Proof that there are no odd perfect numbers

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1. Abstract

For $y$ to be a perfect number, if one of the prime factors is $p$, the exponent of $p$ is an integer $n(n \geq 1)$, the prime factors other than $p$ are $p_1, p_2, p_3, \cdots p_r$, and the even exponent of $p_k$ is $q_k$,

\[ y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) / (2p^n) = \prod_{k=1}^{r} p_k^{q_k} \]

must be satisfied. Let $m$ and $q$ be non-negative integers,

\[ n = 4m + 1 \]
\[ p = 4q + 1 \]

Letting $b$ and $c$ be odd integers, satisfying following expressions,

\[ b = \prod_{k=1}^{r} p_k^{q_k} \]
\[ c = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) / p^n \]
\[ 2b = c(p^n + \cdots + 1) \]

is established. This is a known content. By the consideration of this research paper, let $f$ be an odd integer and $d_r$ be a positive integer and an index of $p_r$ included in $(p + 1)/2$, if following expression

\[(p + 1)/2 = fp_r^{d_r}\]

is satisfied, let $c_r$ be an index of $p_r$ included in $c$,

\[ 2m + 1 = wp_r^{q_r-c_r-d_r} \]

is proved to be the condition in order for $b$ to be divided by $p_r^{q_r}$. By using this conditional expression, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

\[ 1 + 2 + 3 = 6 \]

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.
3. Proof

An odd perfect number is $y$, one of them is an odd prime number $p$, an exponent of $p$ is an integer $n$ ($n > 0$). Let $p_1, p_2, p_3, \ldots, p_r$ be the odd prime numbers of factors other than $p$, $q_k$ the index of $p_k$, and variable $a$ be the sum of product combinations other than prime $p$.

$$a = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \ldots \text{①}$$

The number of terms $N$ of variable $a$ is

$$N = \prod_{k=1}^{r} (q_k + 1) \ldots \text{②}$$

When $y$ is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)$$

is established.

$$a \sum_{k=0}^{n} p^k / 2 = y$$

$$a \sum_{k=0}^{n} p^k / (2p^n) = y / p^n \ldots \text{③}$$

3.1. If $q_k$ has at least one odd integer

Letting the number of terms where $q_k$ is an odd integer be a positive integer $u$, because $y / p^n = \prod_{k=1}^{r} p_k^{q_k}$ is an odd integer, the denominator on the left side of expression ③ has a prime factor 2, from expression ② variable $a$ has more than $u$ prime factor 2 and variable $a$ is an even integer. Therefore $\sum_{k=0}^{n} p^k$ must be an odd integer, $n$ is an even integer and $u$ is 1.

3.2. When all $q_k$ are even integers

$y / p^n$ is an odd integer, the denominator on the left side of expression ③ is an even integer, and since $N$ is an odd integer when $q_k$ are all even integers, variable $a$ is and odd integer. Therefore $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \ (mod \ 2)$ is established, and $n$ must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of $y$ must be an odd integer and variable $a$ must be an odd integer. We consider the case of 3.2 below.
In order for \( y \) to be a perfect number, the following expression must be established.

\[
y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} \left(1 + p_k + p_k^2 + \cdots + p_k^{q_k}\right)/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}
\]

However, \( q_1, q_2, \ldots, q_r \) are all even integers.

Here, let \( b \) be an integer

\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \text{... ④}
\]

A following expression is established.

\[
y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b \\
a(p^{n+1} - 1)/(2(p - 1)p^n) = b \\
(a - 2b)p^{n+1} + 2bp^n - a = 0 \quad \text{... ⑤}
\]

Because it is an \( n + 1 \) order equation of \( p \), the solution of the odd prime \( p \) is \( n + 1 \) at most.

\[
(ap - 2bp + 2b)p^n = a
\]

Since \( ap - 2bp + 2b \) is an odd integer, \( a/p^n \) is an odd integer, which is \( c \).

\[
ap - 2bp + 2b = c \quad (c > 0) \quad \text{... ⑥}
\]

\[
(2b - a)p = 2b - c
\]

Since variable \( a \) is an odd integer, \( 2b - a \) is an odd integer and \( 2b - a \neq 0 \).

\[
p = (2b - c)/(2b - a)
\]
Since \( n \geq 1 \)
\[
a - c = cp^n - c \geq cp - c > 0
\]
a > c
is.

From equation ⑥
\[
2b(p - 1) - (ap - c) = 0
\]
\[
2b - c(p^{n+1} - 1)/(p - 1) = 0
\]
\((p^n + \cdots + 1)/2\) is an odd integer, \( n = 4m + 1 \) is required with \( m \) as an integer.
\[
2b(p - 1) = c(p^{n+1} - 1)
\]
\[
2b = c(p^n + \cdots + 1)
\]
\[
2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) \ldots ⑦
\]
b is an odd integer when \( p + 1 \) is not a multiple of 4. It is necessary that \( p - 1 \) be a multiple of 4. A positive integer is taken as \( q \).
\[
p = 4q + 1
\]
is established.

