adjusts to the known conditions.

11.2 Two particles

The interesting case is that of 2 free particles subject only to their mutual field.

With free particles we can take the first particle as location of the observer. Then

$q_1(t)=\varphi_o(ct,x_1)$ with $x_1=Ct, V_1(t)=c\varepsilon_0, u_1(t), J_1(t)=Ad_{u_1(t)}Q_1\otimes c\varepsilon_0$

$q_2(t)=\varphi_o(ct,x_2(t))$ with $u_2(t), V_2(t)=\frac{dq_2}{dt}, J_2(t)=Ad_{u_2(t)}\widehat{Q}_2\otimes V_2(t)$

The freedom of gauge can be exercised by fixing the value of the field at the location of the observer : $\grave{A}(q_1(t))=A_0, \phi(q_1(t))=\phi_0$

i) We will implement the general results with $O(t)=q_1(t), O(t)=q_2(t)$

$\forall O(t) : \phi(O(t)) = -\frac{1}{\Omega_0} \int_0^{\theta_M} \sum_{k=1}^{p} F_{Y_k \theta *}(J_k)(O(t)) |_{q_k(t-\theta) \in S(O(t-\theta),\theta)} d\theta$

that is the integral over θ of the push forward by Y_k of $J_k(q_k(t-\theta))$ from $q_k(t-\theta) \in S(O(t-\theta),\theta) \subset \Omega_3(t-\theta)$ to $O(t) = \Phi_{Y_k}(\theta, q_k(t-\theta))$. In this integral θ is an argument, but in its computation it becomes a variable $\theta(t)$. It is a measure of the distance between the particles. To keep it simple we will replace $t-\theta(t)$ by $\theta(t) < t$.

First case : $O(t) = q_1(t), O(t-\theta) = \varphi_o(c(t-\theta), x_1) \Leftrightarrow O(\theta(t)) = \varphi_o(c\theta(t), x_1)$

1st sphere $S(q_1(t), 0) : J_1(t)$

2nd sphere $S(q_1(\theta_2(t)), t-\theta_2(t))$: there is a unique propagation curve reaching $q_1(t)$, it starts from $q_2(\theta_2(t))$ with the vector Y_2 :

$q_2(\theta_2(t)) \to q_1(t) = \Phi_{Y_2}(t-\theta_2(t), q_2(\theta_2(t)))$

$J(q_2(\theta_2(t))) \to F_{Y_2(t-\theta_2)*}(J_2(q_2(\theta_2(t))))(q_1(t)) = \left[Ad_{C_2(t)}\right] Ad_{u_2(\theta_2(t))} Q_2 \otimes [L_2(t)] V_2(\theta_2(t))$

where $C_2(t), [L_2(t)] = \left[L\left(\Phi'_{Y_2 m}(t-\theta_2(t), q_2(\theta_2(t)))\right)\right]$ depend on t

$\phi(q_1(t)) = -\frac{1}{\Omega_0}\left(Ad_{u_1(t)} Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(t)}\right] Ad_{u_2(\theta_2(t))} Q_2 \otimes [L_2(t)] V_2(\theta_2(t))\right)$

Second case : $O(t) = q_2(t), O(t-\theta) = \varphi_o(c(t-\theta), x_2(t)) \Leftrightarrow O(\theta(t)) = \varphi_o(c\theta(t), x_1(t))$

1st sphere $S(q_2(t), 0) : J_2(t)$

2nd sphere $S(\varphi_o(c\theta_2(t), x_2(t)), t-\theta_1(t))$: there is a unique propagation curve reaching $q_2(t)$, it starts from $q_1(\theta_1(t))$ with the vector Y_1 :

$q_1(\theta_1(t)) \to q_2(t) = \Phi_{Y_1}(t-\theta_1, q_1(\theta_1(t)))$

$J_1(q_1(\theta_1(t))) \to F_{Y_1(t-\theta_1)*}(J_1(q_1(\theta_1(t))))(q_2(t)) = \left[Ad_{C_1(t)}\right] Ad_{u_1(\theta_1(t))} Q_1 \otimes [L_1(t)] V_1(\theta_1(t)) = \left[Ad_{C_1(t)}\right] Ad_{u_1(\theta_1(t))} Q_1 \otimes c\varepsilon_0$

where $C_1(t), [L_1(t)] = \left[L\left(\Phi'_{Y_1 m}(t-\theta_1, q_1(\theta_1(t)))\right)\right]$ depend on t and $[L_1(t)]\varepsilon_0 = \varepsilon_0$

$\phi(q_2(t)) = -\frac{1}{\Omega_0}\left(\left[Ad_{u_2(t)}\right] Q_2 \otimes V_2(t) + \left[Ad_{C_1(t)}\right] Ad_{u_1(\theta_1(t))} Q_1 \otimes c\varepsilon_0\right)$

ii) ϕ is symmetric on the propagation curves :

$Y_2 : q_2(\theta_2(t)) \to q_1(t) = \Phi_{Y_2}(t-\theta_2, q_2(\theta_2(t)))$

$\Rightarrow \phi(q_1(t)) = \phi(\Phi_{Y_2}(t-\theta_2, q_2(\theta_2(t)))) = F_{Y_2(t-\theta_2)*}(q_2(\theta_2(t)))(q_1(t)) = \left[Ad_{C_2(t)}\right][L_2(t)]\phi(q_2(\theta_2(t)))$

$\phi(q_2(t)) = -\frac{1}{\Omega_0}\left(\left[Ad_{u_2(t)}\right] Q_2 \otimes V_2(t) + \left[Ad_{C_1(t)}\right] Ad_{u_1(\theta_1(t))} Q_1 \otimes c\varepsilon_0\right)$

$\phi(q_2(\theta_2(t))) = -\frac{1}{\Omega_0}\left(\left[Ad_{u_2(\theta_2(t))}\right] Q_2 \otimes V_2(\theta_2(t)) + \left[Ad_{C_1(\theta_2(t))}\right]\left[Ad_{u_1(\theta_1 \circ \theta_2(t))}\right] Q_1 \otimes c\varepsilon_0\right)$

$\phi(q_1(t)) =$

$-\frac{1}{\Omega_0}\left(\left[Ad_{C_2(t)}\right]\left[Ad_{u_2(\theta_2(t))}\right] Q_2 \otimes [L_2(t)] V_2(\theta_2(t)) + \left[Ad_{C_2(t)}\right]\left[Ad_{C_1(\theta_2(t))}\right]\left[Ad_{u_1(\theta_1 \circ \theta_2(t))}\right] Q_1 \otimes c\varepsilon_0\right)$

$Y_1 : q_1(\theta_1(t)) \to q_2(t) = \Phi_{Y_1}(t-\theta_1, q_1(\theta_1(t)))$

$\Rightarrow \phi(q_2(t)) = \phi(\Phi_{Y_1}(t-\theta_1, q_1(\theta_1(t)))) = F_{Y_1(t-\theta_1)*}(\phi(q_1(\theta_1(t))))(q_2(t)) = \left[Ad_{C_1(t)}\right][L_1(t)]\phi(q_1(\theta_1(t)))$

$\phi(q_1(t)) = -\frac{1}{\Omega_0}\left(Ad_{u_1(t)} Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(t)}\right] Ad_{u_2(\theta_2(t))} Q_2 \otimes [L_2(t)] V_2(\theta_2(t))\right)$

$\phi(q_1(\theta_1(t)))$

$= -\frac{1}{\Omega_0}\left(Ad_{u_1(\theta_1(t))} Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(\theta_1(t))}\right]\left[Ad_{u_2(\theta_2 \circ \theta_1(t))}\right] Q_2 \otimes [L_2(\theta_1(t))] V_2(\theta_2 \circ \theta_1(t))\right)$

$\phi(q_2(t)) =$
$-\frac{1}{\Omega_0}(\left[Ad_{C_1(t)}\right]\left[Ad_{u_1(\theta_1(t))}\right]Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_1(t)}\right]\left[Ad_{C_2(\theta_1(t))}\right]\left[Ad_{u_2(\theta_2\circ\theta_1(t))}\right]Q_2$
$\otimes [L_1(t)][L_2(\theta_1(t))]V_2(\theta_2\circ\theta_1(t)))$
Then
$\phi(q_1(t)) = -\frac{1}{\Omega_0}\left(Ad_{u_1(t)}Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(t)}\right]Ad_{u_2(\theta_2(t))}Q_2 \otimes [L_2(t)]V_2(\theta_2(t))\right)$
$= -\frac{1}{\Omega_0}\{\left[Ad_{C_2(t)}\right]\left[Ad_{u_2(\theta_2(t))}\right]Q_2 \otimes [L_2(t)]V_2(\theta_2(t))$
$+ \left[Ad_{C_2(t)}\right]\left[Ad_{C_1(\theta_2(t))}\right]\left[Ad_{u_1(\theta_1\circ\theta_2(t))}\right]Q_1 \otimes c\varepsilon_0\}$
$Ad_{u_1(t)}Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(t)}\right]Ad_{u_2(\theta_2(t))}Q_2 \otimes [L_2(t)]V_2(\theta_2(t))$
$= \left[Ad_{C_2(t)}\right]\left[Ad_{u_2(\theta_2(t))}\right]Q_2 \otimes [L_2(t)]V_2(\theta_2(t))$
$+ \left[Ad_{C_2(t)}\right]\left[Ad_{C_1(\theta_2(t))}\right]\left[Ad_{u_1(\theta_1\circ\theta_2(t))}\right]Q_1 \otimes c\varepsilon_0$
$Ad_{u_1(t)}Q_1 \otimes c\varepsilon_0 = \left[Ad_{C_2(t)}\right]\left[Ad_{C_1(\theta_2(t))}\right]\left[Ad_{u_1(\theta_1\circ\theta_2(t))}\right]Q_1 \otimes c\varepsilon_0$

$$u_1(t) = C_2(t)\cdot C_1(\theta_2(t))\cdot u_1(\theta_1\circ\theta_2(t)) \tag{89}$$

$\phi(q_2(t)) = -\frac{1}{\Omega_0}\left(\left[Ad_{u_2(t)}\right]Q_2 \otimes V_2(t) + \left[Ad_{C_1(t)}\right]Ad_{u_1(\theta_1(t))}Q_1 \otimes c\varepsilon_0\right)$
$= -\frac{1}{\Omega_0}(\left[Ad_{C_1(t)}\right]\left[Ad_{u_1(\theta_1(t))}\right]Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_1(t)}\right]\left[Ad_{C_2(\theta_1(t))}\right]\left[Ad_{u_2(\theta_2\circ\theta_1(t))}\right]Q_2$
$\otimes [L_1(t)][L_2(\theta_1(t))]V_2(\theta_2\circ\theta_1(t)))$
$\left[Ad_{u_2(t)}\right]Q_2 \otimes V_2(t) + \left[Ad_{C_1(t)}\right]Ad_{u_1(\theta_1(t))}Q_1 \otimes c\varepsilon_0 = \left[Ad_{C_1(t)}\right]\left[Ad_{u_1(\theta_1(t))}\right]Q_1 \otimes$
$c\varepsilon_0 + \left[Ad_{C_1(t)}\right]\left[Ad_{C_2(\theta_1(t))}\right]\left[Ad_{u_2(\theta_2\circ\theta_1(t))}\right]Q_2 \otimes [L_1(t)][L_2(\theta_1(t))]V_2(\theta_2\circ\theta_1(t))$
$\left[Ad_{u_2(t)}\right]Q_2 \otimes V_2(t) = \left[Ad_{C_1(t)}\right]\left[Ad_{C_2(\theta_1(t))}\right]\left[Ad_{u_2(\theta_2\circ\theta_1(t))}\right]Q_2 \otimes [L_1(t)][L_2(\theta_1(t))]V_2(\theta_2\circ\theta_1(t))$

$$u_2(t) = C_1(t)\cdot C_2(\theta_1(t))\cdot u_2(\theta_2\circ\theta_1(t)) \tag{90}$$

$$V_2(t) = [L_1(t)][L_2(\theta_1(t))]V_2(\theta_2\circ\theta_1(t)) \tag{91}$$

The obvious solution is :

$$\theta_1\circ\theta_2 = Id = \theta_2\circ\theta_1 \tag{92}$$

$q_1(t) = \Phi_{Y_2}(t-\theta_2(t), q_2(\theta_2(t))) \Rightarrow q_1(\theta_1(t)) = \Phi_{Y_2}(\theta_1(t)-t, q_2(t)) \Leftrightarrow q_2(t) = \Phi_{Y_2}(t-\theta_1(t), q_1(\theta_1(t)))$
$q_2(t) = \Phi_{Y_1}(t-\theta_1(t), q_1(\theta_1(t))) \Rightarrow q_2(\theta_2(t)) = \Phi_{Y_1}(\theta_2(t)-t, q_1(t)) \Leftrightarrow q_1(t) = \Phi_{Y_1}(t-\theta_2(t), q_2(\theta_2(t)))$
then :

$$C_2(t)\cdot C_1(\theta_2(t)) = Id = C_1(t)\cdot C_2(\theta_1(t)) \Leftrightarrow C_2(t) = C_1(\theta_2(t))^{-1} \tag{93}$$

$$[L_1(t)][L_2(\theta_1(t))] = Id \Leftrightarrow [L_2(t)] = [L_1(\theta_2(t))]^{-1} \tag{94}$$

The quantities $c(t-\theta_1(t)), c(t-\theta_2(t))$ are measures of the distance between the particles, they are not necessarily equal (the curves are not the same). From $\theta_1\circ\theta_2(t) = t$ we have $\frac{d\theta_1}{dt}\frac{d\theta_2}{dt} = 1 : \theta_1, \theta_2$ are not constant and vary in the same way with t. So either we have an attraction : $\theta_k(t) \to 0$ or a repulsion. In the first case we have a collision, and in the second case the particles get back their freedom and do not interact any longer.

iii)) Moreover :
$\phi(q_1(t)) = \phi_0 =$
$-\frac{1}{\Omega_0}\left(Ad_{u_1(t)}Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(t)}\right] Ad_{u_2(\theta_2(t))}Q_2 \otimes [L_2(t)] V_2(\theta_2(t))\right)$
$\Rightarrow Ad_{u_1(t)}Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(t)}\right] Ad_{u_2(\theta_2(t))}Q_2 \otimes [L_2(t)] V_2(\theta_2(t)) = -\phi_0\Omega_0|_{q_1(t)}$
At $\theta_1(t)$: $Ad_{u_1(\theta_1)}Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_2(\theta_1)}\right] Ad_{u_2(t)}Q_2 \otimes [L_2(\theta_1)] V_2(t) = -\phi_0\Omega_0|_{q_1(\theta_1(t))}$
$Ad_{u_1(\theta_1)}Q_1 \otimes c\varepsilon_0 + \left[Ad_{C_1(t)^{-1}}\right] Ad_{u_2(t)}Q_2 \otimes [L_1(t)]^{-1} V_2(t) = -\phi_0\Omega_0|_{q_1(\theta_1(t))}$
This tensor is defined at $q_1(\theta_1(t))$. By transport with $Y_1 : q_1(\theta_1(t)) \to q_2(t) = \Phi_{Y_1}(t-\theta_1, q_1(\theta_1(t)))$
$Ad_{u_1(\theta_1)}Q_1 \otimes c\varepsilon_0 \to \left[Ad_{C_1(t)}\right] Ad_{u_1(\theta_1)}Q_1 \otimes [L_1(t)] c\varepsilon_0 = \left[Ad_{C_1(t)}\right]\left[Ad_{u_1(\theta_1)}\right] Q_1 \otimes c\varepsilon_0$
$\left[Ad_{C_1(t)^{-1}}\right] Ad_{u_2(t)}Q_2 \otimes [L_1(t)]^{-1} V_2(t)$
$\to Ad_{u_2(t)}Q_2 \otimes V_2(t)$
$J_2(q_2(t)) = Ad_{u_2(t)}Q_2 \otimes V_2(t)$ is a decomposable tensor and $J_2(q_2(t)) = F_{Y_1(t-\theta_1)*}J_0$ where J_0 is a fixed tensor. That we can write :

$$J_2(q_2(t)) = Ad_{u_2(t)}Q_2 \otimes V_2(t) = \left[Ad_{C_1(t)}\right] Q_0 \otimes [L_1(t)] V_0|_{q_2(t)} \tag{95}$$

There is $T \in U : Q_0 = \epsilon Ad_T Q_2$ where $\epsilon = \pm 1$ to account for antiparticles ($Ad \neq -Id$) and :

$$\begin{array}{c} u_2(t) = C_1(t) \cdot T \\ V_2(t) = [L_1(t)] V_0 \end{array} \tag{96}$$

iv) The potential is similarly transported along the propagation curves.
$\grave{A}(\Phi_V(\tau,m)) = \sum_{a,b=1}^{16}\left[Ad_{C(S(m))}\right]_b^a \sum_{\gamma,\beta=0}^{3}\left[L'\left(\left(\Phi'_{Vm}(\tau,m)\right)\right)\right]_\beta^\gamma \grave{A}_\gamma^b(m)\, d\xi^\beta \otimes \kappa_a(\Phi_V(\tau,m))$
$Y_2 : q_2(\theta_2(t)) \to q_1(t) = \Phi_{Y_2}(t-\theta_2, q_2(\theta_2(t)))$
$\Rightarrow \grave{A}(q_1(t)) = \grave{A}(\Phi_{Y_2}(t-\theta_2, q_2(\theta_2(t)))) = F_{Y_2(t-\theta_2)*}\grave{A}(q_1(t)) = \left[Ad_{C_2(t)}\right]\grave{A}(q_2(\theta_2(t)))[L'_2(t)]$ with $[L'_2(t)] = [L_2(t)]^{-1}$

$$\grave{A}(q_1(t)) = \left[Ad_{C_2(t)}\right]\left[\grave{A}(q_2(\theta_2(t)))\right][L_2(t)]^{-1} \tag{97}$$

$Y_1 : q_1(\theta_1(t)) \to q_2(t) = \Phi_{Y_1}(t-\theta_1, q_1(\theta_1(t)))$
$\Rightarrow \grave{A}(q_2(t)) = \grave{A}(\Phi_{Y_1}(t-\theta_1, q_1(\theta_1(t)))) = F_{Y_1(t-\theta_1)*}\grave{A}(q_2(t)) = \left[Ad_{C_1(t)}\right][L'_1(t)]\grave{A}(q_1(\theta_1(t)))$

$$\left[\grave{A}(q_2(t))\right] = \left[Ad_{C_1(t)}\right]\left[\grave{A}(q_1(\theta_1(t)))\right][L_1(t)]^{-1} \tag{98}$$

and we can check that :
$\grave{A}(q_2(\theta_2(t))) = \left[Ad_{C_1(\theta_2)}\right]\left[\grave{A}(q_1(\theta_1 \circ \theta_2(t)))\right][L_1(\theta_2)]^{-1}$
$\grave{A}(q_1(t)) = \left[Ad_{C_2(t)}\right]\left[Ad_{C_1(\theta_2)}\right]\left[\grave{A}(q_1(\theta_1 \circ \theta_2(t)))\right][L_1(\theta_2)]^{-1}[L_2(t)]^{-1}$

Moreover $\grave{A}\left(q_1\left(t\right)\right)=A_0=Ct$

$$\left[\grave{A}\left(q_2\left(t\right)\right)\right]=\left[Ad_{C_1(t)}\right]\left[A_0\right]\left[L_1\left(t\right)\right]^{-1} \tag{99}$$

v) The equation for the state of the particles imply :
$\left\langle Ad_{u_k}\widehat{Q}_k,\left[\grave{A}\left(V_k\right),\frac{du_k}{dt}\cdot u_k^{-1}\right]+\grave{A}\left(\frac{dV_k}{dt}\right)\right\rangle_H|_{q_k(t)}=0$
For $k=2$:
$Ad_{u_2}\widehat{Q}_2=Ad_{C_1}Ad_T\widehat{Q}_2=Ad_{C_1}Q_0$
$\frac{du_2}{dt}\cdot u_2^{-1}|q_2\left(t\right)=\frac{d}{dt}\left(C_1\left(t\right)\cdot T\right)\cdot T^{-1}\cdot C_1^{-1}=\frac{dC_1}{dt}C_1^{-1}=Ad_{C_1}\left(C_1^{-1}\cdot\frac{dC_1}{dt}\right)$
$\grave{A}\left(V_2\right)=\left[\grave{A}\left(q_2\left(t\right)\right)\right]\left[L_1\right]V_0$
$\left[\grave{A}\left(q_2\left(t\right)\right)\right]=\left[Ad_{C_1(t)}\right]\left[\grave{A}\left(q_1\left(\theta_1\left(t\right)\right)\right)\right]\left[L_1\left(t\right)\right]^{-1}$
$\grave{A}\left(V_2\right)=\left[Ad_{C_1(t)}\right]\left[\grave{A}\left(q_1\left(\theta_1\left(t\right)\right)\right)\right]V_0=\left[Ad_{C_1(t)}\right]\left[A_0\right]V_0$
The derivation $\frac{dV_2}{dt}$ is taken with a constant tetrad :
$V^{\alpha}=cP_0^{\alpha}+2\frac{\sinh\mu_w\cosh\mu_w}{1+2(\cosh\mu_w)^2}\sum_{j=1}^{3}\frac{W^j}{\mu_w}P_j^{\alpha}.$
$\left[V_2\left(t\right)\right]=\left[P\left(q_2\left(t\right)\right)\right]\left[\widetilde{V}_2\left(t\right)\right]=\left[L_1\left(t\right)\right]\left[V_0\right]$
$\left[\widetilde{V}_2\left(t\right)\right]=\left[P'\left(q_2\left(t\right)\right)\right]\left[L_1\left(t\right)\right]\left[V_0\right]$
$\left[\frac{d\widetilde{V}_2}{dt}\right]=\left[P'\left(q_2\left(t\right)\right)\right]\frac{d}{dt}\left[L_1\left(t\right)\right]\left[V_0\right]$
$\frac{dV_2}{dt}=\left[P\left(q_2\left(t\right)\right)\right]\left[\frac{d\widetilde{V}_2}{dt}\right]=\frac{d}{dt}\left[L_1\left(t\right)\right]\left[V_0\right]$
$\grave{A}\left(\frac{dV_k}{dt}\right)|_{q_2(t)}=\left[\grave{A}\left(q_2\left(t\right)\right)\right]\frac{d}{dt}\left[L_1\right]\left[V_0\right]=\left[Ad_{C_1(t)}\right]\left[\grave{A}\left(q_1\left(\theta_1\left(t\right)\right)\right)\right]\left[L_1\left(t\right)\right]^{-1}\frac{d}{dt}\left[L_1\left(t\right)\right]\left[V_0\right]=$
$\left[Ad_{C_1(t)}\right]\left[A_0\right]\left[L_1\left(t\right)\right]^{-1}\frac{d}{dt}\left[L_1\left(t\right)\right]\left[V_0\right]$
The equation reads :
$\left\langle Ad_{C_1}Q_0,\left[\left[Ad_{C_1(t)}\right]\left[A_0\right]V_0,Ad_{C_1}\left(C_1^{-1}\cdot\frac{dC_1}{dt}\right)\right]+\left[Ad_{C_1(t)}\right]\left[A_0\right]\left[L_1\left(t\right)\right]^{-1}\frac{d}{dt}\left[L_1\left(t\right)\right]\left[V_0\right]\right\rangle_H=$
0

The equation for the motion of the second particle sums up to :

$$\left\langle Q_0,\left[\left[A_0\right]V_0,u_2\left(t\right)^{-1}\cdot\frac{du_2}{dt}\right]+\left[A_0\right]\left[L_1\left(t\right)\right]^{-1}\frac{d}{dt}\left[L_1\left(t\right)\right]\left[V_0\right]\right\rangle_H=0 \tag{100}$$

There is a relation between $\left[L_1\right]$ and the restriction $\left[\widehat{N}_1\right]$ of the matrix Ad_{C_k} to the space $Span\left(\varepsilon_j\right)_{j=0}^{3}$

$\left[\Phi'_{Y_1m}\left(t-\theta_1,q_1\left(\theta_1\right)\right)\right]=\left[P\left(\Phi_{Y_1}\left(t-\theta_1,q_1\left(\theta_1\right)\right)\right)\right]\left[\widehat{N}_1\left(t\right)\right]\left[P'\left(q_1\left(\theta_1\right)\right)\right]$
$=\left[P\left(q_2\left(t\right)\right)\right]\left[\widehat{N}_1\left(t\right)\right]\left[P'\left(q_1\left(\theta_1\right)\right)\right]$
$\left[L_1\left(t\right)\right]=\left[P\left(q_2\left(t\right)\right)\right]\left[\widehat{N}_1\left(t\right)\right]\left[P'\left(q_1\left(\theta_1\left(t\right)\right)\right)\right]$

12 Stable systems

As the last example shows a system, even in the continuous picture, is not necessarily lasting. And indeed the only known stable free hadron, composite particles composed of quarks and antiquarks, is the proton with 3 quarks. Meanwhile hadrons can be stable in a nucleus when associated to other particles.

