Proof that there are no odd perfect numbers

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1. Abstract

For y to be a perfect number, if one of the prime factors is p, the exponent of p is an integer $n(n \geqq 1)$, the prime factors other than p are p_1,p_2,p_3,$\cdots p_r$ and the even exponent of p_k is q_k,

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) /(2p^n) = \prod_{k=1}^{r} {p_k}^{q_k}$$

must be satisfied. Let m and q be non-negative integers,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting b and c be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r} {p_k}^{q_k}$$

$$c = \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, let f be an odd integer and d_r be a positive integer and an index of p_r included in $(p + 1)/2$, if following expression

$$(p + 1)/2 = f{p_r}^{d_r}$$

is satisfied, let c_r be an index of p_r included in c,

$$2m + 1 = w{p_r}^{q_r - c_r - d_r}$$

is proved to be the condition in order for b to be divided by ${p_r}^{q_r}$. By using this conditional expression, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n $(n \geqq 1)$. Let $p_1, p_2, p_3, \cdots p_r$ be the odd prime numbers of factors other than p, q_k the index of p_k, and variable a be the sum of product combinations other than prime p.

$$a = \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) \ \ldots ①$$

The number of terms N of variable a is

$$N = \prod_{k=1}^{r} (q_k + 1) \ \ldots ②$$

When y is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)$$

is established.

$$a \sum_{k=0}^{n} p^k / 2 = y$$

$$a \sum_{k=0}^{n} p^k / (2p^n) = y/p^n \ \ldots ③$$

3.1. If q_k has at least one odd integer

Letting the number of terms where q_k is an odd integer be a positive integer u, because $y/p^n = \prod_{k=1}^{r} {p_k}^{q_k}$ is an odd integer, the denominator on the left side of expression ③ has a prime factor 2, from expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore $\sum_{k=0}^{n} p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all q_k are even integers

y/p^n is an odd integer, the denominator on the left side of expression ③ is an even integer, and since N is and odd integer when q_k are all even integers, variable a is and odd integer. Therefore $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \ (mod\ 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

However, $q_1, q_2, \dots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r}{p_k}^{q_k} \quad \dots④$$

A following expression is established.

$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$

$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$

$(a - 2b)p^{n+1} + 2bp^n - a = 0$ ...⑤

Because it is an $n + 1$ order equation of p, the solution of the odd prime p is $n + 1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, a/p^n is an odd integer, which is c.

$ap - 2bp + 2b = c\ (c > 0)$ ...⑥

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From equation ⑥

$2b(p-1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p-1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p-1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p+1)(p^{n-1} + p^{n-3} + \cdots + 1)$ …⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p-1) < p^n/(p-1)$

$p^{n-1} + \cdots + 1 < p^n/(p-1)$ …⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ …⑨

Let a_k and b_k be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,

from inequality ⑧

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From equation ⑦, b has $(p+1)/2$ as a factor. We consider this problem, if $(p+1)/2$ is prime number, $p_r = (p+1)/2$ holds.

When $p_r = (p+1)/2$

$2b = c(p^n + \cdots + 1) = c(p+1)(p^{n-1} + p^{n-3} + \cdots + 1)$

$2b/(p+1) = c(p^{n-1} + p^{n-3} + \cdots + 1)$

Since the index q_r of p_r, the factor of b, b must be divided by $((p+1)/2)^{q_r}$, and $2b/(p+1)$ must be divided by $((p+1)/2)^{q_r-1}$.

Because $2b/(p+1)$ and $(p+1)/2$ are odd numbers, let s be an odd integer.

$2b/(p+1) = c(p^{n-1} + p^{n-3} + \cdots + 1) = s((p+1)/2)^{q_r-1}$

$c(p^{n-1} + p^{n-3} + \cdots + 1) \times 2^{q_r-1} = s(p+1)^{q_r-1}$

$c \times 2^{q_r-1} \equiv s \pmod p$

As $c \times 2^{q_r-1}$ is an even integer and s is odd integer, let t be an odd integer,

$c \times 2^{q_r-1} - s = tp$

$s = c \times 2^{q_r-1} - tp$

is established.

$c(p^{n-1} + p^{n-3} + \cdots + 1) \times 2^{q_r-1} = (c \times 2^{q_r-1} - tp)(p+1)^{q_r-1}$

$tp(p+1)^{q_r-1} = c \times 2^{q_r-1}((p+1)^{q_r-1} - (p^{n-1} + p^{n-3} + \cdots + 1))$

$tp = c \times 2^{q_r-1}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p+1)^{q_r-1})$

Since t is odd integer and p is odd prime, let u be an odd integer, this expression holds in the following cases.