When \( p > 1 \)
\[
p^n - 1 < p^n
\]
\[
(p^n - 1)/(p - 1) < p^n/(p - 1)
\]
\[
p^{n-1} + \cdots + 1 < p^n/(p - 1) \ldots ⑧
\]

Since \( p \) is an odd prime number satisfying \( p = 4q + 1 \) and \( p \geq 5 \)
\[
p^{n-1} + \cdots + 1 < p^n/4
\]
\[
2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)
\]
\[
2b - a < cp^n/4 = a/4
\]
\[
2b < 5a/4
\]
a > 8b/5 \ldots ⑨
Let $a_k$ and $b_k$ be integers and if

$$a_k = 1 + p_k + p_k^2 + \cdots + p_k^q, \quad b_k = p_k^q,$$

from inequality (8)

$$a_k - b_k < b_k/(p_k - 1)$$
$$a_k < b_k p_k/(p_k - 1)$$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$
$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality (9).
From equation ⑦, \( b \) has \( (p + 1)/2 \) as a factor. We consider this problem, if \( (p + 1)/2 \) is prime number, \( p_r = (p + 1)/2 \) holds.

When \( p_r = (p + 1)/2 \)
\[
2b = c(p^n + \cdots + 1) = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1)
\]
\[
2b/(p + 1) = c(p^{n-1} + p^{n-3} + \cdots + 1)
\]
Since the index \( q_r \) of \( p_r \), the factor of \( b \), \( b \) must be divided by \( ((p + 1)/2)^{q_r} \), and \( 2b/(p + 1) \) must be divided by \( ((p + 1)/2)^{q_r-1} \).

Because \( 2b/(p + 1) \) and \( (p + 1)/2 \) are odd numbers, let \( s \) be an odd integer.
\[
2b = c(p^{n-1} + p^{n-3} + \cdots + 1) = s((p + 1)/2)^{q_r-1}
\]
\[
c(p^{n-1} + p^{n-3} + \cdots + 1) \times 2^{q_r-1} = s(p + 1)^{q_r-1}
\]
As \( c \times 2^{q_r-1} \) is an even integer and \( s \) is odd integer, let \( t \) be an odd integer,
\[
c \times 2^{q_r-1} - s = tp
\]
is established.
\[
c(p^{n-1} + p^{n-3} + \cdots + 1) \times 2^{q_r-1} = (c \times 2^{q_r-1} - tp)(p + 1)^{q_r-1}
\]
\[
 tp(p + 1)^{q_r-1} = c \times 2^{q_r-1}((p + 1)^{q_r-1} - (p^{n-1} + p^{n-3} + \cdots + 1))
\]
\[
 tp = c \times 2^{q_r-1}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p + 1)^{q_r-1})
\]
Since \( t \) is odd integer and \( p \) is odd prime, let \( u \) be an odd integer, this expression holds in the following cases.

i. When \( c = up \)
\[
t = u \times 2^{q_r-1}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p + 1)^{q_r-1})
\]
\[
s = c \times 2^{q_r-1} - tp
\]
\[
s = up \times 2^{q_r-1} - tp = (u \times 2^{q_r-1} - t)p
\]
\[
b = sp_r^{q_r} = p(u \times 2^{q_r-1} - t)p_r^{q_r}
\]
is.
\[
b \equiv 0 \pmod{p}
\]
is. Since \( b = \prod_{k=1}^{n} p_k^{q_k} \), it becomes inappropriate contrary to \( b \not\equiv 0 \pmod{p} \).
When \( c \neq up \)

- When \( t \) is not a multiple of \( c \), let \( z \) be a rational number.

\[
z = 2^{q-r-1}(1 - (p^{n-1} + p^{n-3} + \ldots + 1)/(p + 1)^{q_r-1})/t
\]

If \( z \) is an integer, \( p = cz \) when \( z \neq 1 \), contrary to \( p \) is odd prime, and \( c = p \) when \( z = 1 \), contrary to the case classification condition.

If \( z \) is not an integer,

\[
 tp = tcz = c \times 2^{q-r-1}(1 - (p^{n-1} + p^{n-3} + \ldots + 1)/(p + 1)^{q_r-1})
\]

\[
 tp = c(2^{q-r-1} - (p^{n-1} + p^{n-3} + \ldots + 1)/p_r^{q-r-1})
\]

Since \( b = sp_r^{q_r} \),

\[
s = \prod_{k=1}^{r-1} p_k^{q_k}
\]

is established. Therefore \( s \) doesn’t include the factor \( p_r \).

Let the number of factors \( p_r \) included in \( c \) be an integer \( c_r \).

\[
 2b/(p + 1) = b/p_r = sp_r^{q_r-1} = c(p^{n-1} + p^{n-3} + \ldots + 1)
\]

is established. And \( c_r \) can take the value of \( 0 \leq c_r \leq q_r - 1 \).

If \( 2^{q-r-1} - (p^{n-1} + p^{n-3} + \ldots + 1)/p_r^{q_r-1} \) is an integer, let this value be a rational number \( R \).

\[
 tp = cR
\]

is. Because \( c \neq up \), \( R \) must be divisible by \( p \).

\[
 t = cR/p
\]

is established. This contradicts that \( t \) is not a multiple of \( c \). Therefore \( R \) must not be an integer, when \( t \) is not a multiple of \( c \).