With a unified representation of force fields, the first task is then to build a model of matter, and first of nuclei. It is not sensible to represent these very stable objects as systems in perpetuate turmoil. Atoms are represented with an electronic shell, composed of electrons moving around the nucleus, in a continuous process and at equilibrium. Similarly nuclei should be represented by elementary particles interacting together in continuous processes, and at equilibrium. It is usually assumed that nuclei are composed of protons and neutrons, which are themselves composite particles. So we cannot assume that they are the basic bricks : we cannot a priori exclude any particle. Moreover the possibility to represent interacting systems by tensorial products does not exclude the validity of the representations of each object and their interactions. The scope of possible combination can be studied through the equivalence of tensorial representations with sums of representations, but these mathematical operations do not tell us if a possible combination will occur, or is stable.

So it makes sense to study a system of a fixed number of elementary particles, of different types, and the resulting force field, and the conditions for its equilibrium.

We say that a system of particles and their field is stable if it is lasting (the system does not end in collisions or free particles) and behaves as a unique set. Stable systems are ubiquitous : nuclei are extremely resilient, as well as atoms. The strongest materials are based on crystals. At the opposite scale all stellar systems seem to be organized similarly. The processes at work in these systems are continuous, and there seem to be some organizational principle such that they accede to a great stability.

This issue is somewhat the generalization of the old N bodies problem : find the law of evolution of a system composed of N astronomical bodies interacting through the gravitational field. It is well known that this problem has no explicit solution for $N > 3$, and they are usually fragile in the meaning that a small perturbation can lead to catastrophic divergences. Our problem seems much more complicated : we are in General Relativity, the field has 16 components and as many different charges, the laws of interactions are more subtle than the classic $\frac{M_1 M_2}{\|r_1 - r_2\|^2}$. But the fact that a nucleus of iron is very stable shows also that the solutions are common, and robust. And we will see that the Geometry of General Relativity is not an impediment, on the contrary...

Continuous systems show symmetries, for the particles on one hand, and for the field on the other hand. Adding initial conditions, there is no compelling reasons why such a system would be stable, if not for a reconciliation of the symmetries. The symmetry of the field is reflected in the symmetry of the metric. As the example of crystals show, an adequate geometric arrangement is the key to robustness.

12.1 Stable systems of elementary particles

Because the system is stable, we can choose an arbitrary origin of time $t = 0,$.and the system is always at equilibrium. There is a point O of the system which can be considered as its "center of mass", whose trajectory $O(t) = \varphi_o(ct, O(0))$ is representative of the trajectory of the system. If the system is free, that is not subject to an exterior field, O can be taken as the observer, and the gauge is fixed by the value of the field at $O(t)$, then we have $\grave{A}(O(t)) = Ct, \mathcal{F}(O(t)) = \mathcal{F}_0 = Ct, \phi(O(t)) = \phi_0 = Ct$ with non null arbitrary constant.

12.1.1 Symmetries

For a continuous system the variables $u, \widehat{Q}$ are symmetric with respect to the vectors $\mathbf{V}_k$, and their value is fixed by the location $q_j(t) = \varphi_o(ct, x)$ of the particles. The symmetry of a stable system is then defined by the geometric location of the particles at each time t with respect to the point $O(t)$. The particles are located in $\Omega_3(t)$ at increasing spatial distance $c\theta$, and the propagation curves reaching $O(t)$ originate from the particles as they were on the spheres $S(O(t-\theta), \theta)$ with center $O(t-\theta)$. The closest thing to a fixed system is a periodic system. In General Relativity it means that all the variables are periodic with respect to the time. We assume that the particles on $\Omega_3(t)$ are organized in n rings (actually 2 dimensional surfaces) denoted $R_1, ..., R_n$ located at increasing distance $\theta, 2\theta, ...n\theta$ from $O(t)$ and corresponding to the intersection of $\Omega_3(t)$ with the hypercones of propagation curves originating from $O(t-\theta), O(t-2\theta), ..., O(t-n\theta)$.

We denote $q_j(t) = \varphi_o(ct, x_j(t))$ the trajectory of the particles, $V_j = \frac{dq_j}{dt}$ their velocity, Q_j their charge and $\epsilon_j = +1$ for a particle and $\epsilon_j = -1$ for an antiparticle.

We can then use the theorems about the quantization of the system. The state of the particles of the same family belong to the vector space $F(\psi_0)$ spanned by $\{\vartheta(u(t))\psi_0\} \subset Cl(\mathbb{C}, 4)$ and $(F(\psi_0), \vartheta)$ is an irreducible unitary representation of U. It can be collectively represented in the tensorial product $\otimes^n F(\psi_0)$. If we label each of these particles $1...N$ the state of the system is the same for any permutation for the labels, so their collective state belongs to a representation of the symmetric group $\mathfrak{S}(n)$. They are defined by n integers $p_1 \leqslant p_2 ... \leqslant p_n : p_1 + p_2 + ...p_n = N$ which define a class of conjugacy of $\mathfrak{S}(N)$. Each ring is composed of p_k particles belonging to the same family. Moreover their geometric disposition inside the ring is spatially symmetric.

To measure the variables for the ring p in the same gauge we push them forward from $q_j(t-p\theta)$ to $O(t)$.

The propagation curves originating from the particle j of the ring p at $t-p\theta$, that is $q_j(t-p\theta)$, to $O(t)$ have for tangent $Y_j = c\varepsilon_0 + y_j$, the components of y_j do not depend on t, and are fixed by their value at a point on the curve. Then for the ring p :

$q_j(t-p\theta) = \Phi_{Y_j}(-p\theta, O(t)) \Leftrightarrow O(t) = \Phi_{Y_j}(p\theta, q_j(t-p\theta))$

By construct for the vectors of the basis of T_1U : $F_{\theta p*}\left(\kappa_a\left(q_j\left(t-p\theta\right)\right)\right)=\kappa_a\left(O\left(t\right)\right)$

The charges have the same value : $Q_j = Q_p$

The variables are transported along the curve :

$u_j\left(t-p\theta\right)\in P_C\left(q_j\left(t-p\theta\right)\right)\rightarrow F_{Y_jp\theta*}\left(u_j\left(t-p\theta\right)\right)\left(O\left(t\right)\right)=Ad_{C_j(t)}u_j\left(t-p\theta\right)\in P_C\left(O\left(t\right)\right)$

$\grave{A}\left(O\left(t\right)\right)=F_{Y_j\theta p*}\left(\grave{A}\left(q_j\left(t-p\theta\right)\right)\right)\left(O\left(t\right)\right)=Ad_{C_j(t)}\grave{A}\left(q_j\left(t-p\theta\right)\right)\left[L'\left(\Phi'_{Y_jm}\left(p\theta,q_j\left(t-p\theta\right)\right)\right)\right]$

$V_j\left(q_j\left(t-p\theta\right)\right)\in T_{q_j(t-p\theta)}\Omega\rightarrow f_{Y_jp\theta*}V_j\left(q_j\left(t-p\theta\right)\right)$

$=\left[L\left(\Phi'_{Y_jm}\left(p\theta,q_j\left(t-p\theta\right)\right)\right)\right]V_j\left(q_j\left(t-p\theta\right)\right)\in T_{O(t)}\Omega$

the value of $\grave{A}\left(V_j\right)$ is transported as :

$F_{Y_j\theta p*}\grave{A}\left(f_{Y_j\theta p*}V\right)\left(O\left(t\right)\right)=Ad_{C_j(t)}\grave{A}\left(V_j\right)\left(q_j\left(t-p\theta\right)\right)\in T_1U$

In the tetrad $\left[P\left(O\left(t\right)\right)\right]$ the vector $f_{p\theta*}V_j\left(q_j\left(t-p\theta\right)\right)$ has for components :

$\widetilde{V}_j\left(t\right)=\left[P'\left(O\left(t\right)\right)\right]\left[L\left(\Phi'_{Y_jm}\left(p\theta,q_j\left(t-p\theta\right)\right)\right)\right]V_j\left(q_j\left(t-p\theta\right)\right)$

The symmetry with respect to the permutation $\mathfrak{S}\left(n_p\right)$ of the particles of the ring can then be represented by the existence of a spatial rotation at $O\left(t\right)$, represented by $\sigma_p\left(t\right)\in Spin\left(3\right)$ acting in the tetrad at $O(t)$, then in $P_C\left(t\right)$, such that : $\left(\sigma_p\left(t\right)\right)^{n_p}=1$. $\sigma_p\left(t\right)$ is defined up to an arbitrary rotation. If a variable Z_j at $q_j\left(t-p\theta\right)$ has a value $\sum_{a=1}^{16}Z_j^a\left(q_j\left(t-p\theta\right)\right)\kappa_a$, expressed in the orthonormal basis $\kappa_a\left(q_j\left(t-p\theta\right)\right)$ then the variable transported along the curve to $O(t)$ has for value $F_{\theta p*}Z_j=Ad_{C(\sigma_p(t))^j}X_p\left(t\right)$ where $X_p\left(t\right)=\sum_{a=1}^{16}X_p^a\left(t\right)\kappa_a\left(O\left(t\right)\right)$ is common to the ring p. Which is equivalent to

$F_{Y_j\theta p*}\left(\sum_{a=1}^{16}Z_j^a\left(q_j\left(t-p\theta\right)\right)\kappa_a\left(q_j\left(t-p\theta\right)\right)\right)=\sum_{a=1}^{16}Z_j^a\left(q_j\left(t-p\theta\right)\right)F_{Y_j\theta p*}\left(\kappa_a\left(q_j\left(t-p\theta\right)\right)\right)=Ad_{C(\sigma_p(t))^j}\sum_{a=1}^{16}X_p^a\left(t\right)\kappa_a\left(O\left(t\right)\right)$

$F_{Y_j\theta p*}\left(u_j\left(t-p\theta\right)\right)=Ad_{C(\sigma_p(t))^j}u_p\left(t\right)$

The vector $\widetilde{V}_j\left(q_j\left(t-p\theta\right)\right)=\left[P'\left(O\left(\left(q_j\left(t-p\theta\right)\right)\right)\right)\right]V_j\left(q_j\left(t-p\theta\right)\right)$ is transported as $Ad_{C(\sigma_p(t))^j}\widetilde{V}_p\left(t\right)$

$f_{Y_j\theta p*}\widetilde{V}_j=Ad_{C(\sigma_p(t))^j}\widetilde{V}_p\left(t\right)=\left[P'\left(O\left(t\right)\right)\right]Ad_{C(\sigma_p(t))^j}V_p\left(t\right)$

$F_{Y_j\theta p*}\left(\grave{A}\left(V_j\right)\left(q_j\left(t-p\theta\right)\right)\right)=Ad_{C(\sigma_p(t))^j}\left(\grave{A}\left(V_p\right)\left(O\left(t\right)\right)\right)$

with common quantities $u_p\left(t\right)\in U\left(O\left(t\right)\right),\widetilde{V}_p\left(t\right)=\sum_{k=0}^{3}\widetilde{V}_p^k\left(t\right)\varepsilon_k\left(O\left(t\right)\right),V_p\left(t\right)=\left[P\left(O\left(t\right)\right)\right]\widetilde{V}_p\left(t\right)$. The vector $\widetilde{V}_p$ is related to $u_p\left(t\right)$ through the usual formulas for the motion : up to a change of basis V_j is deduced from V_p by a rotation multiple of σ_p.

Then the current for each particle

$J_j\left(q_j\left(t-p\theta\right)\right)=\epsilon_jAd_{u_j}Q_j\otimes\epsilon_jV_j\left(q_j\left(t-p\theta\right)\right)$

has for image at $O\left(t\right)$:

$\epsilon_pAd_{C(\sigma_p(t))^j}Ad_{u_p(t)}Q_p\otimes\epsilon_pAd_{C(\sigma_p(t))^j}V_p\left(t\right)$

The current equation reads at $O(t)$:

$\phi\left(O\left(t\right)\right)=\phi_0=-\frac{1}{\Omega_0}\sum_{p=1}^{n}\sum_{j\in R_p}\epsilon_pAd_{C(\sigma_p(t))^j}Ad_{u_p(t)}Q_p\otimes\epsilon_pAd_{C(\sigma_p(t))^j}V_p\left(t\right)$

The right hand side is the sum of decomposable tensors.

The velocity of the system, considered as a single particle, is $c\varepsilon_0$, its state is 1 by definition and we define the apparent charge of the system as $Q_S \in T_1U$ such that $\phi_0 = Q_S \otimes c\varepsilon_0$. Then :

$$\sum_{p=1}^{n} \sum_{j \in R_p} \epsilon_p Ad_{C(\sigma_p(t))^j} Ad_{u_p(t)} Q_p \otimes \epsilon_p Ad_{C(\sigma_p(t))^j} V_p(t) = Q_S \otimes c\varepsilon_0 \qquad (101)$$

12.1.2 Equations of state

The equations for the state of the particles imply :

$\forall j \in R_p : \left\langle Ad_{u_j} \widehat{Q}_j, \left[\grave{A}(V_j), \frac{du_j}{dt} \cdot u_j^{-1} \right] + \grave{A}\left(\frac{dV_j}{dt} \right) \right\rangle_H |_{q_j(t-p\theta)} = 0$

The hermitian scalar product is preserved by transport along the propagation curve from $q_j(t-p\theta)$ to $O(t)$:

$F_{Y_j\theta p*} Ad_{u_j} \widehat{Q}_j = Ad_{C(\sigma_p(t))^j} Ad_{u_p(t)} Q_p$

$F_{Y_j\theta p*} \left[\grave{A}(V_j), \frac{du_j}{dt} \cdot u_j^{-1} \right] = \left[F_{Y_j\theta p*} \grave{A}(V_j), F_{Y_j\theta p*} \left(\frac{du_j}{dt} \cdot u_j^{-1} \right) \right]$

$= \left[Ad_{C(\sigma_p(t))^j} \grave{A}(V_p)(O(t)), F_{Y_j\theta p*} \left(\frac{du_j}{dt} \cdot u_j^{-1} \right) \right]$

$F_{Y_j\theta p*} u_j = Ad_{C(\sigma_p(t))^j} u_p(t)$

$F_{Y_j\theta p*} \left(\frac{du_j}{dt} \cdot u_j^{-1} \right) = \frac{d}{dt} \left(Ad_{C(\sigma_p(t))^j} u_p(t) \right) \cdot \left(Ad_{C(\sigma_p(t))^j} u_p(t) \right)^{-1}$

$= \frac{d}{dt} \left(C(\sigma_p(t))^j \cdot u_p(t) \cdot C(\sigma_p(t))^{-j} \right) \cdot \left(C(\sigma_p(t))^j \cdot u_p(t)^{-1} \cdot C(\sigma_p(t))^{-j} \right)$

$\frac{d}{dt} \left(C(\sigma_p(t))^j \cdot u_p(t) \cdot C(\sigma_p(t))^{-j} \right)$

$= \frac{d}{dt} \left(C(\sigma_p)^j \right) \cdot u_p \cdot C(\sigma_p)^{-j} + C(\sigma_p)^j \cdot \frac{du_p}{dt} \cdot C(\sigma_p)^{-j} + C(\sigma_p)^j \cdot u_p \cdot \frac{d}{dt} \left(C(\sigma_p)^{-j} \right)$

$F_{Y_j\theta p*} \left(\frac{du_j}{dt} \cdot u_j^{-1} \right)$

$= Ad_{C(\sigma_p)^j} \left(C(\sigma_p(t))^{-j} \cdot \frac{d}{dt} C(\sigma_p(t))^j + \frac{d}{dt} u_p(t) \cdot u_p(t)^{-1} + Ad_{u_p} \left(\frac{d}{dt} \left(C(\sigma_p(t))^{-j} \right) \cdot C(\sigma_p(t))^j \right) \right)$

$F_{Y_j\theta p*} \left[\grave{A}(V_j), \frac{du_j}{dt} \cdot u_j^{-1} \right] =$

$Ad_{C(\sigma_p)^j} \left[\grave{A}(V_p)(O(t)), C(\sigma_p)^{-j} \cdot \frac{d}{dt} C(\sigma_p)^j + \frac{du_p}{dt} \cdot u_p{}^{-1} + Ad_{u_p} \left(\frac{d}{dt} \left(C(\sigma_p)^{-j} \right) \cdot C(\sigma_p)^j \right) \right]$

$\grave{A}\left(\frac{dV_j}{dt} \right) \in T_1U \to Ad_{C(\sigma_p(t))^j} \grave{A}\left(\frac{dV_p}{dt} \right)$

The equation reads :

$0 =< Ad_{C(\sigma_p(t))^j} Ad_{u_p(t)} Q_p,$

$Ad_{C(\sigma_p)^j} \left[\grave{A}(V_p)(O(t)), C(\sigma_p)^{-j} \cdot \frac{d}{dt} C(\sigma_p)^j + \frac{du_p}{dt} \cdot u_p{}^{-1} + Ad_{u_p} \left(\frac{d}{dt} \left(C(\sigma_p)^{-j} \right) \cdot C(\sigma_p)^j \right) \right] + Ad_{C(\sigma_p)^j} \grave{A}\left(\frac{dV_p}{dt} \right) >$

$0 =< Ad_{u_p(t)} Q_p,$

$\left[\grave{A}\left(V_p\right)\left(O\left(t\right)\right), C\left(\sigma_p\right)^{-j} \cdot \frac{d}{dt}C\left(\sigma_p\right)^j + \frac{du_p}{dt} \cdot u_p{}^{-1} + Ad_{u_p}\left(\frac{d}{dt}\left(C\left(\sigma_p\right)^{-j}\right) \cdot C\left(\sigma_p\right)^j\right)\right] +$
$\grave{A}\left(\frac{dV_p}{dt}\right)_H > |_{O(t)}$

For $j = n_p : \frac{d}{dt}C\left(\sigma_p\left(t\right)\right)^j = 0$

$\left\langle Ad_{u_p(t)}Q_p, \left[\grave{A}\left(V_p\right)\left(O\left(t\right)\right), \frac{d}{dt}u_p\left(t\right) \cdot u_p\left(t\right)^{-1}\right] + \grave{A}\left(\frac{dV_p}{dt}\right)\right\rangle_H |_{O(t)} = 0$

$\Rightarrow \forall j :$

$\left\langle Ad_{u_p(t)}Q_p, \left[\grave{A}\left(V_p\right)\left(O\left(t\right)\right), C\left(\sigma_p\left(t\right)\right)^{-j} \cdot \frac{d}{dt}C\left(\sigma_p\left(t\right)\right)^j + Ad_{u_p}\left(\frac{d}{dt}\left(C\left(\sigma_p\left(t\right)\right)^{-j}\right) \cdot C\left(\sigma_p\left(t\right)\right)^j\right)\right]\right\rangle_H =$
0

$\Rightarrow \forall j : \frac{d}{dt}\sigma_p\left(t\right) = 0$

$$\begin{array}{c} \left\langle Ad_{u_p(t)}Q_p, \left[\grave{A}\left(V_p\right)\left(O\left(t\right)\right), \frac{d}{dt}u_p\left(t\right) \cdot u_p\left(t\right)^{-1}\right] + \grave{A}\left(\frac{dV_p}{dt}\right)\right\rangle_H |_{O(t)} = 0 \\ \frac{d}{dt}\sigma_p\left(t\right) = 0 \end{array} \tag{102}$$

The first equation is the motion equation for a particle of charge Q_p, state $\theta\left(u_p\left(t\right)\right)$, velocity V_p, which would be located at $O\left(t\right)$.

12.1.3 Energy-Momentum tensor

The lagrangian for the particles of the ring p at $t - p\theta$ is :

$L\left(j^1u_j\right) = \left\langle Ad_{u_j}Q_j, \frac{du_j}{dt} \cdot u_j^{-1} + \grave{A}\left(V_j\right)\right\rangle_H \left(q_j\left(t - p\theta\right)\right)$

and the same computation as above gives :

$L\left(j^1u_j\right) = \left\langle Ad_{u_p(t)}Q_p, \frac{du_p}{dt} \cdot u_p{}^{-1} + \grave{A}\left(V_p\right)\left(O\left(t\right)\right)\right\rangle_H$

$L\left(j^1u_j\right)$ has the same value on the ring p :

$$L\left(j^1u_j\right) = \left\langle Ad_{u_p(t)}Q_p, \frac{d}{dt}u_p\left(t\right) \cdot u_p\left(t\right)^{-1} + \grave{A}\left(V_p\right)\left(O\left(t\right)\right)\right\rangle_H \tag{103}$$

We can replace the lagrangian for the particles of the system by

$\sum_{j=1}^N L\left(j^1u_j\right) = \sum_{p=1}^n n_p \left\langle Ad_{u_p(t)}Q_p, \frac{d}{dt}u_p\left(t\right) \cdot u_p\left(t\right)^{-1} + \grave{A}\left(V_p\right)\left(O\left(t\right)\right)\right\rangle_H$

The variational derivative with respect to u_p is :

$\frac{\delta L_p}{\delta u_p}\left(\delta u_p\right) = \int_\Omega \left\langle Ad_{u_p}\widehat{Q}_p, \left[\frac{du_p}{dt} \cdot u_p^{-1} + \grave{A}\left(V_p\right), \delta u_p \cdot u_p^{-1}\right] + \grave{A}\left(\delta V_p\right)\right\rangle_H \varpi_4$

and gives the related component of the Energy-Momentum tensor, and then the resistance of the system to a change :

$\delta u_p = \sum_{a=1}^{16} \frac{\partial u_p}{\partial z^a} \cdot u_p^{-1}\delta z^a;$

$\delta V_p^\alpha = \sum_{a=1}^{16} \frac{\partial V_p^\alpha}{\partial z^a}\delta z^a = \sum_{j=1}^3 \left(2\frac{\sinh \mu_w \cosh \mu_w}{1+2(\cosh \mu_w)^2} \sum_{j=1}^3 \frac{\delta W^j}{\mu_w}\right) P_j^\alpha$

The value of the partial derivatives $\frac{\partial u_p}{\partial z^a} \cdot u_p^{-1}$ have been given previously.

12.1.4 Metric

From the metric equation, because $\grave{A}$ is constant at $O(t)$, the metric is also constant at $O(t)$. Along any integral curve of a Killing vector field then $g\left(\Phi_{Y}\left(\tau, O\left(t\right)\right)\right) = f_{Y\tau*}g\left(O\left(t\right)\right)$:

$\left[g\left(\Phi_{Y}\left(\tau, O\left(t\right)\right)\right)\right] = \left[L'\left(\Phi_{Y'm}\left(\tau, O\left(t\right)\right)\right)\right]^{t}\left[g\left(O\left(t\right)\right)\right]\left[L'\left(\Phi_{Y'm}\left(\tau, O\left(t\right)\right)\right)\right]$

We are free to choose the origin of time, there is a unique Killing vector field Y with a given value $Y\left(O\left(0\right)\right)$. We can define a spherical chart of Ω, with center $O\left(0\right)$, the tetrad at $O\left(0\right)$ and the components $\widetilde{Y}^{j}$ of and unitary vector $Y\left(O\left(0\right)\right)$, a point $m = \Phi_{Y}\left(ct, O\left(0\right)\right)$ has then for coordinates $\xi^{0} = ct$, and $\xi^{\beta}, \beta = 1, 2, 3$ are 3 independent parameters defining $Y\left(O\left(0\right)\right)$. The 4 vectors of the holonomic basis $\partial\xi^{\beta}$ are, distinct, Killing vector fields. From $\pounds_{V}g = 0$ with each axis we get $\left[g\left(m\right)\right] = \left[g\left(O\left(0\right)\right)\right] = \left[\eta\right]$.