ⅰ. When $c = up$

$t = u \times 2^{q_r-1}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p+1)^{q_r-1})$

$s = c \times 2^{q_r-1} - tp$

$s = up \times 2^{q_r-1} - tp = (u \times 2^{q_r-1} - t)p$

$b = sp_r^{q_r} = p(u \times 2^{q_r-1} - t)p_r^{q_r}$

is.

$b \equiv 0 \pmod p$

is. Since $b = \prod_{k=1}^{r} {p_k}^{q_k}$, it becomes inappropriate contrary to $b \not\equiv 0 \pmod p$.

ⅱ. When $c \neq up$

・When t is not a multiple of c, let z be a rational number.

$z = 2^{q_r-1}(1-(p^{n-1}+p^{n-3}+\cdots+1)/(p+1)^{q_r-1})/t$

If z is an integer, $p = cz$ when $z \neq 1$, contrary to p is odd prime, and $c = p$ when $z = 1$, contrary to the case classification condition.

If z is not an integer,

$tp = tcz = c \times 2^{q_r-1}(1-(p^{n-1}+p^{n-3}+\cdots+1)/(p+1)^{q_r-1})$

$tp = c(2^{q_r-1}-(p^{n-1}+p^{n-3}+\cdots+1)/{p_r}^{q_r-1})$

Since $b = s{p_r}^{q_r}$,

$$s = \prod_{k=1}^{r-1} {p_k}^{q_k}$$

is established. Therefore s doesn't include the factor p_r.

Let the number of factors p_r included in c be an integer c_r.

$2b/(p+1) = b/p_r = s{p_r}^{q_r-1} = c(p^{n-1}+p^{n-3}+\cdots+1)$

is established. And c_r can take the value of $0 \leqq c_r \leqq q_r - 1$.

If $2^{q_r-1}-(p^{n-1}+p^{n-3}+\cdots+1)/{p_r}^{q_r-1}$ is an integer, let this value be a rational number R.

$tp = cR$

is. Because $c \neq up$, R must be divisible by p.

$t = cR/p$

is established. This contradicts that t is not a multiple of c. Therefore R must not be an integer, when t is not a multiple of c.

If v is an odd number not including the factor p_r,

$c = v{p_r}^{c_r}$

holds.

$tp = v{p_r}^{c_r}(2^{q_r-1}-(p^{n-1}+p^{n-3}+\cdots+1)/{p_r}^{q_r-1})$

$tp = v(2^{q_r-1} \times {p_r}^{c_r}-(p^{n-1}+p^{n-3}+\cdots+1)/{p_r}^{q_r-c_r-1})$

In order for tp to be an integer, the second term on the right side of the above expression must be an integer. Let $2^{q_r-1} \times {p_r}^{c_r}-(p^{n-1}+p^{n-3}+\cdots+1)/{p_r}^{q_r-c_r-1}$ be an integer N.

$R = N/{p_r}^{c_r}$

If N includes the factor p_r, t has factor p_r, as $tp = vN$, and the right side of $s = c \times 2^{q_r-1} - tp$ includes factor p_r, then it becomes inconsistent, since s has factor p_r. Since R is not an integer, $c_r > 0$ holds.

$p^{n-1} + p^{n-3} + \cdots + 1$ must be divided by ${p_r}^{q_r-c_r-1}$.
Let z be an integer, and if
$(p^{n-1} + p^{n-3} + \cdots + 1)/{p_r}^{q_r-c_r-1} = zp_r$
holds,
$N = 2^{q_r-1} \times {p_r}^{c_r} - zp_r$
is established. It becomes contradiction since N is a multiple of p_r. Therefore the index of factor p_r of $p^{n-1} + p^{n-3} + \cdots + 1$ must be $q_r - c_r - 1$.

・When t is a multiple of c
$t = cv$ and $p = 2^{q_r-1}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p+1)^{q_r-1})/v$
$vp = 2^{q_r-1}(1 - (p^{n-1} + p^{n-3} + \cdots + 1)/(p+1)^{q_r-1}$
$vp = 2^{q_r-1} - (p^{n-1} + p^{n-3} + \cdots + 1)/{p_r}^{q_r-1}$
In order for vp is an integer, $p^{n-1} + p^{n-3} + \cdots + 1$ must be divided by ${p_r}^{q_r-1}$.

Let z be an integer, and if
$(p^{n-1} + p^{n-3} + \cdots + 1)/{p_r}^{q_r-1} = zp_r$
holds,
$vp = 2^{q_r-1} - zp_r$
$tp = cvp = c(2^{q_r-1} - zp_r)$
$s = c2^{q_r-1} - tp = c2^{q_r-1} - c(2^{q_r-1} - zp_r) = czp_r$
Since s is a multiple of p_r and it becomes contradiction, the index of factor p_r of $p^{n-1} + p^{n-3} + \cdots + 1$ must be $q_r - 1$.