If \( v \) is an odd number not including the factor \( p_r \),

\[
 c = vp_r^{c_r}
\]

holds.

\[
 tp = vp_r^{c_r}(2^{q_r-1} - (p^{n-1} + p^{n-3} + \ldots + 1)/p_r^{q_r-1})
\]

\[
 tp = v(2^{q_r-1} \times p_r^{c_r} - (p^{n-1} + p^{n-3} + \ldots + 1)/p_r^{q_r-c_r-1})
\]

In order for \( tp \) to be an integer, the second term on the right side of the above expression must be an integer. Let \( 2^{q_r-1} \times p_r^{c_r} - (p^{n-1} + p^{n-3} + \ldots + 1)/p_r^{q_r-c_r-1} \) be an integer \( N \).

\[
 R = N/p_r^{c_r}
\]
If N includes the factor \( p_r \), t has factor \( p_r \), as \( tp = vN \), and the right side of 
\( s = c \times 2^{q_{r-1}} - tp \) includes factor \( p_r \), then it becomes inconsistent, since s has factor 
\( p_r \). Since R is not an integer, \( c_r > 0 \) holds.

\[ p^{n-1} + p^{n-3} + \cdots + 1 \] must be divided by \( p_r^{q_{r-c_r-1}} \).

Let z be an integer, and if 
\[ (p^{n-1} + p^{n-3} + \cdots + 1)/p_r^{q_{r-c_r-1}} = zp_r \] holds, 
\[ N = 2^{q_{r-1}} \times p_r^{c_r} - zp_r \] is established. It becomes contradiction since N is a multiple of \( p_r \). Therefore the 
index of factor \( p_r \) of 
\[ p^{n-1} + p^{n-3} + \cdots + 1 \] must be \( q_r - c_r - 1 \).

When t is a multiple of c 
\[ t = cv \] and \[ p = 2^{q_{r-1}}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p + 1)^{n_{r-1}})/v \]
\[ vp = 2^{q_{r-1}}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p + 1)^{q_r-1}) \]
\[ vp = 2^{q_{r-1}} - (p^{n-1} + p^{n-3} + \cdots + 1)/p_r^{q_{r-1}} \]
In order for \( vp \) is an integer, 
\[ p^{n-1} + p^{n-3} + \cdots + 1 \] must be divided by \( p_r^{q_{r-1}} \).

Let z be an integer, and if 
\[ (p^{n-1} + p^{n-3} + \cdots + 1)/p_r^{q_{r-1}} = zp_r \] holds, 
\[ vp = 2^{q_{r-1}} - zp_r \]
\[ tp = cvp = c(2^{q_{r-1}} - zp_r) \]
\[ s = c2^{q_{r-1}} - tp = c2^{q_{r-1}} - c(2^{q_{r-1}} - zp_r) = czp_r \]
Since s is a multiple of \( p_r \) and it becomes contradiction, the index of factor \( p_r \) of 
\[ p^{n-1} + p^{n-3} + \cdots + 1 \] must be \( q_r - 1 \).
From the above, when \( c_r \) takes the value of \( 0 \leq c_r \leq q_r - 1 \), \( p^{n-1} + p^{n-3} + \cdots + 1 \) must be divisible by \( p_r^{q_r-c_r-1} \).

\[
\begin{align*}
(p^{n-1} + p^{n-3} + \cdots + 1)/p_r^{q_r-c_r-1} & = (p^n + \cdots + 1)/(p + 1)p_r^{q_r-c_r-1} \\
& = (p^n + \cdots + 1)/(2p_r^{q_r-c_r}) \\
& = (p^{n+1} - 1)/(2(p - 1)p_r^{q_r-c_r}) \\
& = ((2p - 1)^{n+1} - 1)/(4(p - 1)p_r^{q_r-c_r})
\end{align*}
\]

When \( f(p_r) = ((2p - 1)^{n+1} - 1)/(4(p - 1)p_r^{q_r-c_r}) \) \( \cdots \)

When \( n = 5 \)

\[
(2p - 1)^6 = \sum_{k=0}^{6} \binom{6}{k} (-1)^k (2p)^{6-k}
\]

\[
(2p - 1)^6 = 64p^6 - 192p^5 + 240p^4 - 160p^3 + 60p^2 - 12p + 1
\]

\[
\begin{align*}
f(p_r) & = (64p_r^6 - 192p_r^5 + 240p_r^4 - 160p_r^3 + 60p_r^2 - 12p_r + 1)/(4(p_r - 1)p_r^{q_r-c_r}) \\
f(p_r) & = (16p_r^4 - 32p_r^3 + 28p_r^2 - 12p_r + 3)/p_r^{q_r-c_r-1}
\end{align*}
\]
We consider the proof that 0 order term of \((2p_r - 1)^{n+1} - 1)/(4p_r(p_r - 1))\) must be 2m + 1, for integer \(m (m \geq 0)\) of \(n = 4m + 1\).

