A Note of Differential Geometry

- Application of the Method of the Repère Mobile to the Ellipsoid of Reference in Geodesy -

by

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Contents

A NOTE OF DIFFERENTIAL GEOMETRY - APPLICATION OF THE METHOD OF THE REPÈRE MOBILE TO THE ELLIPSOID OF REVOLUTION

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1 THE REPÈRE MOBILE

1.1 INTRODUCTION

Let E the geodetic ellipsoid of reference defined by the parameters a, e respectively the semi-major axis and the first eccentricity. Let $\mathcal{R}(O, X, Y, Z)$ the geocentric frame associated to this ellipsoid. A point A est defined by its tridimensional Cartesian coordinates (X, Y, Z) :

$$X = (N + h)cos\varphi cos\lambda \tag{1}$$
$$Y = (N + h)cos\varphi sin\lambda \tag{2}$$
$$Z = (N(1 - e^2) + h)sin\varphi \tag{3}$$

with

$$N = \frac{a}{\sqrt{1 - e^2 sin^2\varphi}} \tag{4}$$

At $A(\varphi, \lambda, h)$, we consider the local frame defined by the orthonormal frame $(e_\lambda, e_\varphi, e_n)$ given in the normed basis (i, j, k) of $\mathcal{R}$ as:

$$e_\lambda = \begin{vmatrix} -sin\lambda \\ cos\lambda \\ 0 \end{vmatrix} ; e_\varphi = \begin{vmatrix} -sin\varphi cos\lambda \\ -sin\varphi sin\lambda \\ cos\varphi \end{vmatrix} ; e_n = \begin{vmatrix} cos\varphi cos\lambda \\ cos\varphi sin\lambda \\ sin\varphi \end{vmatrix} \tag{5}$$

To facilitate the notations, let:

$$e_1 = e_\lambda \tag{6}$$
$$e_2 = e_\varphi \tag{7}$$
$$e_3 = e_n \tag{8}$$

When the geodetic coordinates (φ, λ, h) of A change, the local frame at A moves that it called mobile frame or repère mobile of the point A.

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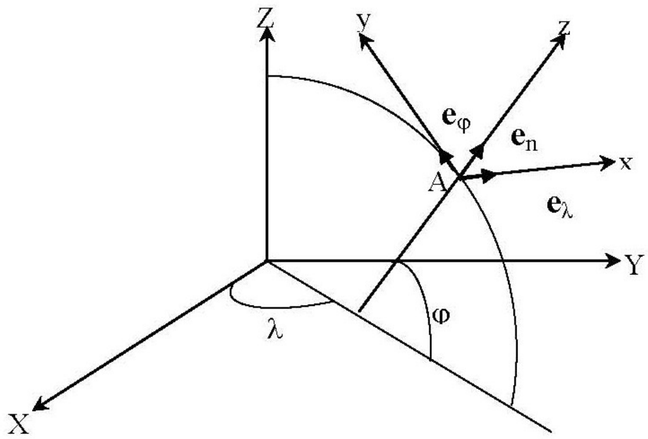

Figure 1: The Mobile Frame

2 THE DIFFERENTIAL EXPRESSIONS OF dA AND de_i

The position of A is given in $\mathcal{R}$ by:

$$OA = Xi + Yj + Zk \tag{9}$$

with (i, j, k) the canonical basis of $\mathcal{R}$. As (e_1, e_2, e_3) is also a orthonormal basis of $\mathcal{R}$, the liaison of (i, j, k) to (e_1, e_2, e_3) (A. Ben Hadj Salem, 2010) is given by:

$$\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \begin{pmatrix} -sin\lambda & -sin\varphi cos\lambda & cos\varphi cos\lambda \\ cos\lambda & -sin\varphi sin\lambda & cos\varphi sin\lambda \\ 0 & cos\varphi & sin\varphi \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{10}$$

The differential of (9) gives :

$$dA = idX + jdY + kdZ$$

Now, we replace i, j et k in function of e_1, e_2 and e_3, we obtain the differential expressions:

$$\boxed{dA = \sum_{i=1}^{i=3} \omega_i e_i} \tag{11}$$

Similarly, the differential of the vector e_i is expressed in (i, j, k), and will be transformed in the basis (e_i) using the inverse of the matrix given by (10), then we obtain:

$$\boxed{de_i = \sum_{j=1}^{j=3} \omega_{ij} e_j} \tag{12}$$

The two formulas (11) and (12) define the infinitesimal shifting of the repère mobile at A when A moves.

