Espil's theorem corollary

By Federico Espil chemistry Engineer (UTN), chemistry technician, and member of the reverse osmosis & arsenic removal research team (UTN)

Author: Espil Federico **(03/2017)**

e-mail: ludhorr@gmail.com // ludhor@hotmail.com

Key words: Dirichlet integral, Espil's theorem, Dirichlet generalization, Espil's law.

Abstract: shortly from the Espil's theorem, we can derive the generalized Dirichlet integral for any natural value when the hole integrand is raised to the n-th power.

1 Proof:

$$\int_{0}^{\infty} \frac{(\mathrm{Sin}[t])^{n}}{t^{n}} dt = \operatorname*{Lim}_{s \to 0} \int_{s}^{\infty} \frac{1}{\Pi_{(n-1)}} \mathcal{L}[D^{n-1}((\mathrm{Sin}[t])^{n})] d\sigma$$

$$= \operatorname*{Lim}_{s \to 0} \int_{s}^{\infty} \frac{1}{(n-1)!} \mathcal{L}[D^{n-1}((\frac{e^{it} - e^{-it}}{2i})^{n})] d\sigma$$

$$= \operatorname*{Lim}_{s \to 0} \int_{s}^{\infty} \frac{1}{(n-1)!} \mathcal{L}[D^{n-1}(\sum_{k=0}^{n} \binom{n}{k} \frac{(e^{it})^{n-k}(-1)^{k}(e^{-it})^{k}}{(2i)^{n}})] d\sigma$$

$$= \operatorname*{Lim}_{s \to 0} \int_{s}^{\infty} \frac{1}{(n-1)!} \mathcal{L}[D^{n-1}(\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k} e^{i(n-2k)t}}{(2i)^{n}})] d\sigma$$

$$= \lim_{s\to 0}\int_{s}^{\infty}\frac{1}{(n-1)!}\mathcal{L}[\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}(i(n-2k))^{n-1}e^{i(n-2k)t}}{(2i)^{n}}]d\sigma$$

$$= \lim_{s\to 0}\int_{s}^{\infty}\frac{1}{(n-1)!}\mathcal{L}[\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}(n-2k)^{n-1}e^{i(n-2k)t}}{2^{n}i}]d\sigma$$

If n is even

$$= \lim_{s\to 0}\int_{s}^{\infty}\frac{1}{(n-1)!}\mathcal{L}[\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{(-1)^{k}(n-2k)^{n-1}(e^{i(n-2k)t}-e^{-i(n-2k)t})}{2^{n-1}(2i)}]d\sigma$$

$$= \lim_{s\to 0}\int_{s}^{\infty}\frac{1}{(n-1)!}\mathcal{L}[\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{(-1)^{k}(n-2k)^{n-1}\mathrm{Sin}[(n-2k)t]}{2^{n-1}}]d\sigma$$

$$= \lim_{s\to 0}\int_{s}^{\infty}\frac{1}{(n-1)!}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{(-1)^{k}(n-2k)^{n-1}}{2^{n-1}}\frac{n-2k}{\sigma^{2}+(n-2k)^{2}}\,d\sigma$$

$$= \lim_{s\to 0}\int_{s}^{\infty}\frac{1}{(n-1)!}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{(-1)^k(n-2k)^{n-1}}{2^{n-1}}\frac{\frac{1}{n-2k}}{\left(\frac{\sigma}{n-2k}\right)^2+1}\,d\sigma$$

$$= \lim_{b\to\infty}\frac{1}{(n-1)!}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{(-1)^k(n-2k)^{n-1}}{2^{n-1}}\operatorname{Arctan}\left[\frac{b}{n-2k}\right]$$

$$= \frac{1}{(n-1)!}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{(-1)^k(n-2k)^{n-1}}{2^{n-1}}\frac{\pi}{2}$$

$$\int_{0}^{\infty}\frac{(\operatorname{Sin}[t])^n}{t^n}\,dt = \frac{\pi}{(n-1)!}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\frac{(-1)^k(n-2k)^{n-1}}{2^{n}}\quad, n\ even$$

If n is odd

$$= \lim_{s\to 0} \int_s^{\infty} \frac{1}{(n-1)!} \mathcal{L}\left[\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{(-1)^k (n-2k)^{n-1} \left(e^{i(n-2k)t} - e^{-i(n-2k)t}\right)}{2^{n-1}(2i)}\right] d\sigma$$

$$= \lim_{s\to 0} \int_s^{\infty} \frac{1}{(n-1)!} \mathcal{L}\left[\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{(-1)^k (n-2k)^{n-1} \mathrm{Sin}[(n-2k)t]}{2^{n-1}}\right] d\sigma$$

$$= \lim_{s\to 0} \int_s^{\infty} \frac{1}{(n-1)!} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{(-1)^k (n-2k)^{n-1}}{2^{n-1}} \frac{n-2k}{\sigma^2 + (n-2k)^2} \, d\sigma$$

$$= \lim_{s\to 0} \int_s^{\infty} \frac{1}{(n-1)!} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{(-1)^k (n-2k)^{n-1}}{2^{n-1}} \frac{\frac{1}{n-2k}}{\left(\frac{\sigma}{n-2k}\right)^2 + 1} \, d\sigma$$

.

$$= \lim_{b\to\infty} \frac{1}{(n-1)!} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{(-1)^k (n-2k)^{n-1}}{2^{n-1}} \mathrm{Arctan}\left[\frac{b}{n-2k}\right]$$

$$= \frac{1}{(n-1)!} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{(-1)^k (n-2k)^{n-1}}{2^{n-1}} \frac{\pi}{2}$$

$$\int_0^\infty \frac{(\mathrm{Sin}[\mathrm{t}])^\mathrm{n}}{\mathrm{t}^\mathrm{n}} \mathrm{dt} = \frac{\pi}{(n-1)!} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \frac{(-1)^k (n-2k)^{n-1}}{2^n} \quad , n \ odd$$

2 Examples:

$$2.1) \int_0^\infty \frac{\mathrm{Sin}[\mathrm{t}]}{\mathrm{t}} \mathrm{dt} = \frac{\pi}{(1-1)!} \sum_{k=0}^{\frac{1-1}{2}} \binom{1}{k} \frac{(-1)^k (1-2k)^{1-1}}{2} = \frac{\pi}{2}$$

$$2.2) \int_0^\infty \frac{(\mathrm{Sin}[\mathrm{t}])^2}{\mathrm{t}^2} \mathrm{dt} = \frac{\pi}{(2-1)!} \sum_{k=0}^{\frac{2}{2}-1} \binom{2}{k} \frac{(-1)^k (2-2k)^{2-1}}{2^2} = \frac{\pi}{2}$$

$$2.3) \int_0^\infty \frac{(\mathrm{Sin}[\mathrm{t}])^3}{\mathrm{t}^3} \mathrm{dt} = \frac{\pi}{(3-1)!} \sum_{k=0}^{\frac{3-1}{2}} \binom{3}{k} \frac{(-1)^k (3-2k)^{3-1}}{2^3} = \frac{\pi}{16} \left(1.3^2 - 3.1^2\right) = \frac{3\pi}{8}$$

$$2.4) \int_0^\infty \frac{(\mathrm{Sin}[\mathrm{t}])^4}{\mathrm{t}^4} \mathrm{dt} = \frac{\pi}{(4-1)!} \sum_{k=0}^{\frac{4}{2}-1} \binom{4}{k} \frac{(-1)^k (4-2k)^{4-1}}{2^4} = \frac{\pi}{6} \left(\frac{1.4^3 - 4.2^3}{2^4}\right) = \frac{\pi}{3}$$