When \(m = 0\), as 0 order term is 1, this proposition holds.
Assuming that this proposition holds when \(m = k\), the first or higher order term of \(p\) is defined as \(g(k)\).

\[
\left(2p_r - 1\right)^{4k+2} - 1 = 4p_r(p_r - 1)(g(k) + 2k + 1)
\]

\[
\left(2p_r - 1\right)^{4k+2} = 4p_r(p_r - 1)(g(k) + 2k + 1) + 1
\]

\[
\left(2p_r - 1\right)^{4k+2}(2p_r - 1)^4 = 4p_r(p_r - 1)(2p_r - 1)^4(g(k) + 2k + 1) + (2p_r - 1)^4
\]

And

\[
\left(2p_r - 1\right)^{4(k+1)+2} - 1)/(4p_r(p_r - 1))
\]

\[
= (2p_r - 1)^4(g(k) + 2k + 1) + ((2p_r - 1)^4 - 1)/(4p_r(p_r - 1))
\]

\[
= (2p_r - 1)^4(g(k) + 2k + 1) + 4p_r^2 - 4p_r + 2
\]

When \(g(k+1) = (2p_r - 1)^4g(k) + ((2p_r - 1)^4 - 1)(2k + 1) + 4p_r^2 - 4p_r\).

\[
\left(2p_r - 1\right)^{4(k+1)+2} - 1)/(4p_r(p_r - 1)) = g(k + 1) + 2k + 3 = g(k + 1) + 2(k + 1) + 1
\]
is holds when \(m = k + 1\). From the above, by mathematical induction, this proposition holds with arbitrary \(m\) satisfying \(m \geq 0\).
When \( n = 9 \)
\[
f(p_r) = \left(256 p_r^8 - 1024 p_r^7 + 1856 p_r^6 - 1984 p_r^5 + 1376 p_r^4 - 640 p_r^3 + 200 p_r^2 - 40 p_r + 5 \right) / p_r^{q_r-c_r-1}
\]

When \( n = 13 \)
\[
f(p_r) = \left(4096 p_r^{12} - 24576 p_r^{11} + 68608 p_r^{10} - 117760 p_r^9 + 138496 p_r^8 - 117760 p_r^7 + 74432 p_r^6 - 35392 p_r^5 + 12656 p_r^4 - 3360 p_r^3 + 644 p_r^2 - 84 p_r + 7 \right) / p_r^{q_r-c_r-1}
\]

When \( n = 17 \)
\[
f(p_r) = \left(65536 p_r^{16} - 524288 p_r^{15} + 1982464 p_r^{14} - 4702208 p_r^{13} + 7831552 p_r^{12} - 9715712 p_r^{11} + 9293824 p_r^{10} - 7000064 p_r^9 + 4201984 p_r^8 - 2021376 p_r^7 + 779136 p_r^6 - 239232 p_r^5 + 57792 p_r^4 - 10752 p_r^3 + 1488 p_r^2 - 144 p_r + 9 \right) / p_r^{q_r-c_r-1}
\]

Let \( g(p_r) \) be the a polynomial that satisfies \( g(p_r) = f(p_r) p_r^{q_r-c_r-1} \), and let the coefficient of \( i \)-th term of \( g(p_r) \) be a rational number \( T_i (i \geq 0) \).
\[
g(p_r) = T_{n-1} p_r^{n-1} + \cdots + T_1 p_r + T_0
\]

\[
(2p_r - 1)^{n+1} - 1 = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i (2p_r)^{n+1-i} - 1
\]

\[
(2p_r - 1)^{n+1} - 1 = \sum_{i=0}^{n} \binom{n+1}{i} (-1)^i (2p_r)^{n+1-i}
\]

\[
4p_r (p_r - 1) g(p_r) = 4p_r (p_r - 1) (T_{n-1} p_r^{n-1} + \cdots + T_1 p_r + T_0)
\]

Comparing the coefficients.
\[
-T_0 = -\left( \frac{n+1}{1} \right) / 2
\]
\[
T_0 - T_1 = \left( \frac{n+1}{2} \right)
\]
\[
T_1 - T_2 = -\left( \frac{n+1}{3} \right) / 2
\]
\[
\cdots
\]
\[
T_{n-2} - T_{n-1} = -\left( \frac{n+1}{n} \right) 2^{n-2}
\]
\[ T_0 = 2m + 1 \]

When \( 1 \leq i \leq n - 1 \)

\[ T_{i-1} - T_i = (-2)^{i-1} \binom{n+1}{i+1} \]

\[ T_i = T_{i-1} - (-2)^{i-1} \binom{n+1}{i+1} \]

\[ T_1 = T_0 - \binom{n+1}{2} \]

\[ T_1 = 2m + 1 - \frac{(n+1)n}{2} = 2m + 1 - (2m + 1)(4m + 1) = -4m(2m + 1) \]

\[ T_2 = T_1 + 2 \binom{n+1}{3} \]

\[ T_2 = -4m(2m + 1) + 2(n + 1)n(n - 1)/6 = -4m(2m + 1) + 2(2m + 1)(4m + 1)4m/3 \]

\[ T_2 = (2m + 1)(-4m + 2(4m + 1)(4m)/3) = 4m(2m + 1)(8m - 1)/3 \]

In the same way,

\[ T_3 = -32m^2(2m - 1)(2m + 1)/3 \]

\[ T_4 = 16m(2m - 1)(2m + 1)(16m^2 - 10m - 1)/15 \]

are established.