12.2 Nuclear interactions

The field has 16 components, the first, the scalar component, associated to the EM field, is special, as well as the components W, R because they are directly related to the geometric motion of material bodies, the most obvious feature of their state. But theoretically all components of the field should act on elementary particles, and then on material bodies, on the same footing. But the weak and strong interactions appear only in some circumstances, at a subatomic level, which is seen usually of the evidence of the existence of distinct force fields with specific characteristics, among them their range.

The field is measured through its interactions with particles, and these interactions depend on the charges of the particles. These charges have been computed from the states of the particles, and actually are just deduced from the behavior of the known elementary particles. The structure of the charges does not proceed from any superior principle : they are experimental facts. Their value, defined up to some universal constant, can be measured with respect to the EM charge, which is just $Q_{1} = \langle\psi_{0}, \psi_{0}\rangle_{H}$. By construct each charge is associated to a component of the field and if a charge, as measured, is small one can expect that the particle interacts weakly with the corresponding components of the field. Indeed this is what we do with the EM charge, which can be null, or the gravitational charge.

Stable systems of elementary particles are special, but they are also common, even at the atomic level. They behave as a unique body, and can be characterized by a charge Q_{S} . And it can happen that, for the known stable systems, that some components of Q_{S} are privileged. Then for all experiments which involve these systems, some interactions are masked. The fact that all systems of quarks, except the proton, are instable leads to this conclusion.

12.3 Deformable solids

A the atomic level and up stable systems behave as single particles, endowed with charges Q_{S} such that they interact only with the EM field, represented by the scalars $T_1U(1) = (iA, 0, 0, 0, 0, 0, 0, 0)$ and the gravitational field, represented by the subalgebra : $T_1U_G = \left\{(0, 0, 0, iW, R, 0, 0, 0), W, R \in \mathbb{R}^3\right\}$. The charges are then $Q = (Q_A, 0, 0, Q_W, Q_R, 0, 0, 0) \in T_1U$ with all the components real.

The state of the particles is then measured by $u \in U(1) \times C(Spin(3,1))$ and the variables related to the field take their value in the subspace

$(iA, 0, 0, iW, R, 0, 0, 0) \subset T_1U$

which is preserved by $Ad_u, u \in U(1) \times C(Spin(3,1))$

All the particles are assumed to belong to the same family with charges Q, the representation of each particle through a common section $u \in \mathfrak{X}(P_U)$ and vector field V. The location of the particle is $q = O(t)$, with velocity $V = V_p$ its state is measured by $u = u_p(t)$ but usually the particle is no longer free and the observer is not $O(t)$ so that for an individual particle $q(t) = \varphi_o(ct, x(t))$ with the usual standard chart. This is a change of observer and the state equation still holds :

$\left\langle Ad_uQ, \left[\grave{A}(V), \frac{d}{dt}u(t) \cdot u(t)^{-1}\right] + \grave{A}(V)\right\rangle_H |q(t) = 0$

It is convenient to introduce a density function $\mu : \Omega \to \mathbb{R}$ such that the flow of charges is constant along the vector field V :

$$\pounds_V(\mu(Q)\varpi_4) = 0 \Longleftrightarrow \frac{d\mu}{dt} + \mu divV = 0 \tag{104}$$

with $divV = \sum_{\beta=0}^{3} \partial_\beta V^\beta + \frac{1}{2\det g} V^\beta \partial_\beta \det g$

We have the usual model of a deformable solid, or a fluid, composed of material points which follow the integral curves of a common vector field, that is without collision or loss of matter.

The lagrangian for the particles is :

$\mu L(j^1u) = \mu\left\langle Ad_uQ, \left[\grave{A}(V), \frac{d}{dt}u(t) \cdot u(t)^{-1}\right] + \grave{A}(V)\right\rangle_H |q(t)$

The state equation comes from the variational derivative with respect to u and one can check that μ is not involved : the density is deduced from the conservation equation above. As noticed before at non relativist speed particles follow geodesics (which are defined by the connection).

The current equation reads :

$\phi(\varphi_o(ct, x)) = -\frac{1}{\Omega_0}\int_0^{\theta_M} \mu(p)\widetilde{J}(p)\,d\theta$

with $\widetilde{J}(p) = F_{Y\theta*}J(\varphi_o(c(t-\theta), x)), J(q(t)) = Ad_{u(q(t))}Q \otimes V(q(t)), Y : q(t) \to \varphi_o(ct, x)$

The PDE for the field and the metric equation still holds.

It is possible to go further. The variables $u, \mu, \widehat{Q}$ are symmetric with respect to the vector field V. If the system is defined at $t = 0$ over some area $\omega(0) \in \Omega_3(0)$, it will be at t over an area $\omega(t) = \Phi_V(t, \omega(0)) \subset \Omega_3(t)$ and the integral $Q_S(t) = \int_{\omega(t)} \mu Q \varpi_3$ is well defined. If there is an isometry on $\Omega_3(t)$ (for the metric g_3) such that its associated Clifford morphism is a morphism for u, then we have a deformation tensor, and the system is a rigid solid it the morphism does not depend on t.

These equations could be useful in Astrophysics, where General Relativity is required and the variation of the metric cannot be neglected. They enable to account for the rotational motion of celestial bodies, which is significant and difficult to model with the traditional representations. But of course some Thermodynamics should be added to account for the collisions.

Part IV
DISCONTINUOUS PROCESSES

There is no totally discontinuous processes, we have discontinuities between continuous processes, with the apparition of a transition phase.

13 Collision

A system of 2 particles is not stable, its fate is either a collision, or a separation, depending on their charges. However if an external field is applied a collision can still happen. The process leading to the collision, as well as the process after the collision, are modeled by the equations given previously, adjusted if there is an external field. But the outcome of the collision itself requires a different approach. We have essentially 2 cases.

13.1 Collision without creation or annihilation of particles

The lagrangian for the particles

$\left\langle Q, \left(\sum_{\alpha=0}^{3}\left(u^{-1}\cdot\delta_\alpha u + Ad_{u^{-1}}\grave{A}_\alpha\left(q\left(t\right)\right)\right)V^\alpha\left(u\right)\right)\right\rangle$

are defined for any section

$j^1u:\mathbb{R}\to J^1U :: j^1u\left(t\right)=\left(q\left(t\right),u\left(t\right),u^{-1}\cdot\delta_\beta u\left(t\right),\beta=0..3\right).$

By definition at the location O of the collision there is a common basis for u, V for all the particles, and $\grave{A}\left(q\left(t\right)\right)$ has the same value.

The energy exchanged by the field is still given by the lagrangian $\langle\mathcal{F},\mathcal{F}\rangle$ and we have seen that $\mathcal{F}$ is a differential operator

$[\mathcal{F}_w]=\left[\begin{array}{ccc}\grave{A}_{01}-\grave{A}_{10}+\left[\grave{A}_0,\grave{A}_1\right] & \grave{A}_{02}-\grave{A}_{20}+\left[\grave{A}_0,\grave{A}_2\right] & \grave{A}_{03}-\grave{A}_{30}+\left[\grave{A}_0,\grave{A}_3\right]\end{array}\right]$

$[\mathcal{F}_r]=\left[\begin{array}{ccc}\grave{A}_{32}-\grave{A}_{23}+\left[\grave{A}_3,\grave{A}_2\right] & \grave{A}_{13}-\grave{A}_{31}+\left[\grave{A}_1,\grave{A}_3\right] & \grave{A}_{21}-\grave{A}_{12}+\left[\grave{A}_2,\grave{A}_1\right]\end{array}\right]$

where

$j^1\grave{A}=\left(m,\grave{A}_\alpha^a,\grave{A}_{\beta\alpha}^a,a=1..16,\alpha,\beta=0..3\right)$

and $\sum_{\alpha=0}^{3}\sum_{a=1}^{16}\grave{A}_{\beta\alpha}^a d\xi^\alpha\otimes\kappa_a$ are 4 independent one form . If $j^1\grave{A}$ is the prolongation of a section then $\grave{A}_{\beta\alpha}^a=\partial_\beta\grave{A}_\alpha^a$ but in a collision it is not necessarily so.

It is subjected to the codifferential equation :

$\exists K\in\Lambda_1\left(TM;T_1U\right):$

$\left[\mathcal{F}_r\left(j^1\grave{A}\right)\right]=\left[dK_w\right]\left[g_3\right]^{-1}\sqrt{\det g_3};\left[\mathcal{F}_w\left(j^1\grave{A}\right)\right]=-\left[dK_r\right]\left[g_3\right]/\sqrt{\det g_3}$

The metric equation still holds : $d\grave{A}=dK$

$\left[d\grave{A}_w\right]=\frac{1}{2}\left(-\left[X_w\right]+\left[X_r\right]\left[g_3\right]/\sqrt{\det g_3}\right)$

$\left[d\grave{A}_r\right]=-\frac{1}{2}\left(\left[X_r\right]+\left[X_w\right]\left[g_3\right]^{-1}\sqrt{\det g_3}\right)$

with
$\left[d\grave{A}_r\right] = \left[\begin{array}{ccc} \grave{A}_{32} - \grave{A}_{23} & \grave{A}_{13} - \grave{A}_{31} & \grave{A}_{21} - \grave{A}_{12} \end{array}\right]$
$\left[d\grave{A}_w\right] = \left[\begin{array}{ccc} \grave{A}_{01} - \grave{A}_{10} & \grave{A}_{02} - \grave{A}_{20} & \grave{A}_{03} - \grave{A}_{30} \end{array}\right]$
but the PDE for $\mathcal{F}$ are no longer valid.
The lagrangian for the field becomes :
$\langle \mathcal{F}, \mathcal{F} \rangle = 2Tr\left(\left[d\grave{A}_r\right]^t [\eta] [dK_w]\right)\sqrt{\det g_3}$
$= 2Tr\left(\left[d\grave{A}_w\right]^t [\eta] [dK_r]\right)\sqrt{\det g_3}$
$= 2Tr\left(\left[d\grave{A}_r\right]^t [\eta] \left[d\grave{A}_w\right]\right)\sqrt{\det g_3}$

But the Chern-Weil theorem tells that $Tr\left[d\grave{A}_r\right]^t [\eta] \left[d\grave{A}_w\right]$ does not depend on the value of the field. So that, if g_3 does not change, that we can assume when there is no creation or annihilation of particles, the field is not involved : we have a case similar to an elastic collision. This holds even for composite particles, as long as they keep their original form, but of course not at the macroscopic level : then the density usually would change.

The conservation of energy sums up to :
$\sum_j \left\langle Ad_{u_j} Q_j, \left(\sum_{\alpha=0}^3 \left(\delta_\alpha u_j \cdot u_j^{-1} + \grave{A}_\alpha (q(t))\right) V_j^\alpha (u_j)\right)\right\rangle_H = Ct$
$\Leftrightarrow$

$$\sum_j \sum_{\alpha=0}^3 \left\langle J_j^\alpha, \delta_\alpha u_j \cdot u_j^{-1} + \grave{A}_\alpha \right\rangle_H |_O = Ct \qquad (105)$$

The processes are continuous, before and after the collision, and $\frac{du_j}{dt} = \sum_{\alpha=0}^3 V_j^\alpha \delta_\alpha u_j$. So that :

$$\sum_j \left\langle Ad_{u_j} Q_j, \left(\frac{du_j}{dt} \cdot u_j^{-1} + \grave{A}(V_j)\right)\right\rangle_H = \sum_j \left\langle Ad_{\widetilde{u}_j} Q_j, \left(\widetilde{\left(\frac{du_j}{dt}\right)} \cdot \widetilde{(u_j)} + \grave{A}\left(\widetilde{V_j}\right)\right)\right\rangle_H \qquad (106)$$

with the variables u_j, V_j for the incoming particles, and $\widetilde{u_j}, \widetilde{V_j}$ for the outgoing particles. The velocity $\widetilde{V_j}$ are fixed by $\widetilde{u_j}$.

We can no longer use variational derivatives. The unknown variables are $\widetilde{(u_j)}, \widetilde{\left(\frac{du_j}{dt}\right)}$ which are independent.

The states equations holds for each particle before and after the collision :
$a = 1...16$:
$\left\langle Ad_u \widehat{Q}_k, \left\{\left[\frac{du_k}{dt} \cdot u_k^{-1} + \grave{A}(V_k), \frac{\partial u_k}{\partial z_a} \cdot u_k^{-1}\right] + \grave{A}\left(\frac{dV_k}{dz_a}\right)\right\}\right\rangle_H |_O = 0$
and together they provide a solution of the problem.

13.2 Creation and annihilation of particles

This is the domain of Particles Physics. In these processes the fundamental state ψ_0 is not preserved. We need to come back to the definition of elementary particles. They are defined, according to the basic theorems of quantization, as specific irreducible representation of the group U : the set $F = \{\vartheta(u)(\psi_0), u \in U\}$ is invariant by U. A system of elementary particles can be represented by a tensor : $\Psi = \sum_1^n \psi_{01} \otimes \psi_{02} ... \otimes \psi_{0n} \in \otimes_1^n F_n$ in another representation of U. But a tensorial representation is usually not irreducible : it can be decomposed in the sum of irreducible representations, which are not necessarily the original ones but are equivalent. In this process new particles can appear.

There are mathematical tools which enable to find equivalent representations. They are based on the Cartan algebra of the Lie algebra. It is simpler here because we have a single Lie algebra (the Clifford algebra P_C itself) whose Cartan algebra is 4 dimensional (see my paper on Clifford algebras).

The lagrangian for the particles is still given by

$\left\langle Ad_{u_j} Q_j, \left(\frac{du_j}{dt} \cdot u_j^{-1} + \grave{A}(V_j)\right)\right\rangle_H$

and because $V_j \neq 0$ their sum is not necessarily preserved. The field is involved. From the Chern-Weil equation it implies that g_3 is not constant : the metric adjusts to the new value of the field, but it is still continuous and the metric equations still holds.

A discontinuity appears in the potential :

$\left[\Delta \grave{A}_r\right] = \left[\begin{array}{ccc} \grave{A}_{32} - \grave{A}_{23} & \grave{A}_{13} - \grave{A}_{31} & \grave{A}_{21} - \grave{A}_{12} \end{array}\right] = -\frac{1}{2}\left([X_r] + [X_w][g_3]^{-1}\sqrt{\det g_3}\right)$

$\left[\Delta \grave{A}_w\right] = \left[\begin{array}{ccc} \grave{A}_{01} - \grave{A}_{10} & \grave{A}_{02} - \grave{A}_{20} & \grave{A}_{03} - \grave{A}_{30} \end{array}\right] = \frac{1}{2}\left(-[X_w] + [X_r][g_3]/\sqrt{\det g_3}\right)$

It is no longer smeared out in the propagation and appears as a boson.

14 Bosons

14.1 Discontinuity of the connection

The propagation of the field is expressed through the derivative of the connection along a projectable vector field $\mathbf{Y} \in TU$:

$\pounds_{\mathbf{Y}} \widehat{\grave{A}}(p_u) = \frac{d}{d\tau}(\Phi_{\mathbf{Y}}(\tau, .))^* \widehat{\grave{A}}(p_u)|_{\tau=0}$

The connection is always continuous, but not necessarily continuously differentiable. If there is a discontinuity of the derivatives (such as expressed in $\Delta\grave{A}$) there can be a right and a left derivative which are not equal :

$\Delta_R(\tau)\widehat{\grave{A}}(p_u) = \lim_{\tau \to 0} \frac{1}{\tau}\left((\Phi_{\mathbf{Y}}(\tau, .))_* \widehat{\grave{A}}(p_u) - \widehat{\grave{A}}(p_u)\right)$

$\Delta_L(\tau)\widehat{\grave{A}}(p_u) = \lim_{\tau \to 0} \frac{1}{\tau}\left((\Phi_{\mathbf{Y}}(-\tau, .))_* \widehat{\grave{A}}(p_u) - \widehat{\grave{A}}(p_u)\right)$

The discontinuity is represented by

$\Delta_{\mathbf{Y}}\widehat{\grave{A}}(m) = \Delta_R(\tau)\widehat{\grave{A}}(p_u) - \Delta_L(\tau)\widehat{\grave{A}}(p_u)$

$= \lim_{\tau \to 0} \frac{1}{\tau}\left((\Phi_{\mathbf{Y}}(\tau, .))^* \widehat{\grave{A}}(p_u) - (\Phi_{\mathbf{Y}}(-\tau, .))_* \widehat{\grave{A}}(p_u)\right)$

$(\Phi_{\mathbf{Y}}(\tau,.))^* \widehat{\grave{A}}(p_u), (\Phi_{\mathbf{Y}}(-\tau,.))_* \widehat{\grave{A}}(p_u)$ are both one forms $\Lambda_1(TU^*; T_1U)$, so $\Delta_{\mathbf{Y}}\widehat{\grave{A}}(m)$ is a one form $\Lambda_1(TU; T_1U)$. Because it is computed by a difference, in a change of gauge on P_U it transforms by $Ad_{\varkappa^{-1}}$ as $\mathcal{F}$ and not an affine law.

It depends on the projectable vector field $\mathbf{Y}$. Similarly to what is done in the definition of $\mathcal{F}$ we can, using the standard gauge $\mathbf{p}_u(m)$, pull back $\Delta_{\mathbf{Y}}\widehat{\grave{A}}(m)$ with a vector $Y \in TM$:

$\mathbf{p}_u^*(y) : TM \to TU :: \mathbf{Y}(\mathbf{p}_u(m)) = \mathbf{p}_u'(m)(Y)$

$\mathbf{p}_u^*\left(\grave{A}\right) : \Lambda_1(TM; T_1U) \to \Lambda_1(TU; T_1U) :: \Delta\widehat{\grave{A}}(\mathbf{p}_u(m)) = \mathbf{p}_u'(m)\Delta\grave{A}(m)$

$\mathbf{p}_u^*\left(\Delta_{\mathbf{Y}}\widehat{\grave{A}}\right) : TM \to T_1U :: \Delta\grave{A}(Y) = \mathbf{p}_u^*\left(\Delta_{\mathbf{Y}}\widehat{\grave{A}}\right)(y) = \Delta\widehat{\grave{A}}(\mathbf{p}_u(m))(\mathbf{p}_u'(m)(y))$

And it is then a one form $\Delta\grave{A} \in \Lambda_1(TM^*; T_1U)$, expressed with the potential $\grave{A}$, depending on the vector field Y.

The phenomenon is similar to a shock wave in a continuous fluid. A continuous fluid can be represented by material points, which follow trajectories $q(t)$ which are integral curves of a common vector field V, and can be identified by their location at $t = 0$. Some physical variable X attached to each material point is represented in a vector bundle by a section which is symmetric with respect to V : $X(\Phi_V(t, q(0))) = f_{Vt*-}X(q(t))$. The fluid is continuous, but a discontinuity can appear in the derivative $\frac{dX}{dt}$: this is a shock wave. It can be represented by a function $\theta : \Omega \to \mathbb{R}$ such that the wave reaches the point q at the time $\theta(q)$, and a section ΔX with the same base Ω, such that $j^1X = \left(q, X, \frac{dX}{dt} + \Delta X\right)$. Which is equivalent to say that we have the superposition of a continuous process, represented by X, and a discontinuous process ΔX which propagates by waves, corresponding to to the sets $\Omega(t) = \{q : \theta(q) = t\}$.

In the case of the field, it is continuous but, as we have seen, its propagation does not follow integral curves of a vector field, but happens in the 3 dimensional folliation $\Omega_3(t)$. As a consequence a discontinuity in the field propagates along curves. It is then similar to the motion of a particle, and leads to the representation of bosons.

A discontinuity is always a transition point between continuous processes, before and after the discontinuity the variation of the field is represented by the Lie derivative $\mathcal{F}$. It propagates with the field, on $\Omega_3(t)$, that is with the spatial speed c. It is then legitimate to assume that it propagates along the same curves, with null, future oriented Killing vectors $Y = c\varepsilon_0 + y$ and $\Delta\grave{A}(\Phi_Y(\tau, q)) = F_{Y\tau*}\Delta\grave{A}(\Phi_Y(\tau, q))$.

Proposition 17 *Discontinuities in the derivative of the field can be represented as one form $\Delta\grave{A} \in \Lambda_1(TM^*; T_1U)$ with support a null, future oriented Killing curve, and they propagate along this curve at the same speed as the field, by transport such that $\Delta\grave{A}(\Phi_Y(\tau, q)) = F_{Y\tau*}\Delta\grave{A}(\Phi_Y(\tau, q))$.*

14.2 Bosons

The vector Y is a characteristic of the discontinuity. The quantity $\Delta\grave{A}(Y)$ is transported along the propagation curve as :

$\Delta\grave{A}(\Phi_Y(\tau,m)) = F_{Y\tau*}\Delta\grave{A}(\Phi_Y(\tau,m)) = \left[Ad_{C(S)}\right]\left[\Delta\grave{A}(m)\right][L'(\Phi_Y(\tau,m))]$

$Y(\Phi_Y(\tau,m)) = f_{Y*\tau}Y(m) = [L(\Phi_Y(\tau,m))][Y(m)]$

$F_{Y*\tau}\Delta\grave{A}(f_{Y*\tau}Y)(\Phi_Y(\tau,m)) = \left[Ad_{C(S)}\right]\Delta\grave{A}(Y)(m)$

where $S \in Spin(3,1)$ depends only on the curve, and so is constant for the discontinuity.

The vector

$$B = \left[Ad_{C(S)}\right]\Delta\grave{A}(Y) \in T_1U \tag{107}$$

is constant, and is the representation of a boson. It is similar to the state ψ of a particle but does not belong to the same vector bundle.

The different types of particles and fields appear, layer after layer, in experiments which involve parts of the unified field identified in T_1U.

The action $\left[Ad_{C(S)}\right]$ preserves the Lie subalgebras corresponding to the different fields, given by $\Delta\grave{A}(Y)$, so bosons manifest themselves when these parts of the fields are involved, and one can associate bosons to each type of field :

Photons : $B = (iA,0,0,0,0,0,0,0)$ are associated to the EM field, in a 1 dimensional vector space, isomorphic to $T_1U(1)$

gravitons $B = (0,0,0,iW,R,0,0,0)$ are associated to the gravitational field, in a 6 dimensional vector space,

weak bosons $B = (0,0,0,0,R,0,0,0)$ are associated to the weak field, in a 3 dimensional vector space, isomorphic to $T_1SU(2)$

gluons $B = (iA,0,iV,0,R,X_0,0,0)$ are associated to the strong field, in a 8 dimensional vector space, isomorphic to $T_1SU(3)$

So there is a part common to the weak and the gravitational field.

In Particles Physics bosons are vectors of a basis of the Lie algebras
$T_1U(1), T_1SU(2), T_1SU(3)$

The action of $Ad_{C(S)}$ is similar to a "spin" : S is a spatial rotation.

The quantity $\langle B,B\rangle_H = \left\langle \Delta\grave{A}(Y), \Delta\grave{A}(Y)\right\rangle_H$ is constant along the propagation curve.

14.3 Interactions of bosons with particles

Bosons are essentially the field, the action of the field on particles is represented by the covariant derivative of the connection, along the trajectory of the particle : $\widehat{\nabla}_V u = Ad_{u^{-1}}\left(\frac{du}{dt}\cdot u^{-1} + \grave{A}(V)\right)$ because the motion of the particle on its trajectory forces the change in the potential. With a boson the change in the potential occurs through $\Delta\grave{A}$. The trajectory of the particle is not involved. So the interaction reads :

$$\Delta u = Ad_{u^{-1}}\left(\frac{du}{dt}\cdot u^{-1}+B\right) \tag{108}$$

The interaction of the field with a particle always changes its trajectory (through $\frac{du}{dt}\cdot u^{-1}$ and the fact that the motion is represented in u). In the interaction with a boson the trajectory is still continuous, but no longer continuously differentiable.