The coefficients ω_i and ω_i respectively in the previous two formulas are differential forms of degree 1 in terms of differential forms $d\lambda, d\varphi$ and dh (A. Ben Hadj salem, 2012).

3 RELATIONS SATISFIED BY DIFFERENTIAL FORMS ω_i AND ω_{ij}

Returning to past formulas (11) and (12), they represent differentials of vector functions, as a result, then:

$$d(dA) = 0 \tag{13}$$

$$d(de_i) = 0 \quad i = 1, 3 \tag{14}$$

The formula (13) gives:

$$d(dA) = d(\sum_i \omega_i e_i) = \sum_i d(\omega_i e_i) = \sum_i (d\omega_i e_i - \omega_i \wedge de_i) = \sum_i d\omega_i e_i - \sum_i \omega_i \wedge de_i = 0 \tag{15}$$

Let:

$$\sum_i d\omega_i e_i = \sum_i \omega_i \wedge de_i$$

Now replace de_i by its expression (12), we get:

$$\sum_i d\omega_i e_i = \sum_i \omega_i \wedge de_i = \sum_i \omega_i \wedge \sum_k \omega_{ik} e_k = \sum_k \sum_i \omega_i \wedge \omega_{ik} e_k$$

Giving by changing the index i and k for the right term:

$$\sum_i d\omega_i e_i = \sum_i \sum_k \omega_k \wedge \omega_{ki} e_i$$

So that:

$$\boxed{d\omega_i = \sum_{k=1}^{k=3} \omega_k \wedge \omega_{ki}} \tag{16}$$

Now back to the formula (14):

$$d(de_i) = 0 = d(\sum_{j=1}^{j=3} \omega_{ij} e_j) = \sum_{j=1}^{j=3} d(\omega_{ij} e_j) \tag{17}$$

But:

$$\sum_j d(\omega_{ij} e_j) = \sum_j d\omega_{ij} e_j - \sum_j \omega_{ij} \wedge d(e_j) = \sum_j d\omega_{ij} e_j - \sum_j \omega_{ij} \wedge (\sum_{k=1}^{k=3} \omega_{jk} e_k) = 0$$

Then:

$$\sum_j d\omega_{ij} e_j = \sum_{j=1}^{j=3} \sum_{k=1}^{k=3} \omega_{ij} \wedge \omega_{jk} e_k \tag{18}$$

The (e_i) form a basis of $\mathcal{R}$, e_j coefficients must be equal to 0, giving after handling:

$$\boxed{d\omega_{ij} = \sum_{k=1}^{k=3} \omega_{ik} \wedge \omega_{kj} \quad 1 \leq i \leq 3,\ 1 \leq j \leq 3} \tag{19}$$

Differential forms given by (16) and (19) constitute the Elie Cartan formulas (H. Cartan, 1979).

3.1 Case of an orthonormal repère mobile

In the case studied in this note, the base (e_i) is an orthonormal basis that is to say:

$$e_i.e_j = \delta_{ij} \qquad \begin{cases} = 1 \text{ si} & i = j = 1 \\ = 0 \text{ si} & i \neq j \end{cases} \tag{20}$$

Differencing (20), we get :

$$de_i.e_j + e_i.de_j = 0 \tag{21}$$

Using the formula giving de_i, that is to say (12), the above expression becomes::

$$(\sum_{k=1}^{k=3} \omega_{ik} e_k).e_j + e_i.(\sum_{k=1}^{k=3} \omega_{jk} e_k) = 0 \tag{22}$$