From the above, when c_r takes the value of $0 \leqq c_r \leqq q_r - 1$, $p^{n-1} + p^{n-3} + \cdots + 1$ must be divisible by $p_r{}^{q_r - c_r - 1}$.

$(p^{n-1} + p^{n-3} + \cdots + 1)/p_r{}^{q_r - c_r - 1}$

$= (p^n + \cdots + 1)/((p+1)p_r{}^{q_r - c_r - 1})$

$= (p^n + \cdots + 1)/(2p_r{}^{q_r - c_r})$

$= (p^{n+1} - 1)/(2(p-1)p_r{}^{q_r - c_r})$

$= ((2p_r - 1)^{n+1} - 1)/(4(p_r - 1)p_r{}^{q_r - c_r})$

When $f(p_r) = ((2p_r - 1)^{n+1} - 1)/(4(p_r - 1)p_r{}^{q_r - c_r})$ …⑩

When $n = 5$

$$(2p_r - 1)^6 = \sum_{k=0}^{6} \binom{6}{k} (-1)^k (2p_r)^{6-k}$$

$(2p_r - 1)^6 = 64p_r{}^6 - 192p_r{}^5 + 240p_r{}^4 - 160p_r{}^3 + 60p_r{}^2 - 12p_r + 1$

$f(p_r) = (64p_r{}^6 - 192p_r{}^5 + 240p_r{}^4 - 160p_r{}^3 + 60p_r{}^2 - 12p_r)/(4(p_r - 1)p_r{}^{q_r - c_r})$

$f(p_r) = (16p_r{}^5 - 48p_r{}^4 + 60p_r{}^3 - 40p_r{}^2 + 15p_r - 3)/(p_r - 1)p_r{}^{q_r - c_r - 1}$

$f(p_r) = (16p_r{}^4 - 32p_r{}^3 + 28p_r{}^2 - 12p_r + 3)/p_r{}^{q_r - c_r - 1}$

We consider the proof that 0 order term of $((2p_r-1)^{n+1}-1)/(4p_r(p_r-1))$ must be $2m+1$, for integer $m(m \geqq 0)$ of $n=4m+1$.

When $m=0$, as 0 order term is 1, this proposition holds.
Assuming that this proposition holds when $m=k$, the first or higher order term of p is defined as $g(k)$.
$((2p_r-1)^{4k+2}-1)/(4p_r(p_r-1)) = g(k)+2k+1$
is assumed to be satisfied.

$(2p_r-1)^{4k+2}-1 = 4p_r(p_r-1)(g(k)+2k+1)$
$(2p_r-1)^{4k+2} = 4p_r(p_r-1)(g(k)+2k+1)+1$

$(2p_r-1)^{4k+2}(2p_r-1)^4 = 4p_r(p_r-1)(2p_r-1)^4(g(k)+2k+1)+(2p_r-1)^4$
And
$$((2p_r-1)^{4(k+1)+2}-1)/(4p_r(p_r-1))$$
$$= (2p_r-1)^4(g(k)+2k+1)+((2p_r-1)^4-1)/(4p_r(p_r-1))$$
$$= (2p_r-1)^4(g(k)+2k+1)+4{p_r}^2-4p_r+2$$

When $g(k+1) = (2p_r-1)^4 g(k)+((2p_r-1)^4-1)(2k+1)+4{p_r}^2-4p_r$.
$((2p_r-1)^{4(k+1)+2}-1)/(4p_r(p_r-1)) = g(k+1)+2k+3 = g(k+1)+2(k+1)+1$
is holds when $m=k+1$. From the above, by mathematical induction, this proposition holds with arbitrary m satisfying $m \geqq 0$.

When $n = 9$

$$f(p_r) = (256{p_r}^8 - 1024{p_r}^7 + 1856{p_r}^6 - 1984{p_r}^5 + 1376{p_r}^4 - 640{p_r}^3 + 200{p_r}^2 - 40p_r + 5)/{p_r}^{q_r - c_r - 1}$$

When $n = 13$

$$f(p_r) = (4096{p_r}^{12} - 24576{p_r}^{11} + 68608{p_r}^{10} - 117760{p_r}^9 + 138496{p_r}^8 - 117760{p_r}^7 + 74432{p_r}^6 - 35392{p_r}^5 + 12656{p_r}^4 - 3360{p_r}^3 + 644{p_r}^2 - 84p_r + 7)/{p_r}^{q_r - c_r - 1}$$