The lagrangian for a particle is :

$L_p\left(j^1u\right)=\frac{1}{i}\left\langle \psi,\nabla_V\psi\right\rangle_H=\frac{1}{i}\left\langle \psi_0,\vartheta'(1)\left(\widehat{\nabla}_V j^1u\right)\psi_0\right\rangle_H$

then in an interaction with a boson :

$L_p\left(j^1u\right)=\frac{1}{i}\left\langle \psi,\Delta\psi\right\rangle_H=\frac{1}{i}\left\langle \psi_0,\vartheta'(1)\left(\Delta u\right)\psi_0\right\rangle_H$

With $\psi_0=(a,v_0,v,w,r,x_0,x,b)$ with complex components in the usual orthonormal basis of $Cl\left(\mathbb{C},4\right)$ and

$\widehat{\nabla}_V j^1u=\delta\kappa=$

$\left\{(iT_A,T_{V_0},iT_V,iT_W,T_R,T_{X_0},iT_X,T_B),T_A,T_{V_0},T_{X_0},T_B\in\mathbb{R},T_V,T_W,T_R,T_X\in\mathbb{R}^3\right\}$

we had

$\frac{1}{i}\left\langle \psi_0,\vartheta'(1)\left(\delta\kappa\right)\left(\psi_0\right)\right\rangle_H=$

$Q_AT_A+Q_{V_0}T_{V_0}+Q_V^tT_V+Q_W^tT_W+Q_R^tT_R+Q_{X_0}T_{X_0}+Q_X^tT_X+Q_BT_B$

that we writes : $\frac{1}{i}\left\langle \psi_0,\vartheta'(1)\left(\delta\kappa\right)\left(\psi_0\right)\right\rangle_H=\left\langle Q,\delta\kappa\right\rangle_H$ so we have similarly a simple expression for the change of energy of particles in an interaction between bosons and particles :

$$L_p\left(j^1u\right)=\left\langle Q,\sum_{a=1}^{16}B^a\kappa_a\right\rangle_H=\left\langle Q,B\right\rangle_H \tag{109}$$

with the components of B in the real basis of T_1U.

By construct the bosons interact with the particles as the corresponding components of the field.

It is then tempting to attribute charges to a boson, they would be B^a, which are constant because B is constant along a propagation curve. In this picture the "mass" of a boson corresponds to the 6 components associated to $T_1Spin\left(3,1\right)$, photons are massless and only "gravitons" and "weak bosons" have a mass. But Bosons do not interact with the field. The deviation of light by stars is explained by the curvature of the space, not by the action of the gravitational field. And in Particle Physics the gravitational field is not considered, and what is called the mass has another definition.

The energy exchanged by a boson in an interaction can be defined by extending the scalar product $\left\langle \mathcal{F},\mathcal{F}\right\rangle$ from 2 form to 1 form.

$$\left\langle \Delta\grave{A},\Delta\grave{A}\right\rangle=\sum_{a=1}^{16}\eta_{aa}G_1\left(\Delta\grave{A}^a,\Delta\grave{A}^a\right)=\sum_{a=1}^{16}\sum_{\lambda,\mu=0}^{3}\eta_{aa}g^{\lambda\mu}\Delta\grave{A}^a_\lambda\Delta\grave{A}^a_\mu \tag{110}$$

and it can express the energy carried by the discontinuity. It is constant along a propagation curve because it is a Killing curve. The energy carried by a boson is constant (the "attenuation" of photons is explained by the expansion of the Universe).

In the action of a boson on a particle the balance of energy then reads :

$$\langle Q, B\rangle_H + \left\langle \Delta\grave{A}, \Delta\grave{A}\right\rangle = Ct \tag{111}$$

14.4 Creation and annihilation of bosons

The picture of the action of the field on particles, given by the currents and the propagation curves, is similar to the usual diagrams, where the interactions are mediated by bosons, seen as the carriers of the field, with the added bonus that the arrows are here clearly identified curves. However bosons are discontinuities in the field, and they appear, or disappear, in special circumstances.

We have seen that in elastic collisions the field is not involved. Bosons are created when and where there is creation or annihilation of particles, which can be composite. In these processes the balance of energy for the particles is the sum $\sum_j \left\langle Ad_{u_j}Q_j, \left(\frac{du_j}{dt}\cdot u_j^{-1} + \grave{A}\left(V_j\right)\right)\right\rangle_H$, and an excess or a deficit is balanced by a discontinuity of the field $\left\langle \Delta\grave{A}, \Delta\grave{A}\right\rangle$.

A more frequent occurrence is the interaction with a composite particle, such an atom. In a stable system the states of elementary particles are quantized. The elementary particles j are located on rings p where their state follows common rules : their lagrangian $L\left(j^1u_j\right)$ has the same value on the ring p :

$L_p = \sum_{j=1}^N L\left(j^1u_j\right) = \sum_{p=1}^n n_p \left\langle Ad_{u_p(t)}Q_p, \frac{d}{dt}u_p\left(t\right)\cdot u_p\left(t\right)^{-1} + \grave{A}\left(V_p\right)\left(O\left(t\right)\right)\right\rangle_H$

In interacting with the field, the state of the particles can be adjusted in two different ways :

i) by a change of the state u_p of the particles in the same ring

ii) by a reorganization of the rings, which implies a change of n_p

The process i) is continuous, but it is bounded by the necessity to keep the stability of the whole system. The process ii) is discontinuous, and as a consequence there is a discrepancy in the field. It works both ways : a stable system is reorganized either following collisions as in the black body radiation and then emits a boson, or after it has absorbed a boson. And in both cases the exchange is quantized : it takes definite incremental values.

The boson disappears in the process but, if the energy that it brings is greater than what is necessary, another boson can be emitted, to absorb the rest, as in the Comton effect.

In these processes the lagrangian for the particles $L_p\left(j^1u\right) = \langle Q, B\rangle_H$ is expressed with $B = \left[Ad_{C(S)}\right]\Delta\grave{A}\left(Y\right)$ and so accounts non only for the energy carried by the boson (in $\Delta\grave{A}$) but also its trajectory (Y) : the latter is involved, and as such the trajectory of an out going boson is fixed.

Part V
CONCLUSION

We have met our objectives, that is a to provide a consistent representation of unified field an elementary particles, with operational computations for systems of particles. We have given a new, consistent representation for bosons and their interactions with particles.

The most important results are about stable systems. The equation
$\sum_{p=1}^{n}\sum_{j\in R_p}\epsilon_p Ad_{C(\sigma_p(t))^j} Ad_{u_p(t)} Q_p \otimes \epsilon_p Ad_{C(\sigma_p(t))^j} V_p(t) = Q_S \otimes c\varepsilon_0$
gives a general condition for their existence, which should be tested with the values for the different elementary particles.

So far we use the nuclear forces in a crude way, based on the natural decay of some nuclei. Other stable systems can exist, and we can hope to go from one to another by stimulating the rearrangement..

Moreover stable systems encompass deformable solids, and the results provided here are usable in Astrophysics.

Part VI
ANNEX

15 Fiber bundles

15.1 Fiber bundle

Definition 18 *A* ***fibered manifold*** $E(M, \pi)$ *is a triple of two Hausdorff manifolds* E, M *and a surjective submersion* $\pi : E \to M$ *.*

which means that $\forall p \in E, \exists m \in M : \pi(p) = m; \pi'(p)$ is surjective

M is called the **base space** and π projects E on M. For any p in E there is a *unique* m in M, but *every* v in V provides a *different* p in E. The set $\pi^{-1}(m) = E(m) \subset E$ is called the **fiber over** m. It is a submanifold of E.

A **fiber bundle** $E(M, V, \pi)$ is a fibered manifold whose fibers are given by a 3d manifold V. Locally $E \sim M \times V$ and they are characterized by the existence of an atlas of consistent charts.

Definition 19 *On a fibered manifold* $E(M, \pi)$ *an* ***atlas of fiber bundle*** *is a set* $\left(V, (O_a, \varphi_a)_{a \in A}\right)$ *where :*

V *is a Hausdorff manifold, called the* ***fiber***

$(O)_{a \in A}$ *is an open cover of* M

$(\varphi_a)_{a \in A}$ *is a family of diffeomorphisms, called* ***trivialization*** *:*

$\varphi_a : O_a \times V \subset M \times V \to \pi^{-1}(O_a) \subset E :: p = \varphi_a(m, v)$

and there is a family of maps $(\varphi_{ab})_{(a,b) \in A \times A}$*, called the* ***transition maps****, defined on* $O_a \cap O_b$ *whenever* $O_a \cap O_b \neq \varnothing$ *, such that* $\varphi_{ab}(m)$ *is a diffeomorphism on* V *and*

$\forall p \in \pi^{-1}(O_a \cap O_b), p = \varphi_a(m, v_a) = \varphi_b(m, v_b) \Rightarrow v_b = \varphi_{ba}(m)(v_a)$

meeting the cocycle conditions, whenever they are defined:

$\forall a, b, c \in A : \varphi_{aa}(m) = Id; \varphi_{ab}(m) \circ \varphi_{bc}(m) = \varphi_{ac}(m)$

$$\begin{array}{ccccc} & E & & & \\ & \pi^{-1}(O_a) & & & \\ & \downarrow & \nwarrow & & \\ \pi & \downarrow & & \varphi_a & \\ & \downarrow & & \nwarrow & \\ & O_a & \longrightarrow \longrightarrow \longrightarrow & & O_a \times V \\ & M & & & M \times V \end{array}$$

A **change of trivialization** on a fiber bundle $E(M, V, \pi)$ with atlas $(O_a, \varphi_a)_{a \in A}$ is the definition of a new, compatible atlas $(O_a, \widetilde{\varphi}_a)_{a \in A}$, the trivialization $\widetilde{\varphi}_a$ is defined by :

$$\varphi_a \to \widetilde{\varphi}_a :: p = \varphi_a(m, v_a) = \widetilde{\varphi}_a(m, \widetilde{v}_a) \Leftrightarrow \widetilde{v}_a = \chi_a(m)(v_a)$$

where $(\chi_a)_{a \in A}$ is a family of diffeomorphisms on V. The new transition maps are : $\widetilde{\varphi}_{ba}(m) = \chi_b(m) \circ \varphi_{ba}(m) \circ \chi_a(m)^{-1}$ and the cocycle conditions are met. χ_a is defined in O_a and valued in the set of diffeomorphisms over V : we have a **local change of trivialization**. When χ_a is constant in O_a (this is the same diffeomorphism whatever m) this is a **global change of trivialization**. When V has an algebraic structure additional conditions are required from χ.

A class r **section** S on a fiber bundle $E(M,V,\pi)$ with trivialization $(O_a,\varphi_a)_{a\in A}$ is a family of maps $(\sigma_a)_{a\in A}, \sigma_a \in C_r(O_a;V)$ such that:

$\forall a \in A, m \in O_a : S(m) = \varphi_a(m,\sigma_a(m))$

In a change of trivialization : $\varphi \to \widetilde{\varphi} :: p = \varphi(m,\sigma(m)) = \widetilde{\varphi}(m,\widetilde{\sigma}(m)) \Leftrightarrow \widetilde{\sigma}(m) = \chi(m)(\sigma)$

Notation 20 $\mathfrak{X}_r(E)$ *is the set of class r sections of the fiber bundle E*

As for manifolds usually one needs more than 1 open O to define a fiber bundle (a sphere requires 2), but the cover $(O)_{a\in A}$ is usually fixed and is not a concern. Meanwhile changes of trivializations are important, notably for physicists. The choice of a trivialization is arbitrary, and in Physics the implementation of the Principle of Relativity tells that a law should not depend on the choice of trivialization. This is similar to what we say for manifolds or vector spaces : a physical property does not depend on the charts or the basis in which it is expressed. If a quantity is tensorial it must change according to precise rules in a change of basis. If an observer uses a trivialization, and another one uses another trivialization, then their measures (the $v \in V$) of the same phenomenon should be related by a precise mathematical transformation : they are equivariant.

It is easy to see that the formulas for a change of trivialization read as the formulas for the transitions by taking $\varphi_{ba}(x) = \chi_a(x)$. So in this paper I will drop the index a labelling the open O_a when it is not required : it is assumed that the formulas hold for a given cover, which does not change.

15.1.1 Vector bundle

Definition 21 *A* ***vector bundle*** $E(M,V,\pi)$ *is a fiber bundle whose standard fiber V is a Banach vector space and transitions maps* $\varphi_{ab}(x) : V \to V$ *are continuous linear invertible maps :* $\varphi_{ab}(x) \in G\mathcal{L}(V;V)$

The fiber over each point of a vector bundle $E(M,V,\pi)$ has a canonical structure of vector space, isomorphic to V. Practically a vector bundle is defined by the choice, at each point $m \in M$ of a **holonomic basis** $(e_i(m))_{i\in I}$ of V and an element of the vector bundle $E(M,V,\pi)$ reads : $V(m) = \varphi\left(m, \sum_{i\in I} v^i e_i\right) = \sum_{i\in I} v^i e_i(m)$

A change of trivialization is a linear change of holonomic basis, which can depend on the point m :

$e_i(m) = \varphi(m,e_i) \to \widetilde{e}_i(m) = \widetilde{\varphi}(m,e_i) = \chi(m)^{-1} e_i(x)$

$V(m) = \sum_{i\in I} v^i e_i(m) = \sum_{i\in I} \widetilde{v}^i \widetilde{e}_i(m) \Leftrightarrow \sum_{i\in I} \widetilde{v}^i \widetilde{e}_i = \chi(m)\left(\sum_{i\in I} v^i e_i\right)$

This is the generalization of the usual tangent bundle over a manifold, defined by frames : the vector space can have different properties and notably a dimension different from the dimension of the manifold.

A **complex vector bundle** $E(M, V, \pi)$ over a real manifold M is a fiber bundle whose standard fiber V is a Banach complex vector space and transitions functions at each point $\varphi_{ab}(m) \in G\mathcal{L}(V;V)$ are complex continuous linear maps.

Using the tensorial product at each point of the vectors $e_i(m)$ any tensor can be defined in the tensorial bundle $\otimes E(M, \otimes V, \pi)$. Its elements is a map from M to the fixed tensorial algebra $\otimes V$.

Usually the bases must meet some additional properties, as being orthonormal, and we have an associated vector bundle.

A morphism between the vector bundles $E_j(M_j, V_j, \pi_j), j = 1, 2$ is a couple (F, f) :

$f : M_1 \to M_2, F : E_1 \to E_2$

$\forall m \in M_1 : F\left(\pi_1(m)^{-1}\right) \in \mathcal{L}\left(\pi_1(m)^{-1}; \pi_2(f(m))^{-1}\right)$

$f \circ \pi_1 = \pi_2 \circ F$

A scalar product on a real vector bundle $E(M, V, \pi)$ is a map defined on M such that $r(m)$ is a bilinear symmetric, non degenerated form on V. It is defined either by a 2nd order covariant tensor on the tensorial bundle $\otimes E(M, \otimes V, \pi)$, or by a a 2 form on V such that the transition maps (or the change of trivilizations) preserve the scalar product.

15.1.2 Principal fiber bundle

Definition 22 *A* ***principal (fiber) bundle*** $P(M, G, \pi)$ *is a fibered manifold, with* G *a Lie group which acts freely on* P *on the right by an action* $\rho : P \times G \to P :: \rho(p, g)$ *such that the orbits of the action* $\{\rho(p, g), g \in G\} = \pi^{-1}(m)$.

Then an atlas of P is defined as follows :

the choice of a set of bijective maps : $\mathbf{p}_a : O_a \to P :: \pi(\mathbf{p}_a(m)) = m$ which are the **gauge** at each point

then

$\varphi_a : O_a \times G \to P :: p = \varphi_a(m, g_a) = \rho(\mathbf{p}_a(m), g_a) \Rightarrow \pi(p) = \pi(p_a) = m$

This is equivalent to define a section $\mathbf{p}_a(m) = \varphi_a(m, 1)$

Then $\rho(\varphi_a(m, g), h) = \varphi_a(m, gh), \pi(\rho(p, g)) = \pi(p)$

The transition maps are :

$\forall p \in \pi^{-1}(O_a \cap O_b), \mathbf{p}_b(m) = \rho(\mathbf{p}_a(m), g_{ba}(m)),$

$p = \varphi_a(m, g_a) = \varphi_b(m, g_b) = \rho(\mathbf{p}_a(m), g_a) = \rho(\mathbf{p}_b(m), g_b) = \rho(\mathbf{p}_a(m), g_{ba}(m) g_b)$

$g_a = g_{ba}(m) g_b \Leftrightarrow g_b = g_{ba}(m)^{-1} \cdot g_a$

This action is free and for $p \in \pi^{-1}(m)$ the map $\rho(p, .) : G \to \pi^{-1}(m)$ is a diffeomorphism, so $\rho'_g(p, g)$ is invertible.

The orbits of the action are the sets $\pi^{-1}(m)$.

The right action of G on a principal bundle $P(M, G, \pi)$ does not depend on the trivialization.

A change of trivialization is a change of gauge :

$\mathbf{p}(m) \to \widetilde{\mathbf{p}(m)} = \rho(\mathbf{p}(m), \varkappa(m))$

$\rho\left(\mathbf{p}\left(m\right),g\right)=\rho\left(\widetilde{\mathbf{p}\left(m\right)},\widetilde{g}\right)=\rho\left(\rho\left(\mathbf{p}\left(m\right),\varkappa\left(m\right)\right),\widetilde{g}\right)=\rho\left(\mathbf{p}\left(m\right),\varkappa\left(m\right)\cdot\widetilde{g}\right)\Leftrightarrow$ $\widetilde{g}=\varkappa\left(m\right)^{-1}\cdot g$

A bundle of linear frames on a vector bundle $E\left(M,V,\pi\right)$ is the set of its linear bases. It has the structure of a principal bundle with base M and group the group to go from one basis to another at the same point (it is included in $GL\left(V;V\right)$).

15.1.3 Associated fiber bundles

Definition 23 *A fiber bundle $E\left(M,V,\pi\right)$ with atlas $\left(O_a,\varphi_a\right)_{a\in A}$ has a **G-bundle structure** if there are :*

- a Lie group G (on the same field as E)

- a left action of G on the standard fiber V : $\lambda:G\times V\rightarrow V$

- a family $\left(g_{ab}\right)_{a,b\in A}$ of maps : $g_{ab}:\left(O_a\cap O_b\right)\rightarrow G$

such that the transition maps read : $\varphi_{ba}\left(m\right)\left(u\right)=\lambda(g_{ba}(m),u)$

$p=\varphi_a(m,u_a)=\varphi_b(m,u_b)\Rightarrow u_b=\lambda(g_{ba}(m),u_a)$

With a G bundle structure the changes of trivialization are given by the left action of the group G

$\varphi\rightarrow\widetilde{\varphi}::p=\varphi\left(x,u\right)=\widetilde{\varphi}\left(x,\widetilde{u}\right)\Leftrightarrow\widetilde{u}=\lambda\left(\varkappa\left(x\right),u\right)$

Notation 24 *$E=M\times_G V$ is a G-bundle with base M, standard fiber V and left action of the group G*

Definition 25 *An **associated bundle** is a structure which consists of :*

i) a principal bundle $P(M,G,\pi_P)$ with right action $\rho:P\times G\rightarrow G$

ii) a manifold V and a left action $\lambda:G\times V\rightarrow V$ which is effective :

$\forall g,h\in G,v\in V:\lambda\left(g,v\right)=\lambda\left(h,v\right)\Rightarrow g=h$

iii) an action of G on $P\times V$ defined as :

$\Lambda:G\times\left(P\times V\right)\rightarrow P\times V::\Lambda\left(g,\left(p,u\right)\right)=\left(\rho\left(p,g\right),\lambda\left(g^{-1},u\right)\right)$

iv) the equivalence relation $\sim$ on $P\times V$: $\forall g\in G:\left(p,u\right)\sim\left(\rho\left(p,g\right),\lambda\left(g^{-1},u\right)\right)$

The associated bundle is the quotient set $E=\left(P\times V\right)/\sim$

The projection $\pi_E:E\rightarrow M::\pi_E\left(\left[p,u\right]_\sim\right)=\pi_P\left(p\right)$

Notation 26 *$P\left[V,\lambda\right]$ is the associated bundle, with principal bundle P, fiber V and action $\lambda:G\times V\rightarrow V$*

The elements of an associated bundle are classes of equivalence located at m. It is equivalent to tell that the trivialization of $P\left[V,\lambda\right]$ is given by $\varphi_E\left(m,u\right)=\left(\mathbf{p}\left(m\right),u\right)$ with the standard gauge $\mathbf{p}\left(m\right)$.

Because λ is effective, $\lambda\left(.,u\right)$ is injective and $\lambda'_g\left(g,u\right)$ is invertible.

In a change of gauge in P :

$\mathbf{p}\left(m\right)\rightarrow\widetilde{\mathbf{p}\left(m\right)}=\rho\left(\mathbf{p}\left(m\right),\varkappa\left(m\right)\right)\Rightarrow\rho\left(\mathbf{p}\left(m\right),g\right)=\rho\left(\widetilde{\mathbf{p}\left(m\right)},\widetilde{g}\right)\Leftrightarrow\widetilde{g}=\varkappa\left(m\right)^{-1}\cdot g$

$\left(\mathbf{p}\left(m\right),u\right)=\left(\widetilde{\mathbf{p}\left(m\right)},\widetilde{u}\right)=\left(\rho\left(\mathbf{p}\left(m\right),\varkappa\left(m\right)\right),\widetilde{u}\right)\sim\left(\mathbf{p}\left(m\right),\lambda\left(\varkappa\left(m\right)^{-1},\widetilde{u}\right)\right)$

$\lambda\left(\varkappa(m)^{-1},\widetilde{u}\right)=u\Leftrightarrow\widetilde{u}=\lambda(\varkappa(m),u)$

For any fiber bundle $E(M,V,\pi)$ endowed with a G structure there is a unique principal bundle $P_G(M,G,\pi_G)$ such that E is the associated bundle $P_G[G,\lambda]$.

It is always possible, and practical, to work locally in the standard gauge $\mathbf{p}(m)$ to express an element of an associated bundle.

A section of an associated bundle is a map $S:M\rightarrow V:(\mathbf{p}(m),S(m))\sim\left(\rho(\mathbf{p}(m),u),\lambda\left(u^{-1},S(m)\right)\right)$

Definition 27 *An* ***associated vector bundle*** *is an associated bundle* $P[V,r]$ *where* (V,r) *is a continuous representation of* G *on a Banach vector space* V *on the same field as* $P(M,G,\pi)$

So we have the equivalence relation on $P\times V:(p,u)\sim\left(\rho(p,g),r\left(g^{-1}\right)u\right)$

In a change of gauge in P :

$\mathbf{p}(m)\rightarrow\widetilde{\mathbf{p}(m)}=\rho(\mathbf{p}(m),\varkappa(m))\Rightarrow(\mathbf{p}(m),u)=\left(\widetilde{\mathbf{p}(m)},r(\varkappa(m))u\right)$

Any vector bundle $E(M,V,\pi)$ has the structure of an associated bundle $P[V,r]$ where $P(M,G,\pi_P)$ is the bundle of its linear frames and r the natural action of the group G on V.

The **adjoint bundle** of a principal bundle $P(M,G,\pi)$ is the associated vector bundle $P[T_1G,Ad]$

15.1.4 Clifford bundle

Let (V,r) any vector space on a field K, endowed with a symmetric form r valued in K, which is not degenerated. Then one can define the Clifford bundle $Cl(V,r)$.

If $E(M,V,\pi)$ is a vector bundle, then one can define similarly a Clifford bundle $E(M,Cl(V,r),\pi)$, through an orthonormal basis at each point. The structure of the Clifford algebra can be real or complex. No relation between the Clifford algebra and the tangent bundle TM is assumed a priori. A morphism between the Clifford bundles $E_j(M_j,Cl(V_j,r_j),\pi_j),j=1,2$ is a couple (F,f) :

$f:M_1\rightarrow M_2,F:E_1\rightarrow E_2$

$\forall m\in M_1:F\left(\pi_1(m)^{-1}\right)\in\mathcal{L}\left(\pi_1(m)^{-1};\pi_2(f(m))^{-1}\right)$

$f\circ\pi_1=\pi_2\circ F$

where F preserves the scalar product and the algebraic structure.

The restriction of $E(M,Cl(V,r),\pi)$ to $Spin(V,r)$ gives a principal bundle $P(M,Spin(V,r),\pi)$ with its natural right action. This is equivalent to define a standard gauge $\mathbf{p}(x)=1\in Cl(V,r)(x)$ with the natural right action $p(x)=\rho(\mathbf{p}(x),g)=g(x)$.