As $e_i.e_j = \delta_{ij}$, we obtain:

$$\boxed{\omega_{ij} + \omega_{ji} = 0 \quad \forall i, j = 1, 2, 3} \tag{23}$$

When $i = j$, we get:

$$\boxed{\omega_{11} = \omega_{22} = \omega_{33} = 0} \tag{24}$$

4 Application to the Ellipsoid of Reference

4.1 Determination of the ω_i

Taking the differential of the formulas (1-3) in the basis (i, j, k), then :

$$dA = idX + jdY + kdZ$$

Replacing i, j and k by its expressions in function of e_1, e_2 and e_3, we obtain:

$$dA = cos\varphi(N + h)d\lambda e_1 + (\rho + h)d\varphi e_2 + dhe_3 \tag{25}$$

Let:

$$\boxed{\omega_1 = cos\varphi(N + h)d\lambda; \quad \omega_2 = (\rho + h)d\varphi; \quad \omega_3 = dh} \tag{26}$$

4.2 DETERMINATION OF THE ω_{ij}

The expression of de_i in function of the vectors e_i of the repère mobile at A is obtained using (5):

$$de_1 = -(icos\lambda + jsin\lambda)d\lambda$$
$$de_2 = (-cos\varphi cos\lambda d\varphi + sin\varphi sin\lambda d\lambda)i + (-cos\varphi sin\lambda d\varphi - sin\varphi cos\lambda d\lambda)j - kcos\varphi d\varphi$$
$$de_3 = (-sin\varphi cos\lambda d\varphi - cos\varphi sin\lambda d\lambda)i + (-sin\varphi sin\lambda d\varphi + cos\varphi cos\lambda d\lambda)j + kcos\varphi d\varphi \quad (27)$$

Starting from (10), we get:

$$de_1 = sin\varphi d\lambda e_2 - cos\varphi d\lambda e_3 \quad (28)$$
$$de_2 = -sin\varphi d\lambda e_1 - d\varphi e_3 \quad (29)$$
$$de_3 = cos\varphi d\lambda e_1 + d\varphi e_2 \quad (30)$$

also in matrix form:

$$\begin{pmatrix} de_1 \\ de_2 \\ de_3 \end{pmatrix} = \begin{pmatrix} 0 & sin\varphi d\lambda & -cos\varphi d\lambda \\ -sin\varphi d\lambda & 0 & -d\varphi \\ cos\varphi d\lambda & d\varphi & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \Omega \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \quad (31)$$

with:

$$\Omega = \begin{pmatrix} 0 & sin\varphi d\lambda & -cos\varphi d\lambda \\ -sin\varphi d\lambda & 0 & -d\varphi \\ cos\varphi d\lambda & d\varphi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \quad (32)$$

Then the elements ω_{ij}:

$$\omega_{11} = \omega_{22} = \omega_{33} = 0 \quad (33)$$
$$\omega_{12} = -\omega_{21} = sin\varphi d\lambda \quad (34)$$
$$\omega_{13} = -\omega_{31} = -cos\varphi d\lambda \quad (35)$$
$$\omega_{23} = -\omega_{32} = -d\varphi \quad (36)$$

4.3 VERIFICATION OF THE FORMULAS $d\omega_i$ AND $d\omega_{ij}$

Let us check the formula (16) that is:

$$d\omega_i = \sum_{k=1}^{k=3} \omega_k \wedge \omega_{ki}$$

For example, we calculate $d\omega_1$:

$$d\omega_1 = d((N+h)cos\varphi d\lambda) = d(Ncos\varphi d\lambda) + d(hcos\varphi d\lambda) \quad (37)$$

but $d(Ncos\varphi) = -\rho sin\varphi d\varphi$, we obtain:

$$d\omega_1 = -\rho sin\varphi d\varphi \wedge d\lambda - hsin\varphi d\varphi \wedge d\lambda + cos\varphi dh \wedge d\lambda \tag{38}$$

or :

$$\boxed{d\omega_1 = -(\rho + h)sin\varphi d\varphi \wedge d\lambda + cos\varphi dh \wedge d\lambda} \tag{39}$$