When $n = 17$

$$f(p_r) = (65536{p_r}^{16} - 524288{p_r}^{15} + 1982464{p_r}^{14} - 4702208{p_r}^{13} + 7831552{p_r}^{12} - 9715712{p_r}^{11} + 9293824{p_r}^{10} - 7000064{p_r}^9 + 4201984{p_r}^8 - 2021376{p_r}^7 + 779136{p_r}^6 - 239232{p_r}^5 + 57792{p_r}^4 - 10752{p_r}^3 + 1488{p_r}^2 - 144p_r + 9)/{p_r}^{q_r - c_r - 1}$$

Let $g(p_r)$ be the a polynomial that satisfies $g(p_r) = f(p_r){p_r}^{q_r - c_r - 1}$, and let the coefficient of i-th term of $g(p_r)$ be a rational number $T_i (i \geqq 0)$.

$$g(p_r) = T_{n-1}p_r^{n-1} + \cdots + T_1 p_r + T_0$$

$$(2p_r - 1)^{n+1} - 1 = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i (2p_r)^{n+1-i} - 1$$

$$(2p_r - 1)^{n+1} - 1 = \sum_{i=0}^{n} \binom{n+1}{i} (-1)^i (2p_r)^{n+1-i}$$

$$4p_r(p_r - 1)g(p_r) = 4p_r(p_r - 1)(T_{n-1}p_r^{n-1} + \cdots + T_1 p_r + T_0)$$

$$4p_r(p_r - 1)g(p_r) = 4T_{n-1}p_r^{n+1} + \cdots + (4T_1 - 4T_2)p_r^3 + (4T_0 - 4T_1)p_r^2 - 4T_0 p_r$$

Comparing the coefficients.

$$-T_0 = -\binom{n+1}{1}/2$$

$$T_0 - T_1 = \binom{n+1}{2}$$

$$T_1 - T_2 = -\binom{n+1}{3}2$$

. . .

$$T_{n-2} - T_{n-1} = -\binom{n+1}{n}2^{n-2}$$

$T_0 = 2m + 1$

When $1 \leqq i \leqq n - 1$

$$T_{i-1} - T_i = (-2)^{i-1}\binom{n+1}{i+1}$$

$$T_i = T_{i-1} - (-2)^{i-1}\binom{n+1}{i+1}$$

$$T_1 = T_0 - \binom{n+1}{2}$$

$T_1 = 2m + 1 - (n + 1)n/2 = 2m + 1 - (2m + 1)(4m + 1) = -4m(2m + 1)$

$$T_2 = T_1 + 2\binom{n+1}{3}$$

$T_2 = -4m(2m + 1) + 2(n + 1)n(n - 1)/6 = -4m(2m + 1) + 2(2m + 1)(4m + 1)4m/3$

$T_2 = (2m + 1)(-4m + 2(4m + 1)(4m)/3) = 4m(2m + 1)(8m - 1)/3$

In the same way,

$T_3 = -32m^2(2m - 1)(2m + 1)/3$

$T_4 = 16m(2m - 1)(2m + 1)(16m^2 - 10m - 1)/15$

are established.

Because $T_0 = 2m + 1$ and $\binom{n+1}{i+1}$ of the recurrence expression of T_i includes a factor $n + 1$ in the range of $1 \leqq i \leqq n - 1$, the factor $2m + 1$ is included in all T_i.

When $n = 9$ and $c_r < q_r - 1$

$$f(p_r) = (256{p_r}^8 - 1024{p_r}^7 + 1856{p_r}^6 - 1984{p_r}^5 + 1376{p_r}^4 - 640{p_r}^3 + 200{p_r}^2 - 40p_r + 5)/{p_r}^{q_r - c_r - 1}$$

Letting $h(p_r)$ be a fractional functions obtained by subtracting a term larger than $q_r - c_r - 2$ from the numerator of $f(p_r)$.

・When $q_r - c_r = 2$

$h(p_r) = T_0/p_r$

$h(p_r) = (2m + 1)/p_r$

$2m + 1$ must be divided by p_r.

・When $q_r - c_r = 3$

$h(p_r) = (T_1 p_r + T_0)/p_r^2$

$h(p_r) = (4m(2m + 1)p_r + 2m + 1)/p_r^2$

$h(p_r) = (2m + 1)(4mp_r + 1)/p_r^2$

$h(p_r)p_r^2/(2m + 1) = 4mp_r + 1 \equiv 1 \pmod{p_r}$

is established. And $h(p_r)p_r^2/(2m + 1)$ isn't divisible by p_r. Therefore $2m + 1$ must be divided by p_r^2.