Because \( T_0 = 2m + 1 \) and \( \binom{n+1}{i+1} \) of the recurrence expression of \( T_i \) includes a factor \( n + 1 \) in the range of \( 1 \leq i \leq n - 1 \), the factor \( 2m + 1 \) is included in all \( T_i \).
When \( n = 9 \) and \( c_r < q_r - 1 \)

\[
f(p_r) = (256p_r^8 - 1024p_r^7 + 1856p_r^6 - 1984p_r^5 + 1376p_r^4 - 640p_r^3 + 200p_r^2 - 40p_r + 5)/p_r^{q_r-c_r-1}
\]

Letting \( h(p_r) \) be a fractional functions obtained by subtracting a term larger than \( q_r - c_r - 2 \) from the numerator of \( f(p_r) \).

- When \( q_r - c_r = 2 \)
  
  \[
h(p_r) = T_0/p_r
  \]

- When \( q_r - c_r = 3 \)
  
  \[
h(p_r) = (T_1p_r + T_0)/p_r^2
\]

- When \( q_r - c_r = 4 \)
  
  \[
h(p_r) = (T_2p_r^2 + T_1p_r + T_0)/p_r^3
\]

\[
4m(8m-1)p_r^2/3 + 4mp_r + 1 = 4m((8m-1)p_r/3 + 1)p_r + 1
\]

When \( 4m((8m-1)p_r/3 + 1) \) is an integer,

\[
4m(8m-1)p_r^2/3 + 4mp_r + 1 \equiv 1 \pmod{p_r}
\]

When \( 4m((8m-1)p_r/3 + 1) \) is not an integer, if \( 4m((8m-1)p_r^2/3 + 4mp_r + 1 \) is a multiple of \( p_r \), assuming that the numerator of \( 4m((8m-1)p_r/3 + 1) \) is an integer

\[A\] and the denominator is an integer \( B \), and letting \( C \) be an integer,

\[
A/B \times p_r + 1 = Cp_r
\]

\[Ap_r + B = BCp_r\]

\[B \equiv 0 \pmod{p_r} \] ... (A)

In order for expression (A) to be satisfied, \( p_r \) must be 3, the denominator of \( B \).

Because the expression (A) is not satisfied, if \( b \) has another prime factor, \( b \) has only 3 as the factor, and it becomes inconsistent, as \( r = 1 \). From the above, \( h(p_r)p_r^3/(2m+1) \) isn’t divided by \( p_r \).
\[ q_r - c_r = 5 \]

\[
h(p_r) = \left( T_3 p_r^3 + T_2 p_r^2 + T_1 p_r + T_0 \right) / p_r^4
\]

\[
h(p_r) = \left( -32 m^2 (2m - 1)(2m + 1)p_r^3 / 3 + 4m(2m + 1)(8m - 1)p_r^7 / 3 + 4m(2m + 1)p_r + 2m + 1 \right) / p_r^4
\]

\[
h(p_r) = (2m + 1)(-32 m^2 (2m - 1)p_r^3 / 3 + 4m(8m - 1)p_r^7 / 3 + 4mp_r + 1) / p_r^4
\]

\[
h(p_r) p_r^4 / (2m + 1) \text{ isn't divided by } p_r \text{ as in the case of } q_r - c_r = 4.
\]

\[
\cdot \text{ When } q_r - c_r \geq 6
\]

When \( h(p_r) p_r^5 \) is an integer, \( h(p_r) p_r^5 / (2m + 1) \) isn’t divisible by \( p_r \). Since the denominator of \( T_4 \) is 15, if the denominator is either 3 or 5, the prime factor of \( b \) is only 3 or 5 and it becomes inappropriate, as \( r = 1 \). If the prime factor of the denominator is 3 and 5, expression (A) holds for both values. If \( p_r = 3 \) and \( p_s = 5 \), since the numerator must be divided by \( p_s \), \( A \equiv 0 \ (\text{mod} \ p_s) \) is required. \( A \) must similarly be a multiple of 15. The numerator and the denominator have the common divisor. Since the form of the formula does not change even if it divides the numerator and the denominator, it becomes contradiction, because it has to be possible indefinitely.

From the above, when \( h(p_r) p_r^{q_r - c_r - 1} \) is an integer, \( h(p_r) p_r^{q_r - c_r - 1} / (2m + 1) \) isn’t divided by \( p_r \). When \( h(p_r) p_r^{q_r - c_r - 1} \) is not an integer, If \( h(p_r) p_r^{q_r - c_r - 1} = A/B \times p_r + 1 \)

In the case of \( B \) has one prime factor, it becomes contradiction, since \( b \) has only one factor and \( r = 1 \). If \( B \) includes plural prime factors, because both numerator and denominator must include the common divisor, the product of two arbitrary prime factors in the prime factors, it becomes inconsistent. Therefore, because for arbitrary \( q_r \) and \( c_r \) where \( c_r < q_r - 1 \), \( h(p_r) p_r^{q_r - c_r - 1} / (2m + 1) \) isn’t divisible by \( p_r \), \( 2m + 1 \) must be a multiple of \( p_r^{q_r - c_r - 1} \).

When \( c_r < q_r - 1 \) in order for \( b \) to be divided by \( p_r^{q_r} \), \( 2m + 1 \) must be divisible by \( p_r^{q_r - c_r - 1} \), let \( w \) be an odd integer,

\[
2m + 1 = wp_r^{q_r - c_r - 1} \ldots \]

is required. However, \( w \) does not include \( p_r \) as a factor.
From equation (7),
\[ 2b(p - 1) = c(p^{n+1} - 1) \]

Letting \( d_k \) be an index of \( p_k \) included in \( (p + 1)/2 \), in order for \( b \) to be divided by \( p_k^{q_k} \), let \( w \) be an odd integer, just as in deriving expression (11), for \( k \) where \( d_k > 0 \),
\[ 2m + 1 = w \prod_{d_k > 0} p_k^{q_k-c_k-d_k} \ldots \text{(B)} \]
must be satisfied.