The Clifford automorphisms on $Cl(V,r)$ are given by the adjoint map Ad_g with $g\in O(Cl(V,r))\equiv K\times Spin(V,r)$. So that, if $G\subset Cl(V,r)$ is a group one can define the associated Clifford bundle $P_G[Cl(V,r),\lambda]$ with the left action $\lambda(g,Z)=Ad_gZ$ at m.

15.2 Tangent bundle to fiber bundles

15.2.1 Fiber bundle

A fiber bundle $E(M,V,\pi)$ is a manifold, its tangent bundle is a vector bundle $TE(TM,TV,T\pi)$

Any vector $Y_p \in T_pE$ of the fiber bundle $E(M,V,\pi)$ has a unique decomposition : $Y_p = \varphi_a'(m,v_a)(y_m,y_{av})$ where : $y_m = \pi'(p)Y_p \in T_{\pi(p)}M$ does not depend on the trivialization. It can be uniquely written: $Y_p = \sum_\beta y_m^\beta \partial m_\beta + \sum_i y_v^i \partial v_i$ with the basis, called a **holonomic** basis,

$\partial m_\alpha = \varphi_{ax}'(m,u)\partial\xi_\alpha, \partial v_i = \varphi_{av}'(m,v)\partial\eta_i$ where $\partial\xi_\beta, \partial\eta_i$ are holonomic bases of T_mM, T_uV

The coordinates of Y_p in this atlas are : $\left(\xi^\beta, \eta^i, y_m^\beta, y_v^i\right)$

$\pi'(p)Y_p = \sum_\beta y_m^\beta \partial\xi_\beta$

Notation 28 *∂m_β (lattin letter, greek indices) is the part of the basis on TE induced by M*

Notation 29 *∂v_i (lattin letter, lattin indices) is the part of the basis on TE induced by V.*

Notation 30 *$\partial\xi_\beta$ (greek letter, greek indices) is a holonomic basis on TM.*

Notation 31 *$\partial\eta_i$ (greek letter, lattin indices) is a holonomic basis on TV.*

The **vertical space** at p is : $V_pE = \ker\pi'(p) = \left\{\sum_i y_v^i \partial v_i\right\}$.This is a vector subspace of T_pE, isomorphic to T_vV, which does not depend on the trivialization. The **vertical bundle** of a fiber bundle $E(M,V,\pi)$ is the vector bundle : $VE(M,TV,\pi) : M \times TV \to VE :: \varphi_u'(m,v)y_v$

A vector field $Y \in \mathfrak{X}(TE)$ is **projectable** if $\pi'(p)(Y) \in \mathfrak{X}(TM)$

15.2.2 Vector bundle

The tangent space to a vector bundle $E(M,V,\pi)$ has the vector bundle structure : $TE(TM, V \times V, T\pi)$. A vector v_p of T_pE at (m,v_m) reads :

$v_p = \sum_{\alpha\in A} v_m^\alpha \partial m_\alpha + \sum_{i\in I} v_u^i \mathbf{e}_i(m)$ where $\partial m_\alpha = \varphi_{Ex}'(m,v)\partial\xi_\alpha$ so the vertical bundle is equivalent to E : $VE = \left\{\sum_{i\in I} v_u^i \mathbf{e}_i(m)\right\}$.

In a change of gauge :

$e_i(m) \to \widetilde{e}_i(m) = \chi(m)e_i(m) \Rightarrow v_p = \sum_{\alpha\in A} v_x^\alpha \partial mx_\alpha + \sum_{i\in I} v_u^i \mathbf{e}_i(m) = \sum_{\alpha\in A} v_x^\alpha \partial m_\alpha + \sum_{i\in I} \widetilde{v}_u^i \widetilde{\mathbf{e}}_i(m)$

$\sum_{i\in I} \widetilde{v}_u^i \widetilde{\mathbf{e}}_i = \sum_{i\in I} v_u^i \mathbf{e}_i$

15.2.3 Lie group

Action of a Lie group on a set

A **left-action** of G on E is a continuous map : $\lambda : G \times E \to E$ such that :

$\forall p \in E, \forall g, g' \in G : \lambda(g, \lambda(g',p)) = \lambda(g \cdot g', p); \lambda(1,p) = p$

A **right-action** of G on E is a continuous map : $\rho : E \times G \to E$ such that :

$\forall p \in E, \forall g, g' \in G : \rho\left(\rho\left(p, g'\right), g\right)\right) = \rho\left(p, g' \cdot g\right); \rho\left(p, 1\right) = p$

If E is a manifold and λ, ρ differentiable, we have the identities :

$\lambda'_p\left(1, p\right) = Id; \rho'_p(p, 1) = Id$

$\lambda'_g\left(g, p\right) = \lambda'_g\left(1, \lambda\left(g, p\right)\right) R'_{g^{-1}}g = \lambda'_p\left(g, p\right) \lambda'_g\left(1, p\right) L'_{g^{-1}}g$

$\rho'_g(p, g) = \rho'_g(\rho(p, g), 1) L'_{g^{-1}}(g) = \rho'_p(p, g)\rho'_g(p, 1)R'_{g^{-1}}(g)$

$\lambda'_p\left(g, p\right)^{-1} = \lambda'_p\left(g^{-1}, \lambda\left(g, p\right)\right); \left(\rho'_p(p, g)\right)^{-1} = \rho'_p(\rho(p, g), g^{-1})$

$\left(\lambda'_p\left(g, p\right)\right)^{-1} = \lambda'_p\left(g^{-1}, \lambda\left(g, p\right)\right)$

$\left(\rho'_p\left(p, g\right)\right)^{-1} = \rho'_p\left(\rho\left(p, g\right), g^{-1}\right)$

Fundamental vector fields

Notation 32 *Let λ be a differentiable action of the Lie group on the manifold N. The **fundamental vector fields** are the vectors fields on N generated by a vector u of the Lie algebra of G: $\zeta_L : T_1G \to TN :: \zeta_L\left(u\right)\left(p\right) = \lambda'_g\left(1, p\right) u$*

We have similarly for a right action : $\zeta_R : T_1G \to TN :: \zeta_R\left(u\right)\left(p\right) = \rho'_g\left(p, 1\right) u$

They have the following properties :

i) the maps ζ_L, ζ_R are linear

ii) $\left[\zeta_L\left(u\right), \zeta_L\left(v\right)\right]_{\mathfrak{X}(TN)} = -\zeta_L\left(\left[u, v\right]_{T_1G}\right)$

$\left[\zeta_R\left(u\right), \zeta_R\left(v\right)\right]_{\mathfrak{X}(TN)} = \zeta_R\left(\left[u, v\right]_{T_1G}\right)$

iii) $\lambda'_p\left(x, q\right)|_{p=q}\zeta_L\left(u\right)\left(q\right) = \zeta_L\left(Ad_x u\right)\left(\lambda\left(x, q\right)\right)$

$\rho'_p\left(q, x\right)|_{p=q}\zeta_R\left(u\right)\left(q\right) = \zeta_R\left(Ad_{x^{-1}} u\right)\left(\rho\left(q, x\right)\right)$

iv) $\zeta_L\left(u\right) = \lambda_*\left(R'_x\left(1\right) u, 0\right), \zeta_R\left(u\right) = \rho_*\left(L'_x\left(1\right) u, 0\right)$

v) the fundamental vector fields span an integrable distribution over N, whose leaves are the connected components of the orbits.

vi) The flow of the fundamental vector fields is :

$\Phi_{\zeta_L(u)}\left(t, p\right) = \lambda\left(\exp tu, p\right)$

$\Phi_{\zeta_L(u)}\left(t, p\right) = \rho\left(p, \exp tu\right)$

Tangent bundle to a Lie group

Notation 33 *$L'_a x$ is the derivative of $L_a\left(g\right) = a \cdot g$ with respect to g, at $g = x$; $L'_a x \in G\mathcal{L}\left(T_xG; T_{ax}G\right)$*

$R'_a x$ is the derivative of $R_a\left(g\right) = g \cdot a$ with respect to g, at $g = x$; $R'_a x \in G\mathcal{L}\left(T_xG; T_{xa}G\right)$

We have the identities :

$\left(L'_g 1\right)^{-1} = L'_{g^{-1}}g; \left(R'_g 1\right)^{-1} = R'_{g^{-1}}g$

$\rho'\left(g\right) = \rho\left(g\right)\rho'\left(1\right) L'_{g^{-1}}g$

$\rho'\left(g\right) R'_g 1\left(X\right) = \rho'\left(1\right)\left(X\right)\rho\left(g\right)$

The adjoint map is $Ad_g = L'_g g^{-1} \circ R'_{g^{-1}}1 = R'_{g^{-1}}g \circ L'_g 1$

The tangent bundle $TG = \cup_{g \in G} T_gG$ of a Lie group is still a Lie group with the actions :

$M:TG\times TG\rightarrow TG::M(X_g,Y_h)=R'_h(g)X_g+L'_g(g)Y_h\in T_{gh}G$

$\Im:TG\rightarrow TG::\Im(X_g)=-R'_{g^{-1}}(1)\circ L'_{g^{-1}}(g)X_g=-L'_{g^{-1}}(g)\circ R'_{g^{-1}}(g)X_g=-Ad_{g^{-1}}X_g\in T_{g^{-1}}G$

Identity : $X_1=0_1\in T_1G$

It is isomorphic to the semi direct product group : $TG\simeq(T_1G,+)\propto_{Ad}G$ with the map $Ad:G\times T_1G\rightarrow T_1G$

$(g,\kappa)\times(g',\kappa')=(gg',\kappa+Ad_g\kappa')$

$(g,\kappa)^{-1}=\left(g^{-1},-Ad_{g^{-1}}\kappa\right)$

Tangent bundle of a Lie group defined in a Clifford algebra

For any group $G\in Cl$ the left invariant vector fields are $Z(\tau)=\exp\tau T$ with $T\in T_1G$

The tangent space at $g\in G$ to the tangent bundle TG is $T_gG=\{g\cdot T,T\in T_1G\}$

The tangent bundle TG is a group, isomorphic to the group product : $TG\simeq G\times(T_1G,+)$

$(g,\kappa)\times(g',\kappa')=(gg',\kappa+\kappa')$

$(g,\kappa)\times(1,0)=(g,\kappa)$

$(g,\kappa)^{-1}=\left(g^{-1},-\kappa\right)$

15.2.4 Tangent bundle on a principal bundle

The fundamental vector fields on the principal fiber bundle P are the vector fields on TP defined, for any fixed κ in T_1G , by :

$\zeta(\kappa):M\rightarrow VP::\zeta(\kappa)(p)=\rho'_g(p,1)\kappa$

They belong to the vertical bundle $\pi'(p)\zeta(\kappa)(p)=0$, and span an integrable distribution over P, whose leaves are the connected components of the orbits

$\forall X,Y\in T_1G:[\zeta(X),\zeta(Y)]_{VP}=\zeta\left([X,Y]_{T_1G}\right)$

In a change of gauge in P :

$\mathbf{p}(x)\rightarrow\widetilde{\mathbf{p}(m)}=\rho(\mathbf{p}(m),\varkappa(m))$

$\zeta(\kappa)(p)\rightarrow\zeta(Ad_{\varkappa^{-1}}\kappa)(p)$

The charts of $P(M,G,\pi)$ are defined over an open cover by :

$\forall p\in\pi^{-1}(O_a\cap O_b),\varphi_a(m,g)=\rho(\mathbf{p}_a(m),g)$

where $\mathbf{p}_a\in C_r(O_a,P)$ defines the gauge. So we assume that it is a differentiable map and $\mathbf{p}'_a(m)v_x$ is a vector of $T_{\mathbf{p}_a(m)}P$.

A vector of T_pP at $p=\varphi_a(m,g)=\rho(\mathbf{p}_a(m),g)$ is then :

$V_p=\rho'_p(\mathbf{p}_a(m),g)\mathbf{p}'_a(m)v_m+\rho'_g(\mathbf{p}_a(m),g)\kappa_g$ with $\kappa_g\in T_gG$

$\rho'_g(\mathbf{p}_a(m),g)\kappa_g=\rho'_g(\rho(\mathbf{p}_a(m),g),1)L'_{g^{-1}}(g)\kappa_g=\zeta\left(L'_{g^{-1}}g(\kappa_g)\right)(\rho(\mathbf{p}_a(m),g))$

A vector $V_p\in T_pP$ of the principal bundle $P(M,G,\pi)$ can be written

$V_p=V_m+\zeta(\kappa)(\rho(\mathbf{p}_a(x),g))=\sum_\alpha v^\alpha_m\partial m_\alpha+\zeta(\kappa)(p)$ where

$\kappa\in T_1G,\partial m_\alpha(p)=\rho'_p(\mathbf{p}_a(m),g)\mathbf{p}'_a(m)\partial\xi_\alpha,\zeta(\kappa)(p)=\zeta\left(L'_{g^{-1}}g(\kappa_g)\right)(\rho(\mathbf{p}_a(m),g))$

The right action of TG on the tangent bundle reads:

$T\rho\left((\varphi(m,h),\varphi'(m,h)v_m+\zeta(v_h)p),\left(g,\left(L'_g1\right)\kappa_g\right)\right)$

$=\left(\rho(p,g),\varphi'_x(m,hg)v_m+\zeta\left(Ad_{g^{-1}}v_h+\kappa_g\right)(\rho(p,g))\right)$

$T\rho:TP\times TG\rightarrow TP::T\rho\left((p,\sum_\alpha v^\alpha_x\partial m_\alpha(p)+\zeta(\kappa)(p)),(\varkappa,\kappa_\varkappa)\right)$

$= (\rho (p, \varkappa), \sum_\alpha v_x^\alpha \partial m_\alpha (\rho (p, \varkappa)) + \zeta (Ad_{\varkappa^{-1}} \kappa + \kappa_\varkappa) (\rho (p, \varkappa)))$

The tangent bundle TP of a principal fiber bundle $P(M, G, \pi)$ is a principal bundle $TP(TM, TG, T\pi)$.

The vertical bundle VP is :

a trivial vector bundle over P : $VP(P, T_1G, \pi) \simeq P \times T_1G$

a principal bundle over M with group TG : $VP(M, TG, \pi)$

The vertical bundle can equivalently be defined as the bundle of the fundamental vector fields, that is the adjoint bundle $P[T_1G, Ad] : (p, \zeta(\kappa)(p)) \sim (\rho(p, \varkappa), \zeta(Ad_{\varkappa^{-1}}\kappa)(\rho(p, \varkappa)))$

In a change of gauge in P :

$\mathbf{p}(m) \rightarrow \widetilde{\mathbf{p}(m)} = \rho(\mathbf{p}(m), \varkappa(m)) \Rightarrow p = \rho(\mathbf{p}(m), g) = \rho\left(\widetilde{\mathbf{p}(m)}, \widetilde{g}\right) \Leftrightarrow \widetilde{g} = \varkappa(m)^{-1} \cdot g$

$v_p = V_m + \zeta(\kappa)(p) = V_m + \zeta(Ad_{\varkappa^{-1}}\kappa)(p)$

15.2.5 Associated bundle

The tangent bundle TE of the associated bundle $E = P[V, \lambda]$ is the associated bundle $TP[TV, T\lambda] \sim TP \times_{TG} TV$

The left action of TG on TV is :

$T\lambda = (\lambda, \lambda') : TG \times TV \rightarrow TV :: T\lambda((\varkappa, \kappa_\varkappa), (u, v_V)) = (\lambda(\varkappa, u), \lambda'(\varkappa, \kappa_\varkappa)(u, v_V))$

The equivalence relation is :

$((p, v_p + \zeta(\kappa)(p)), (u, v_V)) \sim$
$\left((\rho(p, \varkappa), v_p + \zeta(Ad_{\varkappa^{-1}}\kappa + \kappa_\varkappa)\rho(p,\varkappa)), \left(\lambda\left(\varkappa^{-1}, u\right), \lambda'\left(\varkappa^{-1}, -Ad_{\varkappa^{-1}}\kappa_\varkappa\right)(u, v_V)\right)\right)$

The projection is

$\pi'(p)((p, v_p + \zeta(\kappa)(p)), (u, v_V)) = \pi'(p) v_p$

The vertical bundle $\ker T\pi_E$ is generated by the fundamental vector fields : $\left(\zeta(\kappa)(p), -\lambda'_g(1, g)\kappa\right) \in VP \times TV$

16 Jets

In Differential Geometry one avoids as much as possible the coordinates expressions. But this is difficult when dealing with partial derivatives. The r-jet formalism provides a convenient solution, which is an essential tools for differential equations.

16.1 Jets on a manifold

For any r differentiable map $f \in C_r(M; N)$ between manifolds, the partial derivatives $\frac{\partial^s f}{\partial \xi^{\alpha_1} ... \partial \xi^{\alpha_s}}$ at a point m are s symmetric linear maps from the tangent space T_mM to the tangent space T_pN. As any linear map their expression in holonomic bases is a set of scalars $f^i_{\alpha_1 ... \alpha_s}$, *symmetric* in the indices $\alpha_1, ..\alpha_s$.

The relation of equivalence on $C_r(M; N)$:

$f \sim g \Leftrightarrow f(m) = g(m) = p, ... \frac{\partial^s f}{\partial \xi^{\alpha_1} ... \partial \xi^{\alpha_s}}(m) = \frac{\partial^s g}{\partial \xi^{\alpha_1} ... \partial \xi^{\alpha_s}}(m), s = 1...r, \alpha_k = 1.. \dim M$

defines classes of equivalences of maps f, g which have the same value and partial derivative at m up to the order r. They are characterized by the set of scalars :

$j^r = \left(z^i_{\alpha_1...\alpha_s} \in \mathbb{R}, s = 1...r, \alpha_k = 1..\dim M, i = 1..\dim N\right) \in J^r_m (M, N)_p$

$z^i_{\alpha_1...\alpha_s}$ symmetric in the indices $\alpha_1, ..\alpha_s$

The set $J^r_m (M, N)_p$ is a vector space. The $z^i_{\alpha_1...\alpha_s}$ are the components of symmetric tensors belonging to $\odot^s T_m M^* \otimes T_p N$.

A **r jet** with source m and target p is a set $j^r_{m,p} = (m, p, j^r)$ and more generally a r jet is a map $j^r (m) = (m, p (m), j^r (m))$

The **r jet prolongation** of f is the map :

$J^r f (m) = \left(m, f (m), \frac{\partial^s f^i}{\partial \xi^{\alpha_1}...\partial \xi^{\alpha_s}} (m), s = 1...r, \alpha_k = 1..\dim M, i = 1..\dim N\right)$

A key point is that any map f has a r-jet prolongation, which is a r-jet, but conversely in a r-jet *there is a priori no relation between the* $z^i_{\alpha_1...\alpha_s} (m)$: they do not correspond necessarily to the derivatives of the same map f. The distinction between $\frac{\partial f^i}{\partial \xi^{\alpha_1}...\partial \xi^{\alpha_s}}$ and $z^i_{\alpha_1...\alpha_s}$ is useful : a differential equation is a relation between components of a r-jet : $L\left(m, z, z^i_{\alpha_1...\alpha_s}\right) = 0$ and a solution is a map f of $C_r (M; N)$ such that $L\left(m, f (m), \frac{\partial^s f}{\partial \xi^{\alpha_1}...\partial \xi^{\alpha_s}}\right) = 0$.

16.2 Jets on fiber bundles

Fiber bundles $E (M, V, \pi)$ are manifolds, so we can implement the principle above by taking as maps sections on E. They are defined by :

$S : M \to E :: S (m) = \varphi_P (m, z (m))$

and r-jets on E are defined by $r-$jets prolongations of z.

The coordinates of $z (m) \in V$ are $\eta^i, i = 1...\dim V$ in a chart $\left\{\eta^i\right\} = \varphi_V (z)$ of V.

The partial derivatives $\frac{\partial^s z}{\partial \xi^{\alpha_1}...\partial \xi^{\alpha_s}}$ are linear maps whose *components in charts* of M, V are scalars : $\eta^i_{\alpha_1...\alpha_s}$ with the condition that they are symmetric in the indices $\alpha_1, ...\alpha_s$.

The r-jet prolongation $J^r E$ of the fiber bundle $E (M, V, \pi)$ is the vector bundle :

$J^r E \left(E, J^r_0 \left(\mathbb{R}^{\dim M}, V\right)_0, \pi^r\right)$ with basis E, fiber the vector space :

$J^r_0 \left(\mathbb{R}^{\dim M}, V\right)_0 = \left\{\eta^i_{\alpha_1...\alpha_s} \in \mathbb{R}, s = 1...r, \alpha_k = 1..\dim M, i = 1..\dim V\right\}$ and projection : $\pi^r : J^r E \to E$.

So the map :

$\Phi^r_a : (\pi^r)^{-1} (O_a) \to \mathbb{R}^m \times \mathbb{R}^n \times J^r_0 (\mathbb{R}^m, \mathbb{R}^n)_0 ::$

$\Phi^r_a (j^r Z) = \left(\xi^\alpha, \eta^i, \eta^i_{\alpha_1..\alpha_s}, s = 1...r, 1 \leq \alpha_k \leq \alpha_{k+1} \leq m, i = 1..n\right)$

is a chart of $J^r E$ *as manifold* and

$\left(\xi^\alpha, \eta^i, \eta^i_{\alpha_1..\alpha_s}, s = 1...r, 1 \leq \alpha_k \leq \alpha_{k+1} \leq m, i = 1..n\right)$ are the coordinates of $j^r Z$

A section on $J^r E$ is a map : $j^r p (m) = \left(p (m), \eta^i_{\alpha_1...\alpha_s} (m)\right)$ where η^i are coordinates in a chart of V.

The key point is that an element of a jet are expressed through the coordinates in the charts of the manifold V. The definition of the jet bundle is purely local : the transition maps are not involved.

A section S on E gives a section on $J^r E$: $J^r S = \left(S(m), \frac{\partial^s z^i}{\partial \xi^{\alpha_1} ... \partial \xi^{\alpha_s}}(m) \right)$

Two sections S, S' belong to the same $r-$jet if the value of z, z' and their r derivatives are equal.

Because $J^r E$ is a manifold, it has a tangent bundle $TJ^r E$ whose elements are expressed simply by their components :

$\delta j^r Z = \left\{ p(x), \delta \eta^i_{\alpha_1 ... \alpha_s} \in \mathbb{R}, s = 1...r, \alpha_k = 1.. \dim M, i = 1.. \dim V \right\}$

The r-jet prolongation of a vector bundle (associated or not) $E(M; V; \pi)$ is a vector bundle. The holonomic basis of the vector bundle $J^r E$ is the set of vectors $\{e_i, e_i^{\alpha_1 ... \alpha_s}, \alpha_p = 1... \dim M, i = 1... \dim V, s = 0...r\}$ localized at $m \in M$. In a change of basis in V $\mathbf{e}_i^{\alpha_1 ... \alpha_s} \in \odot_s TM^* \otimes V$ changes as a vector of V. So a section can be denoted

$j^r z = \{ z(m), \delta z_{\alpha_1 ... \alpha_s}(m), \alpha_p = 1... \dim M, i = 1... \dim V, s = 0...r \}$

where

$z(m) = \sum_{i=1}^n z^i(m) e_i(m), \delta z_{\alpha_1 ... \alpha_s}(m) = \sum_{i=1}^n \delta z^i_{\alpha_1 ... \alpha_s} e_i(m)$

and a section of $J^r E$ can be considered as a set of independent sections of E (the $z^i_{\alpha_1 ... \alpha_s}$ must be symmetric in α_j). For the r-jet prolongation of a section

$\delta z^i_{\alpha_1 ... \alpha_s} = \partial z^i_{\alpha_1 ... \alpha_s}$.

A section of a principal bundle $P(M, G, \pi)$ reads : $p(m) = \rho(\mathbf{p}(m), \sigma(m))$. The derivatives $\partial_\alpha \sigma = L'_{\sigma^{-1}} \sigma(\delta_\alpha \kappa)$ where $\delta_\alpha \kappa \in T_1 G$ so that a section of $J^1 P$ can be written : $j^1 p = (p, \delta_\alpha \kappa, \alpha = 1... \dim M)$ where $\delta_\alpha \kappa$ are independent sections of the adjoint bundle.