・When $q_r - c_r = 4$

$h(p_r) = (T_2 p_r^2 + T_1 p_r + T_0)/p_r^3$

$h(p_r) = (4m(2m + 1)(8m - 1)p_r^2/3 + 4m(2m + 1)p_r + 2m + 1)/p_r^3$

$h(p_r) = (2m + 1)(4m(8m - 1)p_r^2/3 + 4mp_r + 1)/p_r^3$

$4m(8m - 1)p_r^2/3 + 4mp_r + 1 = 4m((8m - 1)p_r/3 + 1)p_r + 1$

When $4m((8m - 1)p_r/3 + 1)$ is an integer,

$4m(8m - 1)p_r^2/3 + 4mp_r + 1 \equiv 1 \pmod{p_r}$

When $4m((8m - 1)p_r/3 + 1)$ is not an integer, if $4m(8m - 1)p_r^2/3 + 4mp_r + 1$ is a multiple of p_r, assuming that the numerator of $4m((8m - 1)p_r/3 + 1)$ is an integer A and the denominator is an integer B, and letting C be an integer,

$A/B \times p_r + 1 = Cp_r$

$Ap_r + B = BCp_r$

$B \equiv 0 \pmod{p_r}$ ... (A)

In order for expression (A) to be satisfied, p_r must be 3, the denominator of B. Because the expression (A) is not satisfied, if b has another prime factor, b has only 3 as the factor, and it becomes inconsistent, as $r = 1$. From the above, $h(p_r)p_r^3/(2m + 1)$ isn't divided by p_r.

・When $q_r - c_r = 5$

$h(p_r) = (T_3 p_r^3 + T_2 p_r^2 + T_1 p_r + T_0)/p_r^4$

$h(p_r) = (-32m^2(2m-1)(2m+1)p_r^3/3 + 4m(2m+1)(8m-1)p_r^2/3 + 4m(2m+1)p_r + 2m+1)/p_r^4$

$h(p_r) = (2m+1)(-32m^2(2m-1)p_r^3/3 + 4m(8m-1)p_r^2/3 + 4mp_r + 1)/p_r^4$

$h(p_r)p_r^4/(2m+1)$ isn't divided by p_r as in the case of $q_r - c_r = 4$.

・When $q_r - c_r \geqq 6$

When $h(p_r)p_r^5$ is an integer, $h(p_r)p_r^5/(2m+1)$ isn't divisible by p_r. Since the denominator of T_4 is 15, if the denominator is either 3 or 5, the prime factor of b is only 3 or 5 and it becomes inappropriate, as $r = 1$. If the prime factor of the denominator is 3 and 5, expression (A) holds for both values. If $p_r = 3$ and $p_s = 5$, since the numerator must be divided by p_s,

$A \equiv 0 \pmod{p_s}$

is required. A must similarly be a multiple of 15. The numerator and the denominator have the common divisor. Since the form of the formula does not change even if it divides the numerator and the denominator, it becomes contradiction, because it has to be possible indefinitely.

From the above, when $h(p_r)p_r^{q_r-c_r-1}$ is an integer, $h(p_r)p_r^{q_r-c_r-1}/(2m+1)$ isn't divided by p_r. When $h(p_r)p_r^{q_r-c_r-1}$ is not an integer, If

$h(p_r)p_r^{q_r-c_r-1} = A/B \times p_r + 1$

In the case of B has one prime factor, it becomes contradiction, since b has only one factor and $r = 1$. If B includes plural prime factors, because both numerator and denominator must include the common divisor, the product of two arbitrary prime factors in the prime factors, it becomes inconsistent. Therefore, because for arbitrary q_r and c_r where $c_r < q_r - 1$, $h(p_r)p_r^{q_r-c_r-1}/(2m+1)$ isn't divisible by p_r, $2m+1$ must be a multiple of $p_r^{q_r-c_r-1}$.

When $c_r < q_r - 1$ in order for b to be divided by ${p_r}^{q_r}$, $2m+1$ must be divisible by ${p_r}^{q_r-c_r-1}$, let w be an odd integer,

$2m + 1 = w{p_r}^{q_r-c_r-1}$ …⑪

is required. However, w does not include p_r as a factor.

From equation ⑦,

$2b(p-1) = c(p^{n+1}-1)$

Letting d_k be an index of p_k included in $(p+1)/2$, in order for b to be divided by ${p_k}^{q_k}$, let w be an odd integer, just as in deriving expression ⑪, for k where $d_k > 0$,

$2m+1 = w\prod_{d_k>0} {p_k}^{q_k-c_k-d_k}$ ... (B)

must be satisfied.