Since \( d_k > 0 \), according to Fermat's small theorem, let \( t_k \) be a positive rational number,
\[ t_k(n + 1) = p_k - 1 \]
Assuming that the numerator of \( t_k \) is an integer \( N_k \) and the denominator is an integer \( D_k \),
\[ (n + 1)N_k/D_k = p_k - 1 \ldots \text{(C)} \]
\( N_k \) is a divisor of \( p_k - 1 \), \( D_k \) is a divisor of \( n + 1 \).

Substituting expression (B) into expression (C),
\[ 2w \prod_{d_k > 0} p_k^{q_k-c_k-d_k} \times N_k/D_k = p_k - 1 \]
Since \( w \not\equiv 0 \pmod{p_k} \), let \( D_k' \) be an integer,
\[ D_k = D_k' \prod_{d_k > 0} p_k^{q_k-c_k-d_k} \]
\[ 2wN_k/D_k' = p_k - 1 \]
is established.

\[ 2w/D_k' = (p_k - 1)/N_k \]
\( D_k' \) is a divisor of \( 2w \).

\[ (p_k - 1)D_k' = 2wN_k \]
I. If two of the odd prime numbers included in \((p + 1)/2\) are taken as \(p_r, ps\), when

\[ q_r - c_r - d_r > 0, q_s - c_s - d_s > 0, \text{ and } q_k - c_k - d_k = 0 \]

for all other factor \(p_k\)

If \(F(p) = p^{n-1} + p^{n-3} + \cdots + 1\). The factor of \(F(p)\) are \(p_r\) and \(p_s\).

\[ p_r^{q_r - c_r - d_r}p_s^{q_s - c_s - d_s} = p^{n-1} + p^{n-3} + \cdots + 1 \]

\[ p = 2 \prod_{k=1}^{r} p_k^{d_k} - 1 \]

\[ p_r^{q_r - c_r - d_r}p_s^{q_s - c_s - d_s} = (2 \prod_{k=1}^{r} p_k^{d_k} - 1)^{n-1} + (2 \prod_{k=1}^{r} p_k^{d_k} - 1)^{n-3} + \cdots + 1 \]

\((D)\)

From expression (C),

\[ 2m + 1 = w'p_s^{q-s-d_s} \]

is established.

\[ 2m + 1 \geq p_r^{q_r - c_r - d_r}p_s^{q_s - c_s - d_s} \]

\((E)\)

\[ (2 \prod_{k=1}^{r} p_k^{d_k} - 1)^{n-1} + (2 \prod_{k=1}^{r} p_k^{d_k} - 1)^{n-3} + \cdots + 1 \]

monotonically increases in the range of \[2 \prod_{k=1}^{r} p_k^{d_k} - 1 \geq 0\].

\[ 2 \prod_{k=1}^{r} p_k^{d_k} - 1 = 1 \]

When \[ \prod_{k=1}^{r} p_k^{d_k} = 1\],

\[ (2 \prod_{k=1}^{r} p_k^{d_k} - 1)^{n-1} + (2 \prod_{k=1}^{r} p_k^{d_k} - 1)^{n-3} + \cdots + 1 = 2m + 1 \]

is established.

Since the above expression is satisfied, when \(p_r \geq 3\).

\[ p_r^{q_r - c_r - d_r}p_s^{q_s - c_s - d_s} > 2m + 1 \]

must be satisfied. However, since this inequality contradicts expression (E), this inequality is not satisfied and it becomes inappropriate. Since this expression holds when all exponents on the left side of the expression (D) are not 0, this expression is satisfied when \(q_s - c_s - d_s = 0\).

This also holds when three or more prime numbers are included in \((p + 1)/2\). Therefore, it becomes inappropriate if all prime factors contained in \(F(p)\) are those contained in \((p + 1)/2\).
II. If two of the odd prime numbers included in \((p + 1)/2\) are taken as \(p_r, p_s\), when 
\[ q_r - c_r - d_r > 0, \quad q_s - c_s - d_s > 0, \] 
and at least one factor \(p_t\) among factors not included in \((p + 1)/2\) satisfies 
\[ q_t - c_t - d_t > 0, \]
Assuming that \(p_t\) is a factor of \(2m + 1\) at this time, letting its index be a positive integer \(q_t - c_t\), from the expression (E),
\[ 2m + 1 \geq p_t^{a_t - c_t - d_t} p_s^{a_s - c_s - d_s} p_t^{a_t - c_t} \]
is established. However, in this case it becomes inappropriate like proof I.
Therefore, if the prime factor not included in \((p + 1)/2\) is included in \(F(p)\), \(2m + 1\) does not become a multiple of \(p_t^{a_t - c_t}\).