17 Connection on a fiber bundle

17.1 General fiber bundle

The tangent space $T_p E$ at any point p to a fiber bundle $E(M, V, \pi)$ has a preferred subspace : the vertical space corresponding to the kernel of $\pi'(p)$, which does not depend on the trivialization. And any vector v_p can be decomposed between a part $\varphi'_m(m, u) v_m$ related to $T_m M$ and another part $\varphi'_u(m, u) v_u$ related to $T_u V$. If this decomposition is unique for a given trivialization, it depends on the trivialization. A connection is a geometric decomposition, independent from the trivialization.

Definition 34 *A **connection** on a fiber bundle $E(M, V, \pi)$ is a 1-form Φ on E valued in the vertical bundle, which is a projection :*

$\Phi \in \Lambda_1(E; VE) : TE \to VE :: \Phi \circ \Phi = \Phi, \Phi(TE) = VE$

So Φ acts on vectors of the tangent bundle TE, and the result is in the vertical bundle.

Φ has constant rank, and $\ker \Phi$ is a vector subbundle of TE : the **horizontal bundle** HE.

The tangent bundle TE is the direct sum of two vector bundles :

$TE = HE \oplus VE$:$\forall p \in E : T_pE = H_pE \oplus V_pE$

$V_pE = \ker \pi'(p)$

$H_pE = \ker \Phi(p)$

The horizontal bundle can be seen as displacements along the base $M(m)$ and the vertical bundle as displacements along $V(u)$. The key point here is that the decomposition does not depend on the trivialization : it is purely geometric.

A connection Φ on a fiber bundle $E\left(M, V, \pi\right)$ with atlas $\left(O_a, \varphi_a\right)_{a\in A}$ is uniquely defined by a family of maps $\left(\Gamma_a\right)_{a\in A}$ called the **Christoffel forms** of the connection.

$\Gamma_a \in C\left(\pi^{-1}\left(O_a\right); TM^* \otimes TV\right) ::$

$\Phi(p)v_p = \Phi(p)\left(v_m + v_u\right) = \varphi_a'(m, u)\left(0, v_u + \Gamma_a\left(p\right)v_m\right)$

Γ is is a 1-form, defined on E (it depends on p), acting on vectors of $T_{\pi(p)}M$ and valued in T_uV. It must satisfies the condition in a change of gauge :

$\varphi \to \widetilde{\varphi} :: p = \varphi\left(m, u\right) = \widetilde{\varphi}\left(m, \widetilde{u}\right) \Leftrightarrow \widetilde{u} = \chi\left(m, u\right)$

$\Gamma\left(p\right) \to \widetilde{\Gamma}\left(p\right) = -\chi_m'\left(m, u\right) + \chi_u'\left(m, u\right)\left(\Gamma\left(p\right)\right)$

17.1.1 Covariant derivative

A connection acts on vectors of the tangent bundle TE, the covariant derivative ∇ associated to a connection Φ acts on sections of the fiber bundle itself, this is the map : $\nabla : \mathfrak{X}\left(E\right) \to \Lambda_1\left(M; VE\right) :: \nabla S = S^*\Phi$

The covariant derivative along a vector field y on M is a *section* ∇_yS *of the vertical bundle* :

$\nabla_yS(m) = \Phi(S(m))(S'(m)y) \in \mathfrak{X}\left(VE\right)$

If $S\left(m\right) = \varphi\left(x, \sigma\left(m\right)\right) : \nabla_yS(m) = \sum_{\alpha i}(\partial_\alpha\sigma^i + \Gamma(S\left(m\right))^i_\alpha)y^\alpha\partial u_i$

∇S is linear with respect to the vector field y :

$\nabla_{X+Y}S(m) = \nabla_XS(m) + \nabla_YS(m)$,$\nabla_{kX}S(m) = k\nabla_XS(m)$

17.1.2 Horizontal form

The **horizontal form** of a connection Φ on a fiber bundle $E\left(M, V, \pi\right)$ is the 1 form : $\chi \in \Lambda_1\left(E; HE\right) : \chi\left(p\right) = Id_{TE} - \Phi\left(p\right)$

$\chi \circ \chi = \chi$;

$\chi(\Phi) = 0$;

$VE = \ker \chi$;

$\chi\left(p\right)v_p = \varphi'(x, u)\left(v_m, -\Gamma\left(p\right)v_m\right) \in H_pE$

As we can see in the formula above the horizontal form involves only v_m, so we can lift any object defined on TM onto HE by injecting $v_m \in T_mM$ in the formula. The **horizontal lift** of a vector field on M is the map :

$\chi_H : \mathfrak{X}\left(TM\right) \to \mathfrak{X}\left(HE\right) ::$

$Y\left(p\right) = \chi_H\left(p\right)\left(y\left(\pi\left(p\right)\right)\right) = \varphi'(m, u)\left(y\left(m\right), -\Gamma\left(p\right)y\left(m\right)\right)$

$\chi_H\left(p\right)\left(X\right)$ is a horizontal vector field on TE, which is projectable on TM as X.

By lifting the tangent to a curve we can lift the curve itself through the equation :

$\Phi(P\left(c\left(t\right)\right)\left(\frac{dP}{dt}\right) = \Phi(P\left(c\left(t\right)\right)\left(P'\frac{dc}{dt}\right) = \nabla_{c'(t)}P\left(c\left(t\right)\right) = 0$

If S is a section on E, y a vector field on M the map $S'\left(m\right)y \in \mathfrak{X}\left(TE\right)$ and $S'\left(m\right)y$ has a decomposition in a horizontal part $\chi_H\left(S\right)S'\left(m\right)y$ and a vertical part $\nabla_y S : S'\left(m\right)y = \nabla_y S(m) + \chi_H\left(S\left(m\right)\right)\left(y\left(m\right)\right)$

Proof. $S'\left(m\right)y = \varphi'_x(m,u\left(m\right))y + \varphi'_u(m,u\left(m\right))u'\left(m\right)y = \varphi'\left(m,u\left(m\right)\right)\left(y,u'\left(m\right)y\right)$

$\Phi\left(S\left(m\right)\right)S'\left(m\right)y = \nabla_y S(m) = \varphi'_u(m,u\left(m\right))\left(u'\left(m\right)y + \Gamma\left(S\left(m\right)\right)y\right)$

$= \varphi'_u(x,u\left(m\right))\left(u'\left(m\right) + \Gamma\left(S\left(m\right)\right)\right)y$

$\chi_H\left(S\left(m\right)\right)\left(y\left(x\right)\right) = \varphi'(m,u\left(m\right))\left(y\left(m\right), -\Gamma\left(S\left(m\right)\right)y\left(m\right)\right)$

$\nabla_y S(m) + \chi_H\left(S\left(m\right)\right)\left(y\left(m\right)\right) = \varphi'_x(m,u\left(m\right))y + \varphi'_u(m,u\left(m\right)u'\left(m\right)y = S'\left(m\right)y$ ■

The **curvature** of a connection Φ on the fiber bundle $E\left(M,V,\pi\right)$ is the 2-form $\Omega \in \Lambda_2\left(E;VE\right)$ such that for any vector field X,Y on E :

$\Omega(X,Y) = \Phi([\chi X,\chi Y]_{TE})$ where χ is the horizontal form of Φ. Its local components are given by the Maurer-Cartan formula :

$\varphi_a^*\Omega = \sum_i\left(-d_M\Gamma^i + [\Gamma,\Gamma]_V^i\right)\otimes\partial u_i$

$\Omega = \sum_{\alpha\beta}\left(-\partial_\alpha\Gamma_\beta^i + \Gamma_\alpha^j\partial_j\Gamma_\beta^i\right)dm^\alpha\wedge dm^\beta\otimes\partial u_i$

The curvature is zero if one of the vector X,Y is vertical (because then $\chi X = 0$) so the curvature is an horizontal form, valued in the vertical bundle.

The horizontal bundle is integrable if the curvature is null.

17.2 Connection on a vector bundle

A **linear connection** Φ on a vector bundle is a connection such that its Christoffel forms are linear with respect to the vector space structure of each fiber. It can then be defined by maps with domain in M :

$\Gamma\left(m\right) = \sum_{ij\alpha}\Gamma_{\alpha j}^i\left(m\right)d\xi^\alpha\otimes\mathbf{e}_i\left(m\right)\otimes\mathbf{e}^j\left(m\right)$

$\Phi(\varphi\left(m,\sum_{i\in I}u^ie_i\right))\left(v_m^\alpha\partial m_\alpha + v_u^i\mathbf{e}_i\left(m\right)\right) = \sum_i\left(v_u^i + \sum_{j\alpha}\Gamma_{\alpha j}^i\left(m\right)u^jv_m^\alpha\right)\mathbf{e}_i\left(m\right)$

In a change of gauge :

$e_i\left(m\right)\to\widetilde{e_i}\left(m\right) = \chi\left(m\right)e_i\left(m\right)\Rightarrow v_p = \sum_{\alpha\in A}v_m^\alpha\partial m_\alpha + \sum_{i\in I}v_u^i\mathbf{e}_i\left(m\right) = \sum_{\alpha\in A}v_m^\alpha\partial m_\alpha + \sum_{i\in I}\widetilde{v}_u^i\widetilde{\mathbf{e}}_i\left(m\right)$

$\Gamma\left(m\right)\to\widetilde{\Gamma}\left(m\right) = -\chi'\left(m\right)\chi\left(m\right)^{-1} + \Gamma\left(m\right)$

17.2.1 Covariant derivative

The covariant derivative of a section on E reads :

$\nabla : \mathfrak{X}\left(E\right)\to\Lambda_1\left(M;E\right) :: \nabla X(m) = \sum_{\alpha i}(\partial_\alpha X^i + X^j\Gamma_{j\alpha}^i(m))d\xi^\alpha\otimes\mathbf{e}_i\left(m\right)$

Because the tangent bundle to a vector bundle is equivalent to E, the covariant derivative along a vector field can be seen as another section of the vector bundle: $\nabla_y X(m) = \sum_{\alpha i}(\partial_\alpha X^i + X^j\Gamma_{j\alpha}^i(m))y^\alpha\mathbf{e}_i\left(m\right)$

For any section $S\in\mathfrak{X}\left(E\right)$: $\nabla_X S = \pounds_{\chi_H(X)}S$

It can be extended to any tensor defined on E :

$\nabla\left(S\otimes T\right) = \left(\nabla S\right)\otimes T + S\otimes\left(\nabla T\right)$

If $f\in C_1\left(M;\mathbb{R}\right), X\in\mathfrak{X}\left(TM\right), Y\in\mathfrak{X}\left(E\right) : \nabla_X fY = df\left(X\right)Y + f\nabla_X Y$

17.2.2 Exterior covariant derivative

The **exterior covariant derivative** ∇_e of r-forms ϖ *on* M valued in the vector bundle $E(M,V,\pi)$, is a map : $\nabla_e : \Lambda_r(M;E) \to \Lambda_{r+1}(M;E)$ given in a holonomic basis by the formula : $\nabla_e \varpi = \sum_i \left(d\varpi^i + \left(\sum_j \left(\sum_\alpha \Gamma^i_{j\alpha} d\xi^\alpha \right) \wedge \varpi^j \right) \right) \otimes \mathbf{e}_i(m)$

If $\varpi \in \Lambda_0(M;E) : \nabla_e \varpi = \nabla \varpi$ (we have the usual covariant derivative of a section on E)

$\forall \mu_r \in \Lambda_r(M;\mathbb{R}), \forall \varpi_s \in \Lambda_s(M;E) : \nabla_e(\mu_r \wedge \varpi_s) = (d\mu_r) \wedge \varpi_s + (-1)^r \mu_r \wedge \nabla_e \varpi_s$

If $f \in C_\infty(N;M), \varpi \in \Lambda_r(N;f^*E) : \nabla_e(f^*\varpi) = f^*(\nabla_e \varpi)$

$\nabla_e(\nabla X) = \sum_{\{\alpha\beta\}} R^i_{j\alpha\beta} X^j d\xi^\alpha \wedge d\xi^\beta \otimes \mathbf{e}_i(x)$

$\nabla_e(\nabla_e \varpi) = R \wedge \varpi$

where R is the Riemann curvature tensor

$R = \sum_{\{\alpha\beta\}} \sum_{ij} R^i_{j\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes \mathbf{e}^j(m) \otimes \mathbf{e}_i(m)$

$R^i_{j\alpha\beta} = \partial_\alpha \Gamma^i_{j\beta} - \partial_\beta \Gamma^i_{j\alpha} + \sum_k \left(\Gamma^i_{k\alpha} \Gamma^k_{j\beta} - \Gamma^i_{k\beta} \Gamma^k_{j\alpha} \right)$

17.3 Connections on principal bundles

A connection on a principal bundle $P(M,G,\pi)$ is a 1-form Φ on TP valued in the vertical bundle VP, which is a projection :

$\Phi \in \Lambda_1(P;VP) : \Phi(p)v_p = \Phi(p)(y_m + \zeta(\kappa)) = \zeta(\kappa + \Gamma(p) y_m)(p)$

The connection form of the connection Φ is the form on TP, valued in the fixed vector space T_1G :

$\widehat{\Phi}(p) : TP \to T_1G :: \Phi(p)(Y_p) = \zeta\left(\widehat{\Phi}(p)(Y_p)\right)(p)$

A connection on a principal bundle $P(M,G,\pi)$ is said to be principal if it is equivariant under the right action of G :

$\forall g \in G, p \in P : \rho(p,g)^* \Phi(p) = \rho'_p(p,g)\Phi(p)$

$\Leftrightarrow \Phi(\rho(p,g))\rho'_p(p,g) Y_p = \rho'_p(p,g)\Phi(p) Y_p$

$\rho(p,g)_* \widehat{\Phi}(p) = Ad_{g^{-1}} \widehat{\Phi}(p)$

The potential of the connection is : $\grave{A}(m) = \widehat{\Phi}(\mathbf{p}(m)) = \mathbf{p}^* \widehat{\Phi}$. It is then uniquely defined by a family $\left(\grave{A}_a\right)_{a \in A}$ of maps : $\grave{A}_a \in \Lambda_1(O_a;T_1G)$:

$\Phi(\rho(\mathbf{p}(m),g)) \left(\sum_\beta y^\beta_m \partial m_\beta(p) + \zeta(\kappa)(p) \right) = \zeta\left(\kappa + Ad_{g^{-1}} \grave{A}(m) y_m\right)(p)$

In a change of gauge in P :

$\mathbf{p}(m) \to \widetilde{\mathbf{p}(m)} = \rho(\mathbf{p}(m), \varkappa(m)) \Rightarrow \grave{A}(m)$

$\to \widetilde{\grave{A}}(m) = Ad_\chi \left(\grave{A}(m) - L'_{\chi^{-1}}(\chi) \chi'(m) \right)$

The fundamental vectors are invariant by a principal connection: $\forall X \in T_1G : \Phi(p)(\zeta(X)(p)) = \zeta(X)(p)$

For any fixed $\kappa \in T_1G$ along the fundamental vector field : $\pounds_{\zeta(\kappa)} \widehat{\Phi} = -ad(\kappa)\left(\widehat{\Phi}\right) = \left[\widehat{\Phi}, \kappa\right]_{T_1G}$

The covariant derivative of a *section* $S = \rho(\mathbf{p}(m), g(m))$ is :

$\nabla_y S(m) = \zeta\left(L'_{g^{-1}} g\left(g' + R'_g 1\left(\grave{A}(m)\right)\right)\right)(S(m)) \in \mathfrak{X}(VP)$

The horizontal lift of a vector field on M is the map : $\chi_H : \mathfrak{X}(TM) \to \mathfrak{X}(HP) :: \chi_H(p)(y) = \varphi'_m(m,g) y(m) - \zeta\left(Ad_{g^{-1}}\left(\grave{A}_a(m) y(m)\right)\right)(p)$

The curvature form of the principal connection is the 2 form $\widehat{\Omega} \in \Lambda_2(P; T_1G)$ such that : $\Omega = \zeta\left(\widehat{\Omega}\right)$. It has the following expression in an holonomic basis of TP and basis (κ_i) of T_1G

$\widehat{\Omega}(\rho(\mathbf{p}(m), g))$

$= -Ad_{g^{-1}} \sum_i \sum_{\alpha\beta}\left(\partial_\alpha \grave{A}^i_\beta + \left[\grave{A}_a, \grave{A}_\beta\right]^i_{T_1G}\right) dm^\alpha \Lambda dm^\beta \otimes \kappa_i \in \Lambda_2(P; T_1G)$

The **strength of the principal connection** is the 2 form $\mathcal{F} \in \Lambda_2(M; T_1G)$ such that : $\mathcal{F} = -\mathbf{p}(m)^* \widehat{\Omega}$. It has the following expression in an holonomic basis of TM and basis (κ_i) of T_1G :

$\mathcal{F}(m) = \sum_i \left(d\grave{A}^i + \sum_{\alpha\beta}\left[\grave{A}_a, \grave{A}_\beta\right]^i_{T_1G} d\xi^\alpha \Lambda d\xi^\beta\right) \otimes \kappa_i$

In a change of gauge in P :

$\mathbf{p}(m) \to \widetilde{\mathbf{p}(m)} = \rho(\mathbf{p}(m), \varkappa(m))$

$\mathcal{F}(m) \to \widetilde{\mathcal{F}}(m) = Ad_{\chi(m)} \mathcal{F}(m)$

The covariant derivative of a section $S \in \mathfrak{X}(P)$ along a vector field $y \in \mathfrak{X}(TM)$ is a fundamental vector field on TP : $S(m) = \rho_a(\mathbf{p}_a(m), \sigma(m))$: $\nabla_y S = \zeta\left(L'_{\sigma^{-1}}(\sigma)\left(\sigma'(y) + R'_\sigma(1)\grave{A}(m)(y)\right)\right)(S(m))$

It is *invariant* in a change of gauge but its expression with a map σ varies the usual way for a section in a change of gauge.

17.4 Connection on associated bundles

A principal connection Φ on a principal bundle $P(M, G, \pi)$ induces on any associated bundle $E = P[V, \lambda]$ a connection :

$Y_p = \sum_\beta y^\beta_m \partial m_\beta + \sum_i y^i_v \partial v_i \in TE$

$\Phi(\mathbf{p})(Y_p) = \left(\mathbf{p}, \sum_i\left(y^i_v + \lambda'_g(1,p)\left(\sum_\beta \grave{A}^i_\beta(m) y^\beta_m\right)\right)\partial v_i\right) \in VE$

defined by the set of Christoffel forms : $\Gamma_\beta(p) = \lambda'_g(1,p)\left(\grave{A}_\beta(m)\right)$

In a change of gauge in P :

$\mathbf{p}(m) \to \widetilde{\mathbf{p}(m)} = \rho(\mathbf{p}(m), \varkappa(m)) \Rightarrow \Gamma(p)$

$\to \widetilde{\Gamma}(p) = -\lambda'_g(\chi(m), g) + \lambda'_v(\chi(m), v)(\Gamma(p))$

For a section $S(m) = (\mathbf{p}(m), S(m)) \sim \left(\rho(\mathbf{p}(m), g), \lambda\left(g^{-1}, S(m)\right)\right)$

$\nabla_y S = \left(\mathbf{p}(m), \sum_\beta\left(\partial_\beta S + \lambda'_g(1,S)\left(\grave{A}_\beta(m)\right) y^\beta\right)\right)$

18 Categories

18.0.1 The category of Clifford bundles

Vector bundles $E(M,V,\pi)$ with vector spaces V on the same field K and their morphisms constitute a category.

Vector bundles such that the vector space is endowed with a symmetric 2 form r, and morphisms which preserves the scalar product constitute a subcategory.

Similarly Clifford bundles $E(M,Cl(V,r),\pi)$ with Clifford algebras on the same field K and their morphisms constitute a category.

Real Clifford bundles $Cl(\mathbb{R},p,q)$ on vector spaces of same dimension endowed with a symmetric form of signature (p,q) are Clifford isomorphic : there is a linear isomorphism, preserving both the scalar product and the algebraic operations. Similarly complex Clifford algebras $Cl(\mathbb{C},n)$ on complex vector spaces of same dimension endowed with a symmetric form are Clifford isomorphic. So that we have the category $\mathcal{C}Cl(\mathbb{R},p,q)$ of real Clifford bundles $E(M,Cl(\mathbb{R},p,q),\pi)$ and the category $\mathcal{C}Cl(\mathbb{C},n)$ of complex Clifford bundles $E(M,Cl(\mathbb{C},n),\pi)$.

Any linear map $F\in\mathcal{L}(V_1;V_2)$ between the vector spaces (V_j,r_j) on the same field, which preserves the scalar product r_j, can be extended to a Clifford morphism $\widehat{F}\in\mathcal{L}(Cl(V_1,r_1);Cl(V_2,r_2))$ between the Clifford algebras. As a consequence there are functors :

between the category $\mathcal{CV}(\mathbb{R},p,q)$ of real vector bundles $E(M,V,\pi)$ endowed with a scalar product of signature (p,q) and the linear maps which preserve the scalar product on one hand, and the category $\mathcal{C}Cl(\mathbb{R},p,q)$ of real Clifford bundles $E(M,Cl(\mathbb{R},p,q),\pi)$ with their Clifford morphisms on the other hand

between the category $\mathcal{CV}(\mathbb{C},n)$ of complex vector bundles $E(M,V,\pi)$ endowed with a scalar product and the linear maps which preserve the scalar product on one hand, and the category $\mathcal{C}Cl(\mathbb{C},n)$ of complex Clifford bundles $E(M,Cl(\mathbb{C},n),\pi)$ with their Clifford morphisms on the other hand.