Since $d_k > 0$, according to Fermat's small theorem, let t_k be a positive rational number,

$t_k(n+1) = p_k - 1$

Assuming that the numerator of t_k is an integer N_k and the denominator is an integer D_k,

$(n+1)N_k/D_k = p_k - 1$ ... (C)

N_k is a divisor of $p_k - 1$, D_k is a divisor of $n+1$.

Substituting expression (B) into expression (C),

$2w\prod_{d_k>0} {p_k}^{q_k-c_k-d_k} \times N_k/D_k = p_k - 1$

Since $w \not\equiv 0 \pmod{p_k}$, let $D_k{}'$ be an integer,

$D_k = D_k{}'\prod_{d_k>0} {p_k}^{q_k-c_k-d_k}$

$2wN_k/D_k{}' = p_k - 1$

is established.

$2w/D_k' = (p_k-1)/N_k$

$D_k{}'$ is a divisor of $2w$.

$(p_k-1)D_k' = 2wN_k$

Ⅰ. If two of the odd prime numbers included in $(p+1)/2$ are taken as p_r, p_s, when $q_r - c_r - d_r > 0$, $q_s - c_s - d_s > 0$, and $q_k - c_k - d_k = 0$ for all other factor p_k
If $F(p) = p^{n-1} + p^{n-3} + \cdots + 1$. The factor of $F(p)$ are p_r and p_s.

$${p_r}^{q_r - c_r - d_r} = p^{n-1} + p^{n-3} + \cdots + 1$$

$$p = 2\prod_{k=1}^{r} {p_k}^{d_k} - 1$$

$${p_r}^{q_r - c_r - d_r}{p_s}^{q_s - c_s - d_s} = (2\prod_{k=1}^{r} {p_k}^{d_k} - 1)^{n-1} + (2\prod_{k=1}^{r} {p_k}^{d_k} - 1)^{n-3} + \cdots + 1 \ldots \text{(D)}$$

From expression (C),

$$2m + 1 = w'{p_s}^{q_s - c_s - d_s}$$

is established.

$$2m + 1 \geqq {p_r}^{q_r - c_r - d_r}{p_s}^{q_s - c_s - d_s} \ \ldots \text{(E)}$$

$(2\prod_{k=1}^{r} {p_k}^{d_k} - 1)^{n-1} + (2\prod_{k=1}^{r} {p_k}^{d_k} - 1)^{n-3} + \cdots + 1$ monotonically increases in the range of $2\prod_{k=1}^{r} {p_k}^{d_k} - 1 \geqq 0$.

$$2\prod_{k=1}^{r} {p_k}^{d_k} - 1 = 1$$

When $\prod_{k=1}^{r} {p_k}^{d_k} = 1$,

$$(2\prod_{k=1}^{r} {p_k}^{d_k} - 1)^{n-1} + (2\prod_{k=1}^{r} {p_k}^{d_k} - 1)^{n-3} + \cdots + 1 = 2m + 1$$

is established.

Since the above expression is satisfied, when $p_r \geqq 3$.

$${p_r}^{q_r - c_r - d_r}{p_s}^{q_s - c_s - d_s} > 2m + 1$$

must be satisfied. However, since this inequality contradicts expression (E), this inequality is not satisfied and it becomes inappropriate. Since this expression holds when all exponents on the left side of the expression (D) are not 0, this expression is satisfied when $q_s - c_s - d_s = 0$.
This also holds when three or more prime numbers are included in $(p+1)/2$. Therefore, it becomes inappropriate if all prime factors contained in $F(p)$ are those contained in $(p+1)/2$.

Ⅱ. If two of the odd prime numbers included in $(p+1)/2$ are taken as p_r, p_s, when $q_r - c_r - d_r > 0$, $q_s - c_s - d_s > 0$, and at least one factor p_t among factors not included in $(p+1)/2$ satisfies $q_t - c_t - d_t > 0$

Assuming that p_t is a factor of $2m+1$ at this time, letting its index be a positive integer $q_t - c_t$, from the expression (E),

$2m + 1 \geqq {p_r}^{q_r - c_r - d_r} {p_s}^{q_s - c_s - d_s} {p_t}^{q_t - c_t}$

is established. However, in this case it becomes inappropriate like proof Ⅰ. Therefore, if the prime factor not included in $(p+1)/2$ is included in $F(p)$, $2m+1$ does not become a multiple of ${p_t}^{q_t - c_t}$.

Ⅲ. If one of the prime numbers included in $(p+1)/2$ is set as p_r, when $q_r - c_r - d_r > 0$, and a prime factor other than p_1 to p_r is set as p_t

From $a = cp^n$ and expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1$ ...(F)

If p_t is a prime number not included in $(p+1)/2$ and p_t does not exist, this expression does not hold from the proof of Ⅰ.