III. If one of the prime numbers included in \((p + 1)/2\) is set as \(p_r\), when 
\[ q_r - c_r - d_r > 0, \] 
a prime factor other than \(p_t\) to \(p_r\) is set as \(p_t\)
From \(a = cp^n\) and expression (7),
\[ 2bp^n = a(p^n + \cdots + 1) \]
\[ a(p^n + \cdots + 1)/(2bp^n) = 1 \ldots \text{(F)} \]
If \(p_t\) is a prime number not included in \((p + 1)/2\) and \(p_t\) does not exist, this expression does not hold from the proof of I.

Let \(R\) be a rational number,
\[ R = a(p^n + \cdots + 1)/(2bp^n) \]
Let \(b'\) be a rational number and let \(A\) and \(B\) to be an integer,
\[ b' = (p_t^{q_t+1} - 1)/(p_t^{q_t}(p_t - 1)) > 1 \]
\[ A = (p_t^{q_t+1} - 1)/(p_t - 1) \]
\[ B = p_t^{q_t} \]
Multiplying \(R\) by \(b'\), there are both cases that \(p_r\) increases \(p\) or does not change.
When multiplied by \(b'\), the rate of change of \(R\) is
\[ Ap^n(p^n + \cdots + 1)/(Bp^{q_t}(p^n + \cdots + 1)) \]
if \(p\) after variation is \(p'\). If the rate of change of \(R\) is 1,
\[ Ap^n(p^n + \cdots + 1)/(Bp^{q_t}(p^n + \cdots + 1)) = 1 \]
\[ Ap^n(p^n + \cdots + 1) = Bp^{q_t}(p^n + \cdots + 1) \]
This expression does not hold, since the right side is not a multiple of \(p\) when \(p' > p\), and \(A > B\) holds when \(p' = p\). Due to this operation, \(R\) may be larger or smaller than the original value, since the rate of change of \(R\) does not become 1.
From $R \neq 1$ and $a = cp^n$ for some $r$, also multiplying fractions of prime numbers, if

\[ a(p^n + \cdots + 1)/\left(2bp^n\right) \times A_1p^n(p_1^n + \cdots + 1)/(B_1p_1^n(p^n + \cdots + 1)) \times A_2p_1^n(p_2^n + \cdots + 1)/(B_2p_2^n(p^n + \cdots + 1)) \times A_xp_{x-1}^n(p_x^n + \cdots + 1)/(B_xp_x^n(p^n + \cdots + 1)) = 1 \]

\[ a/(2b) \times A_1/B_1 \times A_2/B_2 \cdots A_x(p^n + \cdots + 1)/(B_xp^n) = 1 \]

\[ a(p_x^n + \cdots + 1)A_1A_2 \cdots A_x = 2bp_x^nB_1B_2 \cdots B_x \]

\[ cp^n(p_x^n + \cdots + 1)A_1A_2 \cdots A_x = 2bp_x^nB_1B_2 \cdots B_x \]

It becomes inconsistent when $p_x > p$, since the right side of this expression does not include $p$ as a factor.

When $p_x = p$,

\[ cp^n(p^n + \cdots + 1)A_1A_2 \cdots A_x = c(p^n + \cdots + 1)p^n \]

\[ A_1A_2 \cdots A_x = 1 \]

It becomes contradiction, since this expression is not established. Therefore, $a = cp^n$

holds at one point where $R = 1$.

Assuming that $F(p)$ contains prime factor $p_t$,

\[ p^{n-1} + p^{n-3} + \cdots + 1 = (2m + 1)/w \times p_t^{q_t-c_t} \]

holds.

\[ p^{n+1} - 1 = (p^2 - 1)(2m + 1)/w \times p_t^{q_t-c_t} \]

\[ 2m + 1 = w(p^{n+1} - 1)/(p^2 - 1)p_t^{q_t-c_t} \ldots (F) \]

Let $N_t$ and $D_t$ be integers. From Euler's theorem,

\[ (n + 1)N_t/D_t = (p_t - 1)p_t^{q_t-c_t} \]

\[ 2(2m + 1)/D_t = (p_t - 1)p_t^{q_t-c_t-1}/N_t \]

$N_t$ is a divisor of $(p_t - 1)p_t^{q_t-c_t-1}$, and $D_t$ is a divisor of $n + 1$.

Substituting expression (F),

\[ 2w(p^{n+1} - 1)/(p^2 - 1)p_t^{q_t-c_t}D_t = (p_t - 1)p_t^{q_t-c_t-1}/N_t \]

\[ 2w(p^{n+1} - 1)/(p^2 - 1)p_tD_t = (p_t - 1)/N_t \]

\[ 2w(p^{n+1} - 1) = (p_t - 1)/N_t \times (p^2 - 1)p_tD_t \]

\[ 2w(p^{n-1} + p^{n-3} + \cdots + 1) = (p_t - 1)/N_t \times p_tD_t \]
Assuming that \( w_t \) is the index of \( p_t \) contained in \( w \) and \( \delta_t \) is the index of \( p_t \) contained in \( D_t \). The term containing \( p_t \) on the left side is

\[ p_t^{q_t-c_t} \times p_t^{w_t} \]

The term that includes \( p_t \) on the right side is

\[ p_t^{\delta_t+1} \]

\[ q_t - c_t + w_t = \delta_t + 1 \]
\[ \delta_t = q_t - c_t + w_t - 1 \]

Since the exponents of \( p_t \) included in \( 2m + 1 \) and \( w \) are the same, it becomes contradiction in the case of \( \delta_t > w_t \). Therefore,

\[ q_t - c_t - 1 \leq 0 \]
\[ q_t - c_t \leq 1 \]

must be satisfied.