By the formula (16):

$$d\omega_1 = \sum_{k=1}^{k=3} \omega_k \wedge \omega_{k1} = \omega_1 \wedge \omega_{11} + \omega_2 \wedge \omega_{21} + \omega_3 \wedge \omega_{31} =$$

$$0 + \omega_2 \wedge \omega_{21} + \omega_3 \wedge \omega_{31} = (\rho + h)d\varphi \wedge (-sin\varphi d\lambda) + dh \wedge cos\varphi d\lambda =$$

$$-(\rho + h)sin\varphi d\varphi \wedge d\lambda + cos\varphi dh \wedge d\lambda \tag{40}$$

Which is identical to the equation (39) above. Now we check the formulas of the $d\omega_{ij}$:

$$d\omega_{ij} = \sum_{k=1}^{k=3} \omega_{ik} \wedge \omega_{kj}$$

and we calculate for example $d\omega_{12}$:

$$\boxed{d\omega_{12} = d(sin\varphi d\lambda) = cos\varphi d\varphi \wedge d\lambda} \tag{41}$$

On the other hand, according to the formula (19), we get:

$$d\omega_{12} = \sum_{k=1}^{k=3} \omega_{1k} \wedge \omega_{k2} = \omega_{11} \wedge \omega_{12} + \omega_{21} \wedge \omega_{22} + \omega_{13} \wedge \omega_{32}$$

but:

$$\omega_{11} = \omega_{22} = 0$$

Finally, we obtain:

$$d\omega_{12} = \sum_{k=1}^{k=3} \omega_{1k} \wedge \omega_{k2} = \omega_{13} \wedge \omega_{32} = -cos\varphi d\lambda \wedge d\varphi = cos\varphi d\varphi \wedge d\lambda \tag{42}$$

It is the result of (41).

5 CASE WHERE $h = 0$

If $h = 0$, then the point A is on the tangent plane on the ellipsoid, in this case, we obtain:

$$\omega_1 = Ncos\varphi d\lambda \tag{43}$$

$$\omega_2 = \rho d\varphi \tag{44}$$

$$\omega_{12} = sin\varphi d\lambda \tag{45}$$

However, according to the fundamental theorem of the local Riemannian geometry, there exists a unique differential form ω_{12} defined in the tangent plane which satisfies the equations (S.S Chern, 1985):

$$d\omega_1 = \omega_{12} \wedge \omega_2 \tag{46}$$

$$d\omega_2 = \omega_1 \wedge \omega_{12} \tag{47}$$

$$\text{et} \quad d\omega_{12} = -K\omega_1 \wedge \omega_2 \tag{48}$$

with K is Gauss curvature or the total curvature at the point A.

Let us check the equations (46-48). From (43-45), we obtain:

$$d\omega_1 = -\rho sin\varphi d\varphi \wedge d\lambda \tag{49}$$

$$d\omega_2 = \rho' d\varphi \wedge d\varphi = 0 \tag{50}$$

$$d\omega_{12} = cos\varphi d\varphi \wedge d\lambda \tag{51}$$

then:

$$d\omega_1 = \omega_{12} \wedge \omega_2 = sin\varphi d\lambda \wedge \rho d\varphi = \rho sin\varphi d\lambda \wedge d\varphi \tag{52}$$

$$d\omega_2 = \omega_1 \wedge \omega_{12} = N cos\varphi d\lambda \wedge sin\varphi d\lambda = 0 \tag{53}$$

$$\text{et} \quad d\omega_{12} = cos\varphi d\varphi \wedge d\lambda \tag{54}$$

but :

$$\omega_1 \wedge \omega_2 = N cos\varphi d\lambda \wedge \rho d\varphi = -\rho N cos\varphi d\varphi \wedge d\lambda \tag{55}$$

Comparing (51) and (55), we find that :

$$d\omega_{12} = -\frac{1}{\rho N}\omega_1 \wedge \omega_2 = -K\omega_1 \wedge \omega_2 \tag{56}$$

with:

$$K = \frac{1}{\rho N} = \text{the total curvature or Gauss curvature}$$

It is the inverse of the product of two radii of curvature of the ellipsoid. (57)

Q.E.D

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