Let R be a rational number,

$R = a(p^n + \cdots + 1)/(2bp^n)$

Let b' be a rational number and let A and B to be an integer,

$b' = ({p_t}^{q_t+1} - 1)/({p_t}^{q_t}(p_t - 1)) > 1$

$A = ({p_t}^{q_t+1} - 1)/(p_t - 1)$

$B = {p_t}^{q_t}$

Multiplying R by b', there are both cases that p_r increases p or does not change. When multiplied by b', the rate of change of R is $Ap^n({p'}^n + \cdots + 1)/(B{p'}^n(p^n + \cdots + 1))$, if p after variation is p'. If the rate of change of R is 1,

$Ap^n({p'}^n + \cdots + 1)/(B{p'}^n(p^n + \cdots + 1)) = 1$

$Ap^n({p'}^n + \cdots + 1) = B{p'}^n(p^n + \cdots + 1)$

This expression does not hold, since the right side is not a multiple of p when $p' > p$, and $A > B$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value, since the rate of change of R does not become 1.

From $R \neq 1$ and $a = cp^n$ for some r, also multiplying fractions of prime numbers, if $b' = A_1/B_1$, $b'' = A_2/B_2$, $\cdots b''^{\cdots\prime} = A_x/B_x$,

$$a(p^n + \cdots + 1)/(2bp^n) \times A_1p^n(p_1{}^n + \cdots + 1)/(B_1p_1{}^n(p^n + \cdots + 1)) \times A_2p_1{}^n(p_2{}^n + \cdots + 1)/(B_2p_2{}^n(p_1{}^n + \cdots + 1)) \ldots A_xp_{x-1}{}^n(p_x{}^n + \cdots + 1)/(B_xp_x{}^n(p_{x-1}{}^n + \cdots + 1)) = 1$$

$$a/(2b) \times A_1/B_1 \times A_2/B_2 \ldots A_x(p_x{}^n + \cdots + 1)/(B_xp_x{}^n) = 1$$

$$a(p_x{}^n + \cdots + 1)A_1A_2 \ldots A_x = 2bp_x{}^nB_1B_2 \ldots B_x$$

$$cp^n(p_x{}^n + \cdots + 1)A_1A_2 \ldots A_x = 2bp_x{}^nB_1B_2 \ldots B_x$$

It becomes inconsistent when $p_x > p$, since the right side of this expression does not include p as a factor.

When $p_x = p$,

$$cp^n(p^n + \cdots + 1)A_1A_2 \ldots A_x = c(p^n + \cdots + 1)p^n$$

$$A_1A_2 \ldots A_x = 1$$

It becomes contradiction, since this expression is not established. Therefore, $a = cp^n$ holds at one point where $R = 1$.

Assuming that $F(p)$ contains prime factor p_t,

$$p^{n-1} + p^{n-3} + \cdots + 1 = (2m + 1)/w \times p_t{}^{q_t - c_t}$$

holds.

$$p^{n+1} - 1 = (p^2 - 1)(2m + 1)/w \times p_t{}^{q_t - c_t}$$

$$2m + 1 = w(p^{n+1} - 1)/((p^2 - 1)p_t{}^{q_t - c_t}) \ \ldots (F)$$

Let Nt and Dt be integers. From Euler's theorem,

$$(n + 1)N_t/D_t = (p_t - 1)p_t{}^{q_t - c_t - 1}$$

$$2(2m + 1)/D_t = (p_t - 1)p_t{}^{q_t - c_t - 1}/N_t$$

N_t is a divisor of $(p_t - 1)p_t{}^{q_t - c_t - 1}$, and D_t is a divisor of $n + 1$.

Substituting expression (F),

$$2w(p^{n+1} - 1)/((p^2 - 1)p_t{}^{q_t - c_t}D_t) = (p_t - 1)p_t{}^{q_t - c_t - 1}/N_t$$

$$2w(p^{n+1} - 1)/((p^2 - 1)p_tD_t) = (p_t - 1)/N_t$$

$$2w(p^{n+1} - 1) = (p_t - 1)/N_t \times (p^2 - 1)p_tD_t$$

$$2w(p^{n-1} + p^{n-3} + \cdots + 1) = (p_t - 1)/N_t \times p_tD_t$$

Assuming that w_t is the index of p_t contained in w and δ_t is the index of p_t contained in D_t, The term containing p_t on the left side is

$p_t{}^{q_t - c_t} \times p_t{}^{w_t}$

The term that includes p_t on the right side is

$p_t{}^{\delta_t + 1}$

$q_t - c_t + w_t = \delta_t + 1$

$\delta_t = q_t - c_t + w_t - 1$

Since the exponents of p_t included in $2m + 1$ and w are the same, it becomes contradiction in the case of $\delta_t > w_t$. Therefore,

$q_t - c_t - 1 \leqq 0$

$q_t - c_t \leqq 1$

must be satisfied.