For simplifying prime numbers from 1 to \( r \), express them with two primes \( p_r \) and \( p_s \),

\[ 2m + 1 = w p_r^{q_r-c_r-d_r} p_s^{q_s-c_s-d_s} \]

When \( q_t - c_t = 1 \),

\( F(p) = (2m + 1)p_t/w \)

is established.

\[ (2m + 1)p_t/w = p^{n-1} + p^{n-3} + \cdots + 1 \]
\[ (2m + 1)p_t/w \times c(p + 1) = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) = c(p^n + \cdots + 1) \]
\[ b = c(p + 1)/2 \times (2m + 1)p_t/w \]

Let \( b' \) and \( c' \) be integers. If

\[ b' = b/p_t^{q_t} \]
\[ c' = c/p_t^{c_t} = c/p_t^{q_t-1} \]

is satisfied. By dividing both sides of the above equation by \( p_t^{q_t} \),

\[ b' = c'(p + 1)/2 \times (2m + 1)/w \]

must be established. However, the value of \( p \) changes as \( F(p) = (2m + 1)/w \). It becomes contradiction, since this expression does not hold. Therefore, it is inappropriate when \( n \geq 5 \).
IV. When $q_k - c_k - d_k = 0$ for all $k$

Since $F(p) = 1$,

$1 = p^{n-1} + p^{n-3} + \cdots + 1$

· When $n \geq 5$

$p^{n-1} + p^{n-3} + \cdots + p^2 = 0$

$p^2(p^{n-3} + p^{n-5} + \cdots + 1) = 0$

It becomes inappropriate, since there are no real solution in the range of $p > 0$.

· When $n = 1$

From $a = cp$ and expression (7),

$2bp = a(p + 1)$

$a(p + 1)/(2bp) = 1 \ldots (G)$

When $r = 1$, if

$a = (p_1q_1+1 - 1)/(p_1 - 1)$

$b = p_1q_1$

holds, since there are no odd perfect number when $r = 1$, expression (G) does not hold.

Let $R$ be a rational number,

$R = a(p + 1)/(2bp)$

If $p_r$ is an odd prime number other than $p_1$ and let $b'$ to be a rational number, and $A$ and $B$ to be integers,

$b' = (p_rq_{r+1}^1 - 1)/(p_rq(r - 1)) > 1$

$A = (p_rq_{r+1}^1 - 1)/(p_r - 1)$

$B = p_rq^r$
Multiplying $R$ by $b'$, there are both cases that $p_r$ increases $p$ or does not change. When multiplied by $b'$, the rate of change of $R$ is $A p' + 1)/(B p' + 1)$, if $p$ after variation is $p'$. If the rate of change of $R$ is 1, $A p' + 1)/(B p' + 1) = 1$ $A p' + 1 = B p' + 1$

This expression does not hold, since the right side is not a multiple of $p$ when $p' > p$, and $A > B$ holds when $p' = p$. Due to this operation, $R$ may be larger or smaller than the original value, since the rate of change of $R$ does not become 1. Assuming that $R = 1$ in some $r$ by multiplying fractions of prime numbers $b' = A_1/B_1$, $b'' = A_2/B_2$, $\ldots b'\ldots' = A_x/B_x$ from $r = 1$ and $R \neq 1$, $a(p + 1)/(2bp) \times A_1 p_1 + 1)/(B_1 p_1 + 1) \times A_2 p_2 + 1)/(B_2 p_2 + 1) \times A_3 p_3 \times A_{x-1}(px + 1)/(B_x p_x(p_{x-1} + 1)) = 1$

$a/(2b) \times A_1 B_1 \times A_2 B_2 \ldots A_x(p_x + 1)/(B_x p_x) = 1$

$a(p_x + 1)A_1 A_2 \ldots A_x = 2bp_x B_1 B_2 \ldots B_x$

cp(p_x + 1) = 2bp_x B_1 B_2 \ldots B_x

It becomes inconsistent when $p_x > p$, since the right side of this expression does not include $p$ as a factor. Therefore the value of $p$ is invariable and the number of prime factors included in $(p + 1)/2$ is one.

Assuming that $R = 1$ in some $r$ and

$b = \prod_{k=1}^{r} p_k^{q_k}$

$c = \prod_{k=1}^{r} p_k^{c_k}$

$b = c(p + 1)/2$

By fixing $p$ and dividing both sides of the above equation by $\prod_{k=2}^{r} p_k^{q_k}$, $b' = b/\prod_{k=2}^{r} p_k^{q_k} = p_1^{q_1}$

$c' = c/\prod_{k=2}^{r} p_k^{q_k}$

$b' = c'(p + 1)/2$

are established. However, it becomes contradiction since it is inappropriate when $r = 1$.

From the above I, II, III, IV, there are no odd perfect numbers.
4. Acknowledgement

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5. References

Hiroyuki Kojima "The world is made of prime numbers" Kadokawa Shoten, 2017
Fumio Sairaiji Kenichi Shimizu "A story that prime is playing" Kodansha, 2015