There is a Clifford isomorphism $C:Cl(\mathbb{R},p,q)\to Cl(\mathbb{C},p+q)$ such that $C(Cl(\mathbb{R},p,q))$ is a real form of $Cl(\mathbb{C},p+q)$:

$Cl(\mathbb{C},p+q)=C(Cl(\mathbb{R},p,q))+iC(Cl(\mathbb{R},p,q))$

$\forall Z\in Cl(\mathbb{C},p+q):Z=\operatorname{Re}Z+i\operatorname{Im}Z=C(Z_1)+iC(Z_2)$

A Clifford morphism F on $Cl(\mathbb{R},p,q)$ can be extended to a Clifford morphism $\widehat{F}$ on $Cl(\mathbb{C},p+q)$:

$F:Cl(\mathbb{R},p,q)\to Cl(\mathbb{R},p,q)$ is such that :

$F(aZ+bZ')=aF(Z)+bF(Z');F(Z\cdot Z')=F(Z)\cdot F(Z'),\langle F(Z),F(Z')\rangle=\langle Z,Z'\rangle$

Define $\widehat{F}(\operatorname{Re}Z+i\operatorname{Im}Z)=\widehat{F}(C(Z_1)+iC(Z_2))=C(F(Z_1))+iC(F(Z_2))=C\left(F\left(C^{-1}(\operatorname{Re}Z)\right)\right)+iC\left(F\left(C^{-1}(\operatorname{Im}Z)\right)\right)$

The map $\widehat{F}$ is complex linear :

$\widehat{F}((a+ib)\operatorname{Re}Z+i(a+ib)\operatorname{Im}Z)=\widehat{F}(a\operatorname{Re}Z-b\operatorname{Im}Z+i(a\operatorname{Im}Z+b\operatorname{Re}Z))=$
$C(F(a\operatorname{Re}Z-b\operatorname{Im}Z))+iC(F(a\operatorname{Im}Z+b\operatorname{Re}Z))$
$=aC(F(\operatorname{Re}Z))-bC(F(\operatorname{Im}Z))+aiC(F(\operatorname{Im}Z))+biC(F(\operatorname{Re}Z))$
$=(a+bi)C(F(\operatorname{Re}Z))+i(a+ib)C(F(\operatorname{Im}Z))$
$=(a+bi)(C(F(\operatorname{Re}Z))+iC(F(\operatorname{Im}Z)))=(a+bi)\widehat{F}(\operatorname{Re}Z+i\operatorname{Im}Z)$

It is an algebra morphism :

$\widehat{F}(Z\cdot Z')=\widehat{F}((\operatorname{Re}Z+i\operatorname{Im}Z)\cdot(\operatorname{Re}Z'+i\operatorname{Im}Z'))$
$=\widehat{F}(\operatorname{Re}Z\cdot\operatorname{Re}Z')+\widehat{F}(\operatorname{Re}Z\cdot i\operatorname{Im}Z')+\widehat{F}(i\operatorname{Im}Z\cdot\operatorname{Re}Z')+\widehat{F}(i\operatorname{Im}Z\cdot i\operatorname{Im}Z')$
$=\widehat{F}(\operatorname{Re}Z\cdot\operatorname{Re}Z')+i\widehat{F}(\operatorname{Re}Z\cdot\operatorname{Im}Z')+i\widehat{F}(\operatorname{Im}Z\cdot\operatorname{Re}Z')-\widehat{F}(\operatorname{Im}Z\cdot\operatorname{Im}Z')$
$=C(F(\operatorname{Re}Z\cdot\operatorname{Re}Z'))+iC(F(\operatorname{Re}Z\cdot\operatorname{Im}Z'))+iC(F(\operatorname{Im}Z\cdot\operatorname{Re}Z'))-C(F(\operatorname{Im}Z\cdot\operatorname{Im}Z'))$
$=C(F(\operatorname{Re}Z)\cdot F(\operatorname{Re}Z'))+iC(F(\operatorname{Re}Z)\cdot F(\operatorname{Im}Z'))+iC(F(\operatorname{Im}Z)\cdot F(\operatorname{Re}Z'))-$
$C(F(\operatorname{Im}Z)\cdot F(\operatorname{Im}Z'))$
$=CF(\operatorname{Re}Z)\cdot CF(\operatorname{Re}Z')+iCF(\operatorname{Re}Z)\cdot CF(\operatorname{Im}Z')+iCF(\operatorname{Im}Z)\cdot CF(\operatorname{Re}Z')-$
$CF(\operatorname{Im}Z)\cdot CF(\operatorname{Im}Z')$
$=(CF(\operatorname{Re}Z)+iCF(\operatorname{Im}Z))\cdot CF(\operatorname{Re}Z')+i(CF(\operatorname{Re}Z)+iCF(\operatorname{Im}Z))\cdot$
$CF(\operatorname{Im}Z')$
$=\widehat{F}(Z)\cdot CF(\operatorname{Re}Z')+\widehat{F}(Z)\cdot iCF(\operatorname{Im}Z')=\widehat{F}(Z)\cdot\widehat{F}(Z')$

It preserves the scalar product :

$\left\langle\widehat{F}(Z),\widehat{F}(Z')\right\rangle_{\mathbb{C}}=\left\langle\widehat{F}(\operatorname{Re}Z+i\operatorname{Im}Z),\widehat{F}(\operatorname{Re}Z'+i\operatorname{Im}Z')\right\rangle_{\mathbb{C}}$
$=\left\langle\widehat{F}(\operatorname{Re}Z),\widehat{F}(\operatorname{Re}Z')\right\rangle_{\mathbb{C}}+i\left\langle\widehat{F}(\operatorname{Re}Z),\widehat{F}(\operatorname{Im}Z')\right\rangle_{\mathbb{C}}+i\left\langle\widehat{F}(\operatorname{Im}Z),\widehat{F}(\operatorname{Re}Z')\right\rangle_{\mathbb{C}}-$
$\left\langle\widehat{F}(\operatorname{Im}Z),\widehat{F}(\operatorname{Im}Z')\right\rangle_{\mathbb{C}}$
$=\langle C(F(\operatorname{Re}Z)),C(F(\operatorname{Re}Z'))\rangle_{\mathbb{C}}+i\langle C(F(\operatorname{Re}Z)),C(F(\operatorname{Im}Z'))\rangle_{\mathbb{C}}$
$+i\langle C(F(\operatorname{Im}Z)),C(F(\operatorname{Re}Z'))\rangle_{\mathbb{C}}-\langle C(F(\operatorname{Im}Z)),C(F(\operatorname{Im}Z'))\rangle_{\mathbb{C}}$
$=\langle F(\operatorname{Re}Z),F(\operatorname{Re}Z')\rangle_R+i\langle F(\operatorname{Re}Z),F(\operatorname{Im}Z')\rangle_R+i\langle F(\operatorname{Im}Z),F(\operatorname{Re}Z')\rangle_R-$
$\langle F(\operatorname{Im}Z),F(\operatorname{Im}Z')\rangle_R$
$=\langle\operatorname{Re}Z,\operatorname{Re}Z'\rangle_R+i\langle\operatorname{Re}Z,\operatorname{Im}Z'\rangle_R+i\langle\operatorname{Im}Z,\operatorname{Re}Z'\rangle_R-\langle\operatorname{Im}Z,\operatorname{Im}Z'\rangle_R$
$=\langle C(\operatorname{Re}Z),C(\operatorname{Re}Z')\rangle_{\mathbb{C}}+i\langle C(\operatorname{Re}Z),C(\operatorname{Im}Z')\rangle_{\mathbb{C}}+i\langle C(\operatorname{Im}Z),C(\operatorname{Re}Z')\rangle_{\mathbb{C}}-$
$\langle C(\operatorname{Im}Z),C(\operatorname{Im}Z')\rangle_{\mathbb{C}}$
$=\langle Z,Z'\rangle_{\mathbb{C}}$

So :

to any real Clifford bundle $E(M,Cl(\mathbb{R},p,q),\pi)\in\mathcal{C}Cl(\mathbb{R},p,q)$ one can associate a complex Clifford bundle $E(M,Cl(\mathbb{C},p+q),\pi)\in\mathcal{C}Cl(\mathbb{C},p+q)$ such that $C(E(M,Cl(\mathbb{R},p,q),\pi))$ is a real form of $E(M,Cl(\mathbb{C},p+q),\pi)$

to any real morphism in $\mathcal{C}Cl(\mathbb{R},p,q)$ a complex morphism in $\mathcal{C}Cl(\mathbb{C},p+q)$

and there is a functor $C:\mathcal{C}Cl(\mathbb{R},p,q)\mapsto\mathcal{C}Cl(\mathbb{C},p+q)$.

By product there is a functor between the category of real vector bundles $\mathcal{CV}(\mathbb{R},p,q)$ and the category of complex Clifford bundles $\mathcal{C}Cl(\mathbb{C},p+q)$.

19 Symmetries on a manifold

19.1 Pull back and push forward of tensors fields

Let M,N be smooth manifolds, and f a differentiable map : $f:M\to N$.

The push forward of a vector field $y \in \mathfrak{X}(TM)$ by f is the linear map :

$f_* : \mathfrak{X}(TM) \to \mathfrak{X}(TN) :: f_*(y(m))(f(m)) = Y(f(m)) = f'(m)(y(m))$

The pull back of a of a form $\Lambda \in \mathfrak{X}(TN^*)$ by f is the linear map :

$f^* : \mathfrak{X}(TN^*) \to \mathfrak{X}(TM^*) :: f^*(\Lambda(m))(f(m)) = \Lambda(f(m)) \circ f'(m)$

If f is invertible we have similarly :

The pull back of a vector field $Y \in \mathfrak{X}(TN)$ by f is the linear map :

$f^* : \mathfrak{X}(TN) \to \mathfrak{X}(TM) ::$

$f^*(Y(n))\left(f^{-1}(n)\right) = y\left(f^{-1}(n)\right) = \left(f^{-1}(n)\right)'(Y(n))$

The push forward of a form $\lambda \in \mathfrak{X}(TM^*)$ by f is the linear map :

$f_* : \mathfrak{X}(TM^*) \to \mathfrak{X}(TN^*) ::$

$f_*(\lambda(m))(f(m)) = \Lambda(n) = \lambda\left(f^{-1}(n)\right) \circ \left(f^{-1}(n)\right)'$

f^*, f_* are linear maps, which are inverse of each other if f is invertible : $f_* = (f^*)^{-1}$

By construct, if $\lambda \in \mathfrak{X}(TM^*), Y \in \mathfrak{X}(TN)$:

$f_*(\lambda)(f_*(y))(f(m)) = \Lambda(f(m))(Y(f(m))) = \lambda(m) \circ \left(f^{-1}(n)\right)' f'(m)(y(m)) = \lambda(m)(y(m))$

The operations hold for functions, considered as 0 forms.

Vector fields on a manifold have a structure of Lie algebra with the commutator as bracket, and f^* is a morphism of Lie algebra (it preserves the commutator).

Smooth manifolds and differentiable maps constitute a category. With the functor which associates to each manifold its tangent bundle, these maps f^*, f_*, defined with a unique $f \in C_1(M;N)$, can be extended to linear maps F^*, F_* between the tensor bundles $\mathfrak{X}(\otimes TM), \mathfrak{X}(\otimes TN)$.

$$f \quad \to \quad f_* \in \mathcal{L}(\mathfrak{X}(TM);\mathfrak{X}(TM)) \quad \rightsquigarrow \quad F_* \in \mathcal{L}(\mathfrak{X}(\otimes TM);\mathfrak{X}(\otimes TM))$$

They are the inverse of each other, preserve the type of the tensors, the product of tensors and the exterior product of forms, commute with the exterior differential, and can be composed : $(F \circ G)^* = G^* \circ F^*; (F \circ G)_* = F^* \circ G_*$.

The change of charts $\varphi \to \psi$ in a manifold can be expressed as the push forward $\left(\psi \circ \varphi^{-1}\right)_*$.

The components expression of F^*, F_* is given by the product of the jacobians and its inverse (Maths.1440).

19.2 Diffeomorphisms on a manifold

A diffeomorphism on a manifold M is a bijective map $f : M \to M$ such that its derivative f' is itself invertible. On a relatively compact $\Omega \subset M$ there is an homeomorphism between diffeomorphisms and vector fields : diffeomorphisms constitute a Lie group $\mathcal{D}(M)$, with Lie algebra the vector space of vector fields endowed with the commutator as bracket.

A one parameter group on $\mathcal{D}(M)$ is a map

$F : \mathbb{R} \to \mathcal{D}(M) :: F(\tau + \tau') = F(\tau) \circ F(\tau'), F(0) = Id$

Its infinitesimal generator is a vector field $X \in \mathfrak{X}(TM)$:

$X(F(\theta, m)) = F'_m(\theta, m) X(m) \Leftrightarrow X \circ F = F_* X$

The one parameter group generated by any vector field is defined by the differential equation $\frac{d}{d\tau}\Phi_V(\tau,m)|_{\tau=\theta}=V(\Phi_V(\theta,m))$. Then for τ fixed $\Phi_V(\tau,m)$ defines a diffeomorphism :

$f_\tau: M\to M :: f_\tau(m)=\Phi_V(\tau,m)$ by taking the integral curve going through m. The derivative $f'_\tau(m)=\Phi'_{Vm}(\tau,m)$.

So there is a close association between the Lie group $\mathcal{D}(M)$ of diffeomorphisms and its Lie algebra $T_1\mathcal{D}(M)\simeq\mathfrak{X}(TM)$ of vector fields, a one parameter group of diffeomorphisms can be characterized by a vector field, which justifies the common notation : $\Phi_V(\tau,m)=\exp\tau V(m)$.

The map : $f_*:\mathfrak{X}(TM)\to\mathfrak{X}(TM)$ preserves the commutator $[f_*X,f_*Y]=f_*[X,Y]$. For any diffeomorphism f_* it is an automorphism on the Lie algebra of vector fields.

The vector field V is transported by $\Phi_V(\tau,m)$:

$f_{\tau*}:\mathfrak{X}(TM)\to\mathfrak{X}(TM)::$

$f_{\tau*}(V)=\widetilde{V}:\widetilde{V}(f_\tau(m))=f'_\tau(m)(V(m))=V(f_\tau(m))$

The vector field V is eigen vector with eigen value 1 of the map $f_{\tau*}:\mathfrak{X}(TM)\to\mathfrak{X}(TM)$. But it is not the unique one with this property :

$\forall W\in\mathfrak{X}(TM):[V,W]=0\Leftrightarrow\Phi_V^*(W)=W$

We have the identities :

$\Phi_V(\tau,\Phi_V(\tau',m))=\Phi_V(\tau+\tau',m)$

$\Phi'_{Vm}(\tau+\tau',m)=\Phi'_{Vm}(\tau,\Phi_V(\tau',m))\circ\Phi'_{Vm}(\tau',m)$

$=\Phi'_{Vm}(\tau',\Phi_V(\tau,m))\circ\Phi'_{Vm}(\tau,m)$

$\Rightarrow\Phi'_{Vm}(0,m)=Id=\Phi'_{Vm}(\tau,\Phi_V(-\tau,m))\circ\Phi'_{Vm}(-\tau,m)$

$=\Phi'_{Vm}(-\tau,\Phi_V(\tau,m))\circ\Phi'_{Vm}(\tau,m)$

$\Phi'_{Vm}(-\tau,\Phi_V(\tau,m))=(\Phi'_{Vm}(\tau,m))^{-1}$

$\Phi'_{Vm}(\tau,\Phi_V(-\tau,m))=\Phi'_{Vm}(-\tau,m)^{-1}$

The flow of a vector field on a relatively compact domain is complete, and we can define the operations :

The push forward of a vector field $y\in\mathfrak{X}(TM)$ by f is the linear map :

$f_*:\mathfrak{X}(TM)\to\mathfrak{X}(TM)::$

$f_*(y(m))(\Phi_V(\tau,m))=Y(\Phi_V(\tau,m))=\Phi'_{Vm}(\tau,m)(y(m))$

The pull back of a form $\lambda\in\mathfrak{X}(TM^*)$ by f is the linear map :

$f^*:\mathfrak{X}(TM^*)\to\mathfrak{X}(TM^*)::$

$f^*(\lambda(\Phi_V(\tau,m)))(m)=\Lambda(m)=\lambda(\Phi_V(\tau,m))\circ\Phi'_{Vm}(\tau,m)$

The pull back of a vector field $y\in\mathfrak{X}(TM)$ by f is the linear map :

$f^*:\mathfrak{X}(TM)\to\mathfrak{X}(TM)::$

$f^*(y(m))(\Phi_V(-\tau,m))=Y(\Phi_V(-\tau,m))=\Phi'_{Vm}(-\tau,m)(y(m))$

The push forward of a form $\lambda\in\mathfrak{X}(TM^*)$ by f is the linear map :

$f_*:\mathfrak{X}(TM^*)\to\mathfrak{X}(TM^*)::$

$f_*(\lambda(m))(\Phi_V(\tau,m))=\Lambda(\Phi_V(\tau,m))=\lambda(m)\circ\Phi'_{Vm}(-\tau,\Phi_V(\tau,m))$

These operations can be extended to the tensor bundle with the same properties as above.

If f is a diffeomorphism and V a vector field : $f\circ\Phi_{f^*V}=\Phi_V\circ f$ and for two vector fields : $\frac{\partial}{\partial\tau}(\Phi_V(\tau,.)_*W)|_{\tau=0}=\Phi_V(\tau,.)_*[V,W]$

The following are equivalent :

$[V,W]=0$
$\Phi_{V*}W=W$
$\Phi_V(\tau,\Phi_W(s,m))=\Phi_W(s,\Phi_V(\tau,m))$

19.3 Lie derivative of a tensor field

The Lie derivative of a tensor field T along a vector V is then the tensor field :
$\pounds_V T(m)=\frac{\partial}{\partial\tau}\left(\Phi_V(\tau,.)_* T\right)(m)|_{\tau=0}$
$=\lim_{\tau\to 0}\frac{1}{\tau}\left(\left(\Phi_V(\tau,.)_* T(\Phi_V(\tau,.))\right)(m)-T(m)\right)$
$=\lim_{\tau\to 0}\frac{1}{\tau}\left(T(m)-\Phi_V(-\tau,.)_* T(\Phi_V(-\tau,.))(m)\right)$

The Lie derivative is a derivation on the Lie algebra of tensors fields, which preserves the type (order, symmetric or antisymmetric) of the tensor, is linear with respect to the vector field V, commutes with the contraction and external differentiation of tensors, follows the Leibnitz rule with the tensorial or external product of tensors. Its computation is done using the previous rules (Maths.4.3.3).

For vector fields :
$\pounds_{[V,W]}=\pounds_V\circ\pounds_W-\pounds_W\circ\pounds_V=[\pounds_V,\pounds_W]$

19.4 Symmetries of tensors and diffeomorphisms

Let f be a diffeomorphism on M, then we can say that the tensor T is symmetric with respect to f if :
$F_*(T(m))(f(m))=T(f(m))\Leftrightarrow F^*\left(T\left(\left(f^{-1}(m)\right)\right)\right)(m)=T\left(f^{-1}(m)\right)$

Then : $T(f(m))=(F(m))_{\otimes}T(m)$ the tensor at $f(m)$ is the image of the tensor at m, by the linear map F deduced from the derivative $f'(m)$.

Because of the relation between diffeomorphisms and vector fields, the same relation happens with the one parameter groups along an integral curve. So we have a symmetry for a vector field V along an integral curve of y if :
$\forall\tau:(\Phi_y(\tau,\cdot))_*(V(m))=\Phi'_{ym}(\tau,m)V(m)=V(\Phi_y(\tau,m))$

We have : $\Phi_V^*(-\tau,.)\pounds_V T=-\frac{\partial}{\partial\tau}\left(\Phi_V^*(-\tau,.)T\right)|_{\tau=0}$ so the simplest test of the existence of a symmetry is through the Lie derivative. A tensor field T is symmetric [3] by the flow of a vector field y iff $\pounds_y T=0$. Then $T(\Phi_y(\tau,m))=\left[\Phi'_{ym}(\tau,m)\right]_{\otimes TM}T(m)$. The value of the tensor at $\Phi_y(\tau,m)$ on the integral curve is the image of its value at m by the linear map $\Phi'_{ym}(\tau,m)$ extended to tensors.

$$\pounds_y T=0\Leftrightarrow(\Phi_y(\tau,\cdot))_*(T)=T\circ\Phi_y(\tau,\cdot)\Leftrightarrow T(\Phi_y(\tau,m))=\left[\Phi'_{ym}(\tau,m)\right]_{\otimes TM}T(m)$$

Conversely the transport of a tensor T along an integral curve is the tensor $\widetilde{T}(\tau)$ defined by the conditions $\pounds_y\widetilde{T}=0,\widetilde{T}(0)=T$. This is similar to the transport of tensors by covariant derivative.

[3] it is said "invariant" in Mathematics, but as this may cause confusion we will say symmetric

Because there is always an integral curve going through a point, if $\pounds_y T = 0$ the value of the map $T : M \to \otimes TM$ is closely linked to the vector field y.

Time translation : this is the transport by the vector field $c\varepsilon_0$.

$m = \varphi_o(ct, x) \to \widetilde{m}(\tau) = \varphi_o(ct + c\tau, x) = \Phi_{c\varepsilon_0}(\tau, \varphi_o(ct, x))$

$\Phi'_{c\varepsilon_{0m}}(\tau, \varphi_o(ct, x)) = I$

so

$V(\varphi_o(ct, x)) \to \widetilde{V}(\varphi_o(c(t+\tau), x)) = \Phi_{c\varepsilon_{0*}}(\tau, \varphi_o(ct, x)) V = V(\varphi_o(c(t+\tau), x))$

19.5 Lie derivative of a section of a fiber bundle[4]

i) The **Lie derivative** of a section $X \in \mathfrak{X}(E)$ along a vector field $W \in \mathfrak{X}(TE)$ is defined as for any other manifold by the section

$\pounds_W X(p) = \frac{d}{d\tau}(\Phi_W(\tau, \cdot)_* X(p))|_{\tau=0}$

$= \lim_{\tau \to 0} \frac{1}{\tau}(X(\Phi_W(\tau, p)) - X((\Phi_W(-\tau, p)))) \in \mathfrak{X}(E)$

The flow of a projectable vector field $W \in \mathfrak{X}(TE)$ on any fiber bundle $E(M, V, \pi)$ is a fibered manifold morphism :

$\pi'(p) W(p) = Y(\pi(p)) \in T_{\pi(p)}M$

$\frac{\partial}{\partial \tau}\Phi_W(\tau, p)|_{\tau=\theta} = W(\Phi_W(\theta, p))$

$\pi(\Phi_W(\tau, p)) = \Phi_Y(\tau, \pi(p))$

ii) If E is a fiber bundle, S a section, Y a vector field on TE projectable on y on TM, the map :

$F_Y : \mathfrak{X}(E) \to \mathfrak{X}(E) :: F_Y(m, \tau) = \Phi_Y(\tau, S(\Phi_y(-\tau, m)))$ is fiber preserving : $\pi(F_Y(m, \tau)) = m$

Proof. $\frac{\partial}{\partial \tau}\pi(F_Y(m, \tau))|_{\tau=\theta} = \pi'(F_Y(m, \theta)) \frac{\partial}{\partial \tau}(\Phi_Y(\tau, S(\Phi_y(-\tau, m))))|_{\tau=\theta} =$

$= \pi'(F_Y(m, \theta))$

$\times \Big(Y(\Phi_Y(\theta, S(\Phi_y(-\theta, m)))) - \frac{\partial}{\partial p}(\Phi_Y(\theta, S(\Phi_y(-\theta, m))) S'(\Phi_y(-\theta, m)) y((-\theta, m)))\Big)$

but :

$\pi'(F_Y(m, \theta)) Y(\Phi_Y(\theta, S(\Phi_y(-\theta, m)))) = \pi'(F_Y(m, \theta)) Y(F_Y(m, \theta)) = y(\pi(F_Y(m, \theta)))$

$\frac{\partial}{\partial p}(\Phi_Y(\theta, S(\Phi_y(-\theta, m))) S'(\Phi_y(-\theta, m)) y((-\theta, m)))$

$= \frac{\partial}{\partial \tau}\Phi_Y(\tau, S(\Phi_y(-\tau, m)))|_{\tau=\theta} = Y(\Phi_Y(\theta, S(\Phi_y(-\theta, m))))$

thus : $\forall \theta : \frac{\partial}{\partial \tau}\pi(F_Y(m, \tau))|_{\tau=\theta} = 0 \Rightarrow \pi(F_Y(m, \tau)) = \pi(F_Y(m, 0)) = \pi(S(m)) = m$ ∎

The image by F_Y of a section is another section on E, which is "deformed" according to the parameter τ. When τ varies $F_Y(m, \tau)$ moves in the fiber $\pi^{-1}(m)$ starting at $S(m)$ for $\tau = 0$.

The Lie derivative of the section S along Y (Maths.1930 and Kolar p.377) is then defined as follows :

$\pounds_Y S(F_Y(m, \tau)) = \frac{\partial}{\partial s}F_Y(m, s)|_{s=\tau} = \frac{\partial}{\partial \tau}\Phi_Y(s, S(\Phi_y(-s, m)))|_{s=\tau}$

This is a vertical vector (because one stays in the same fiber), it belongs to the vertical bundle VE at the point $F_Y(m, \tau)$.

If E is a vector bundle $\pounds_Y S$ is a section of the vector bundle.

If Y is vertical $\pounds_Y S = Y(S)$.

$\pounds_Y S = 0 \Leftrightarrow \forall \theta : S(\Phi_y(\theta, m)) = \Phi_Y(\theta, S(m))$

[4] These results, inspired by Kolar, are new.

Proof. i) if $\pounds_Y S = 0$: then $\forall \tau : F_Y(m,\tau) = S(m)$.

$S(\Phi_y(\theta,m)) = F_Y(\Phi_y(\theta,m),\tau) = \Phi_Y(\tau, S(\Phi_y(-\tau,\Phi_y(\theta,m))))$

$= \Phi_Y(\tau, S(\Phi_y(\theta-\tau,m)))$

$\tau = \theta : S(\Phi_y(\theta,m)) = \Phi_Y(\theta, S(\Phi_y(0,m))) = \Phi_Y(\theta, S(m))$

ii) if $\forall \theta : S(\Phi_y(\theta,m)) = \Phi_Y(\theta, S(m))$

$F_Y(m,\tau) = \Phi_Y(\tau, S(\Phi_y(-\tau,m))) = \Phi_Y(\tau, \Phi_Y(-\tau, S(m))) = S(m) \Rightarrow \frac{\partial}{\partial s} F_Y(m,s)|_{s=\tau} = 0$ ■

Then the section S changes along Y as the morphism $\Phi_Y : S(\Phi_y(\theta,m)) = \Phi_Y(\theta, S(m))$. The value of the section is closely linked to the projectable vector field Y and we say that the section S is symmetric with respect to Y.

If there is a connection Φ on E with horizontal lift χ_H :

$S'(x)y = \nabla_y S(x) + \chi_H(S(x))(y(x))$

Take $Y = S'(x)y$, it is projectable on y.

$\pounds_Y S = \pounds_{\nabla_y S} S + \pounds_{\chi_H(y)} = \nabla_y S + \pounds_{\chi_H(y)} S$

$\pounds_Y S = 0 \Leftrightarrow \forall \theta : S(\Phi_y(\theta,m)) = \Phi_Y(\theta, S(m)) \Leftrightarrow \nabla_y S + \pounds_{\chi_H(y)} S = 0$

iii) The Lie derivative of a section of a principal bundle along a fundamental vector field is $\pounds_{\zeta(X)} S = \zeta(X)(S)$

20 Distributions on a manifold

A r dimensional distribution (no rapport with linear functionals) on a manifold is a map : $D : M \to (TM)^r$ such that $D(m)$ is a r dimensional *vector subspace* of $T_m M$. A connected submanifold L of M is an integral of D if $\forall m \in M$: $D(m) = T_m L$. A distribution is integrable if $\exists (L_i)_{i\in I}$ submanifolds of M such that $\forall m \in M : \exists i \in I : D(m) = T_m L_i$. A map $f \in C_1(M;M)$ is a morphism for D if $f'(m) D(m) \subset D(f(m))$. A vector field is the generator of a morphism for D if its flow is a morphism for D.

D is integrable if (alternate propositions) :

for each vector field V such that $\forall m \in M : V(m) \subset D(m)$ the flow of V is a morphism for D

there is a family $V = (V_i)_{i\in I} \in \mathfrak{X}(TM)$ which generates D and such that $\forall i,j : [V_i, V_j] \in V$

If $\lambda \in \Lambda_1(TM;V)$ such that $\ker\lambda$ has a constant finite dimension, then it is a distribution, which is integrable if $\forall u,v \in \ker\lambda : d\lambda(u,v) = 0$

So a function on M such that $\dim\ker f'(m) = Ct$ has the integrable distribution $f(m) = Ct$.

The fundamental vector fields over a principal bundle span an integrable distribution whose leaves are the connected components of the orbits : $\forall X, Y \in T_1 G : [\zeta(X), \zeta(Y)] = \zeta([X,Y])$

21 Isometries

21.1 Isometries

The metric is the physical part of the geometry of the universe. It is defined everywhere. It is a tensor, whose expression depends on the chart and varies from one point to another. A symmetry is then a diffeomorphism : $f : M \to M$ called isometry which preserves the metric : $\forall u, v \in T_m M$: $g(f(m))(f'(m)u, f'(m)v) = g(m)(u,v) \Leftrightarrow g \circ f = f_* g$

With the matrix $[J]$ of $[f'(m)]$ in a chart : $\left([J(m)]^{-1}\right)^t [g(m)][J(m)]^{-1} = [g(f(m))] \Leftrightarrow [g(f(m))] = [K(m)]^t [g(m)][K(m)]$ with $[J(m)]^{-1} = [K(m)]$.

The image of an orthonormal basis is an orthonormal basis. As any diffeomorphism they transport tensors and in this operation the coordinates of tensors in an orthonormal basis are conserved. As a consequence an isometry preserves the scalar product of forms : $G_r(\lambda, \mu)$ does not depend on the basis, expressed in an orthonormal basis it is a simple expression which is preserved by $f'(m)$. Thus the Hodge dual of $F^*\lambda$ is the image of the Hodge dual : $*(F^*\lambda) = F^*(*\lambda)$. An isometry preserves the scalar product $\langle \mathcal{F}, K \rangle$ of scalar forms as well as the volume form ϖ_4.

The isometries are a subgroup of diffeomorphisms and its Lie algebra are the Killing vector fields, vector fields V on TM such that their flow $f(m) = \Phi_V(\tau, m)$ is an isometry. There is a bijective correspondance between one parameter groups of isometries and Killing vector fields. The transport along a Killing vector field preserves the scalar product.

The condition is ;

$\pounds_V g = 0 \Leftrightarrow \alpha, \beta = 0...3 : \sum_{\gamma=0}^{3} V^\gamma [\partial_\gamma g]_\beta^\alpha + [g]_\gamma^\beta [\partial_\alpha V]^\gamma + [g]_\gamma^\alpha [\partial_\beta V]^\gamma = 0$

Vectors of a Killing field have a constant length :

$\pounds_V (g(V,V)) = \pounds_V g(V,V) + 2g(\pounds_V V, V) = 0$

If $V_i, i = 1...p$ are Killing vector fields,$V = \sum_{i=1}^{n} a_i V_i$ with fixed scalars is a Killing vector field. $V = 0$ corresponds to the isometry $f(m) = m$. As a consequence if a Killing vector field is null at a point, it is null everywhere, and if the vector fields $(V_i)_{i=1...N}$ are linearly independent at a point, they are linearly independent everywhere, and conversely, if they are linearly dependent at some point, they are linearly dependent everywhere. Using the condition above it is easy to see that the space of Killing vector fields on a manifold of dimension n is at most $\frac{n(n+1)}{2}$ dimensional.

With the identity : $\pounds_{[V,W]} = \pounds_V \circ \pounds_W - \pounds_W \circ \pounds_V = [\pounds_V, \pounds_W]$ if V, W are Killing vectors, then $[V, W]$ is a Killing vector. The vector space K of Killing vectors defines an integrable distribution : $\exists (K_i)_{i \in I}$ submanifolds of M such that $\forall m \in M, \exists i \in I : \forall V \in K : V(m) \in T_m K_i$. For 2 points $m, m' \in M$ the relation of equivalence $m \sim m' \Leftrightarrow (m \in K_i) \& (m' \in K_i)$ defines a partition of M, called a foliation, whose connected components are the leaves. If 2 points m, m' belong to the same leave i then all Killing vector fields are tangent to K_i at m and $m\prime$. The manifold M and the domain Ω are path connected, then the leaves are path connected (Maths.570). The integral curves of Killing vector fields are Killing curves, there are infinitely many Killing curves going through a given point. For any 2 points $m, m' \in K_i$ there is a Killing curve going from m to $m\prime$.

If there is a Killing curve joining m, m', that is $\exists V \in K : m' = \Phi_V(\tau, m) \Rightarrow \forall \lambda > 0 : \lambda V \in K$ the vector field λV joins $m, m' : m' = \Phi_{\lambda V}\left(\frac{\tau}{\lambda}, m\right)$ and $g(\lambda V, \lambda V) = \lambda^2 g(V, V) = Ct$. So there is a Killing vector field joining m, m' with a given length $g(V, V) > 0$.

21.2 Isometries as Clifford morphims

A linear map which preserves the scalar product on vector spaces can be extended to a Clifford morphism. An isometry on M can be extended to a Clifford isomorphism on P_{Cl} :

f maps $m \to f(m)$

$f'(m)$ maps $T_m M \to T_{f(m)} M$ and preserves the scalar product, so there is a Clifford morphism $\widehat{f}(m) : Cl(\mathbb{R}, 3, 1) \to Cl(\mathbb{R}, 3, 1)$ such that $\widehat{f}(m) = f'(m)$. (Maths.494).

$\forall u, v \in T_m M : \widehat{f}(m)(u \cdot v) = f'(m)(u) \cdot f'(m) v$

$\widehat{f}(m)(\varepsilon_j \cdot \varepsilon_k) = f'(m)(\varepsilon_j) \cdot f'(m)(\varepsilon_k)$

$\widehat{f}(m)(F_a(m)) = \sum_{b=1}^{16} \left[\widehat{f}(m)\right]_a^b F_b(f(m))$

A Clifford isomorphism which preserves the vector space spanned by $(\varepsilon_j)_{j=0}^3$ can be written as Ad_S where S is the product of at most 4 vectors : it is either the scalar multiple of a fixed vector, or an element of $Spin(3, 1)$. There is a section $S \in \mathfrak{X}(P_{Cl})$ such that :

$\widehat{f}(m)(F_a(m)) = \sum_{b=1}^{16} \left[Ad_{S(f(m))}\right]_a^b F_b(f(m))$

A real automorphism F on P_{Cl} can be extended to a real automorphism $\widehat{F}$ on P_C (see Categories) :

$\widehat{F}(\operatorname{Re} Z + i \operatorname{Im} Z) = \widehat{F}(C(Z_1) + iC(Z_2)) = C(F(Z_1)) + iC(F(Z_2)) = C\left(F\left(C^{-1}(\operatorname{Re} Z)\right)\right) + iC\left(F\left(C^{-1}(\operatorname{Im} Z)\right)\right)$

F and $\widehat{F}$ preserves the scalar product.

F preserves the transposition : take a vector of the basis at m : $\varepsilon_{a_1} \cdot ... \cdot \varepsilon_{a_p}$

$F\left(\left(\varepsilon_{a_1} \cdot ... \varepsilon_{a_p}\right)^t\right) = F\left(\varepsilon_{a_p} \cdot ... \varepsilon_{a_1}\right) = F\left(\varepsilon_{a_p}\right) \cdot ... F\left(\varepsilon_{a_1}\right) = \left(F\left(\varepsilon_{a_1}\right) \cdot ... F\left(\varepsilon_{a_p}\right)\right)^t$

$F(Z^t) = (F(Z))^t$

and so does $\widehat{F}$ which, then preserves the hermitian product :

$CC\left(\widehat{F}(\operatorname{Re} Z + i \operatorname{Im} Z)\right) = CC(C(F(Z_1)) + iC(F(Z_2))) = C(F(Z_1)) - iC(F(Z_2)) = \widehat{F}(CC(Z))$

$\left\langle \widehat{F}(Z) ; , \widehat{F}(Z') \right\rangle_H = \left\langle CC\left(\widehat{F}(Z)\right), \widehat{F}(Z') \right\rangle_{Cl} = \left\langle \widehat{F}(CC(Z)), \widehat{F}(Z') \right\rangle_{Cl} = \langle CC(Z), Z' \rangle_{Cl} = \langle Z, Z' \rangle_H$

Indeed $\widehat{F}(m)(Z) = Ad_{C(S(m))} Z$.

So, with the isometry f we can define a Clifford bundle isomorphism (f, F) on P_C

$f : M \rightarrow M$
$F{:}P_C \rightarrow P_C : F(m)(Z) = F(m)\left(\sum_{a=1}^{16} Z^a F_a(m)\right)$
$= \sum_{a,b=1}^{16} \left[Ad_{C(S(f(m)))}\right]_b^a Z^b F_a(f(m))$

$$\begin{array}{cccc} & & f & \\ M & m & \rightarrow & f(m) \\ & \downarrow & & \downarrow \\ TM & \varepsilon_j(m) & \rightarrow & f'(m)\,\varepsilon_j(m) \\ & \downarrow & & \downarrow \\ P_C & F_a(m) & \rightarrow & Ad_{C(S(f(m)))}(F_a(m)) \\ & & F & \end{array}$$

Now, if we have a section $Z \in \mathfrak{X}(P_C)$ we can define its pull back :
$F^* : \mathfrak{X}(P_C) \rightarrow \mathfrak{X}(P_C) :: F^*(S(f(m)))(m) = \sum_{a,b=1}^{16} \left[Ad_{C(S(m))}\right]_b^a Z^b(f(m)) F_a(m)$
and its push forward :
$F_* : \mathfrak{X}(P_C) \rightarrow \mathfrak{X}(P_C) :: F_*(S(m))(f(m)) = \sum_{a,b=1}^{16} \left[Ad_{C(S(f(m)))}\right]_b^a Z^b(m) F_a(f(m))$

21.3 One parameter group of isometries

A one parameter group of isometries is defined by the flow of a Killing vector field V, it defines a one parameter group of morphisms on P_C :
$f_t : M \rightarrow M :: f_\tau(m) = \Phi_V(\tau, m)$
$F_\tau : P_C \rightarrow P_C : F_\tau(m)\left(\sum_{a=1}^{16} Z^a F_a(m)\right) = \sum_{a,b=1}^{16} \left[Ad_{C(S(\Phi_V(\tau,m)))}\right]_b^a Z^b F_a(\Phi_V(\tau, m))$
where $C(S(\Phi_V(\tau, m)))$ is given by $\Phi'_{Vm}(\tau, m)$.
Because we have a semi-group : $F_{\tau+h}(m) = F_h(F_\tau(m))$
$Ad_{C(S(\Phi_V(\tau+h,m)))} = Ad_{C(S(\Phi_V(h,m)))} \circ Ad_{C(S(\Phi_V(\tau,m)))}$
by derivating at $h = 0$
$\frac{d}{dh} Ad_{C(S(\Phi_V(\tau+h,m)))}|_{h=0} = ad(C(V)) = Ad_{C(S)} \circ ad(C(V))$
Using : $\frac{d}{d\tau} Ad_{\exp \tau T} = Ad_{\exp \tau T} \circ adT$ we deduce :

$$C(S(\Phi_V(\tau, m))) = \exp \tau C(V) \cdot C(S(m)) \tag{112}$$

With $V = v_0\varepsilon_0 + v$ where $v_0 \in \mathbb{R}, v \in \mathbb{R}^3$ the components of V in the tetrad
$C(V) = iv_0\varepsilon_0 + v$
$(0, v_0, v, 0, 0, 0, 0, 0) \cdot (0, v'_0, v', 0, 0, 0, 0, 0)$
$= (A, V_0, V, W, R, X_0, X, B) = (v_0 v'_0 + v^t v', 0, 0, v_0 v' - v'_0 v, -j(v) v', 0, 0, 0)$
$C(V) \cdot C(V) = \left(-v_0^2 + v^t v, 0, 0, 0, 0, 0, 0, 0\right)$
If V is future oriented : $-v_0^2 + v^t v' = \langle V, V\rangle < 0$:
$\exp \tau V = \cos \tau\mu_v + \frac{\sin \tau\mu_v}{\mu_v} V = \left(\cos \tau\mu_v, \frac{\sin \tau\mu_v}{\mu_v} v_0, \frac{\sin \tau\mu_v}{\mu_v} v, 0, 0, 0, 0, 0\right)$ with $\mu_v = \sqrt{v_0^2 - v^t v}$
If $-v_0^2 + v^t v' = \langle V, V\rangle = 0 : \exp \tau V = 1$
If $V = v$ (vector in $\Omega_3(t)$:
$\exp \tau V = \cosh \tau\mu_v + \frac{\sinh \tau\mu_v}{\mu_v} V = \left(\cosh \tau\mu_v, \frac{\sinh \tau\mu_v}{\mu_v} v_0, \frac{\sinh \tau\mu_v}{\mu_v} v, 0, 0, 0, 0, 0\right)$
with $\mu_v = \sqrt{v^t v}$

$Ad_{C(S(\Phi_V(\tau+h,m)))} = Ad_{\exp\tau C(V)} Ad_{C(S(m))}$
$Ad_{\exp\tau C(V)} = \exp\tau ad\left(C\left(V\right)\right)$

The vector V is tranported along the curve :
$\Phi'_{Vm}\left(\tau,m\right)\left(\sum_{j=0}^{3} V^j\left(m\right)\varepsilon_j\left(m\right)\right) = V\left(\Phi_V\left(\tau,m\right)\right)$
S is such that : $\forall\tau : V\left(\Phi_V\left(\tau,m\right)\right) = Ad_{S(\Phi_V(\tau,m))}V\left(m\right)$ by taking the restriction of $Ad_{S(\Phi_V(\tau,m))}$ to the space generated by $\left(\varepsilon_j\right)_{i=0..3}$ in $Cl\left(3,1\right)$
$V\left(\Phi_V\left(\tau,m\right)\right) = Ad_{\exp\tau V} Ad_{S(m)}V\left(m\right)$
At $\forall m, \tau = 0 : V\left(m\right) = Ad_{S(m)}V\left(m\right)$
The vector V is eigen vector of Ad_S with eigen value 1, which can be written
$Ad_{S(m)}V\left(m\right) = V\left(m\right) \Leftrightarrow S\cdot V = V\cdot S$
This is a necessary condition for V, but V is not the only vector which is transported by Φ_V.
If $\exp\tau V = 1$ then the components of V in the tetrad of P_C are constant along an integral curve originating at m.
S is either the multiple of a vector S or $S\in Spin\left(3,1\right)$.

i) If S is the multiple of a vector S: $S = s_0\varepsilon_0 + s$
$S\cdot V = V\cdot S$
Using the rules for the procuct in $Cl\left(3,1\right)$:
$S\cdot V = V\cdot S \Leftrightarrow$
$\left(-s_0v_0 - s^tv, 0, 0, s_0v - v_0s, -j\left(s\right)v, 0, 0, 0\right)$
$= \left(-v_0s_0 - v^ts, 0, 0, v_0s - s_0v, -j\left(v\right)s, 0, 0, 0\right)$
$s_0v - v_0s = v_0s - s_0v = 0$
$-j\left(s\right)v = -j\left(v\right)s = 0 \Rightarrow s = \lambda v$
gives : $S = \lambda\left(v_0\varepsilon_0 + v\right) = \lambda V$
The vector V is such that $\langle V,V\rangle = Ct = v^tv - v_0^2$ and $\langle S,S\rangle = \lambda^2\langle V,V\rangle$. The map $Ad_S = Ad_V$.
But, because $V^{-1} = V/\langle V,V\rangle$, this solution requires that $\langle V,V\rangle \neq 0$.

ii) If $S\in Spin\left(3,1\right) : S\left(m\right) = \left[a,0,0,w,r,0,0,b\right], a,b\in\mathbb{R}, w,r\in\mathbb{R}^3$
$w^tr = -ab$
$a^2 - b^2 - w^tw + r^tr = 1$
Using the formulas for the product in $Cl\left(3,1\right)$: we get the necessary relations for S,V :
$S\left(m\right)\cdot V = V\cdot S\left(m\right)$
$\Leftrightarrow$
$av_0 + w^tv = v_0a - v^tw$
$av + v_0w + j\left(r\right)v = va - v_0w + j\left(v\right)r$
$bv_0 - r^tv = -v_0b - v^tr$
$-bv + v_0r - j\left(w\right)v = bv + v_0r + j\left(v\right)w$
$w^tv = 0$
$v_0w + j\left(r\right)v = 0$
$bv_0 = 0$
$bv = 0$

$V \neq 0 \Rightarrow b = 0$
$w^t r = -ab = 0$
$a^2 - b^2 - w^t w + r^t r = 1 = a^2 - w^t w + r^t r$
So $S(m)$ is the product of a spatial rotation and a translation, such that :
$S = [a, 0, 0, w, r, 0, 0, 0]$
$w^t v = 0$
$v_0 w + j(r) v = 0$
$w^t r = 0$
$a^2 - w^t w + r^t r = 1$
$C([a, v_0, v, w, r, x_0, x, b]) = (a, iv_0, v, iw, r, x_0, ix, ib)$
$C(S) = (a, 0, 0, iw, r, 0, 0, 0)$
where w, r are defined by $\Phi'_{Vm}(0, m)$: these parameters are not given by the components of V but must be compatible with V. Usually S has a 2 dimensional eigen space with eigen value 1.

If $v_0 = c$ then
$w = -\frac{1}{c} j(r) v$
$S(m)$ on $Cl(3,1)$ is necessarily the product of a spatial rotation and a translation, defined by $V = c\varepsilon_0 + \widetilde{v}$ with the components $\widetilde{v}$ of v in the tetrad, and a parameter r which can be seen as a polarisation :

The matrix of the adjoint map $Ad_{C(S(m))}$ is then :

$$[Ad_{C(S)}]_{16 \times 16} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & [N]_{4\times 4}(V_0, V) & 0 & 0 & 0 \\ 0 & 0 & [M]_{6\times 6}(W, R) & 0 & 0 \\ 0 & 0 & 0 & [N]_{4\times 4}(X_0, X) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $[N], [M]$ are the matrices, acting on the vector subspaces
$(0, V_0, V, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, X_0, X, 0), (0, 0, 0, W, R, 0, 0, 0)$
$[N(v_0, v)]_{4\times 4} = [N(x_0, x)]_{4\times 4}$
$$= (1 + 2w^t w) I_4 + 2 \begin{bmatrix} 0 & iw^t(a - j(r)) \\ -i(a + j(r)) w & aj(r) + j(r) j(r) + j(w) j(w) \end{bmatrix}$$
$[M(w, r)]_{6\times 6} = I_6$
$$+2 \begin{bmatrix} aj(r) + j(r) j(r) - j(w) j(w) & i(r^t w + aj(w) + j(r) j(w) + j(w) j(r)) \\ i(r^t w + aj(w) + j(w) j(r) + j(r) j(w)) & aj(r) + j(r) j(r) - j(w) j(w) \end{bmatrix}$$

In the holonomic basis of the chart the map f^*_τ is expressed, in coordinates, through the jacobian $[J] = [\Phi'_{Vm}(\tau, m)]$:

$$\begin{array}{cccc} & & \Phi_V(\tau, m) & \\ M & m & \rightarrow & \Phi_V(\tau, m) \\ & \downarrow & & \downarrow \\ TM & \partial\xi_\beta(m) & \rightarrow & \Phi'_{Vm}(\tau, m)(\partial\xi_\beta(m)) \\ & \downarrow & & \downarrow \\ P_C & \varepsilon_j(m) & \rightarrow & Ad_{C(S(f(m)))}(\varepsilon_j(m)) \\ & & F & \end{array}$$

The tetrad which defines P_C is transported as another orthonormal basis in P_C :

$\Phi'_{Vm}(\tau,m)(\varepsilon_j(m)) = \Phi'_{Vm}(\tau,m)\left(\sum_{\beta=0}^3 P_j^\beta(m)\,\partial\xi_\beta(m)\right)$

$= \sum_{\beta,\gamma=0}^3 [J(\tau,m)]_\gamma^\beta P_j^\gamma(m)\,\partial\xi_\beta(\Phi_V(\tau,m))$

$= \sum_{k=0}^3 \left[Ad_{S(\Phi_V(\tau,m))}\right]_j^k \varepsilon_k(\Phi_V(\tau,m)) = \sum_{k=0}^3 \left[Ad_{S(\Phi_V(\tau,m))}\right]_j^k \sum_{\beta=0}^3 P_k^\beta(\Phi_V(\tau,m))\,\partial\xi_\beta$

$\sum_{\gamma=0}^3 [J(\tau,m)]_\gamma^\beta P_j^\gamma(m) = \sum_{k=0}^3 \left[Ad_{S(\Phi_V(\tau,m))}\right]_j^k P_k^\beta(\Phi_V(\tau,m))$

$\sum_{\gamma,j=0}^3 [J(\tau,m)]_\gamma^\beta P_j^\gamma(m)\,P'^j_\alpha(m) = \sum_{k,j=0}^3 \left[Ad_{S(\Phi_V(\tau,m))}\right]_j^k P_k^\beta(\Phi_V(\tau,m))\,P'^j_\alpha(m)$

$[J(\tau,m)]_\alpha^\beta = \sum_{jk=0}^3 \left[Ad_{S(\Phi_V(\tau,m))}\right]_j^k P_k^\beta(\Phi_V(\tau,m))\,P'^j_\alpha(m)$

so the matrix $[J]$ is related to the restriction of $\left[Ad_{S(\Phi_V(\tau,m))}\right]$ to the space spanned by $(\varepsilon_j)_{j=0}^3$

$$[J(\tau,m)] = [P(\Phi_V(\tau,m))]\left[Ad_{S(\Phi_V(\tau,m))}\right][P'(m)] \qquad (113)$$

At $\tau = 0 : [J(0,m)] = [P(m)]\left[Ad_{S(m)}\right][P'(m)]$

Part VII
BIBLIOGRAPHY

JC.Dutailly *Mathematics for theoretical Physics* (2016) Amazon E-Book

J.C Dutailly *Theoretical Physics* (2016) Amazon

J.C.Dutailly *Clifford Algebras* (2018) Hal 018 265 551

I.Kolar, P.Michor, J.Slovak *Natural operations in differential geometry* (1991) Springer-Verlag