For simplifying prime numbers from 1 to r, express them with two primes p_r and p_s,

$2m + 1 = wp_r{}^{q_r - c_r - d_r} p_s{}^{q_s - c_s - d_s}$

When $q_t - c_t = 1$,

$F(p) = (2m + 1)p_t/w$

is established.

$(2m + 1)p_t/w = p^{n-1} + p^{n-3} + \cdots + 1$

$(2m + 1)p_t/w \times c(p + 1) = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) = c(p^n + \cdots + 1)$

$b = c(p + 1)/2 \times (2m + 1)p_t/w$

Let b 'and c' be integers. If

$b' = b/p_t{}^{q_t}$

$c' = c/p_t{}^{c_t} = c/p_t{}^{q_t - 1}$

is satisfied. By dividing both sides of the above equation by $p_t{}^{q_t}$,

$b' = c'(p + 1)/2 \times (2m + 1)/w$

must be established. However, the value of p changes as $F(p) = (2m + 1)/w$. It becomes contradiction, since this expression does not hold. Therefore, it is inappropriate when $n \geqq 5$.

Ⅳ. When $q_k - c_k - d_k = 0$ for all k

Since $F(p) = 1$,

$1 = p^{n-1} + p^{n-3} + \cdots + 1$

・When $n \geqq 5$

$p^{n-1} + p^{n-3} + \cdots + p^2 = 0$

$p^2(p^{n-3} + p^{n-5} + \cdots + 1) = 0$

It becomes inappropriate, since there are no real solution in the range of $p > 0$.

・When $n = 1$

From $a = cp$ and expression ⑦,

$2bp = a(p + 1)$

$a(p + 1)/(2bp) = 1$ ... (G)

When $r = 1$, if

$a = (p_1{}^{q_1+1} - 1)/(p_1 - 1)$

$b = p_1{}^{q_1}$

holds, since there are no odd perfect number when $r = 1$, expression (G) does not hold.

Let R be a rational number,

$R = a(p + 1)/(2bp)$

If p_r is an odd prime number other than p_1 and let b' to be a rational number, and A and B to be integers,

$b' = (p_r{}^{q_r+1} - 1)/(p_r{}^{q_r}(p_r - 1)) > 1$

$A = (p_r{}^{q_r+1} - 1)/(p_r - 1)$

$B = p_r{}^{q_r}$

Multiplying R by b', there are both cases that p_r increases p or does not change. When multiplied by b', the rate of change of R is $Ap(p'+1)/(Bp'(p+1))$, if p after variation is p'. If the rate of change of R is 1,

$$Ap(p'+1)/(Bp'(p+1)) = 1$$

$$Ap(p'+1) = Bp'(p+1)$$

This expression does not hold, since the right side is not a multiple of p when $p' > p$, and $A > B$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value, since the rate of change of R does not become 1. From $R \neq 1$ and $a = cp$ for some r, also multiplying fractions of prime numbers, if $b' = A_1/B_1$, $b'' = A_2/B_2$, $\cdots b''^{\cdots\prime} = A_x/B_x$,

$$a(p+1)/(2bp) \times A_1p(p_1+1)/(B_1p_1(p+1)) \times A_2p_1(p_2+1)/(B_2p_2(p_1+1)) \dots A_xp_{x-1}(p_x + 1)/(B_xp_x(p_{x-1}+1)) = 1$$

$$a/(2b) \times A_1/B_1 \times A_2/B_2 \dots A_x(p_x+1)/(B_xp_x) = 1$$

$$a(p_x+1)A_1A_2 \dots A_x = 2bp_xB_1B_2 \dots B_x$$

$$cp(p_x+1)A_1A_2 \dots A_x = 2bp_xB_1B_2 \dots B_x$$

It becomes inconsistent when $p_x > p$, since the right side of this expression does not include p as a factor.

When $p_x = p$,

$$cp(p+1)A_1A_2 \dots A_x = c(p+1)p$$

$$A_1A_2 \dots A_x = 1$$

It becomes contradiction, since this expression is not established. Therefore, $a = cp$ holds at one point where $R = 1$.

Assuming that $R=1$ holds in some r, if the above operation is performed,

$$1 \times A_1 p(p_1+1)/(B_1 p_1(p+1)) \times A_2 p_1(p_2+1)/(B_2 p_2(p_1+1)) \dots A_y p_{y-1}(p_y + 1)/(B_y p_y(p_{y-1}+1)) = 1$$

$$p/(p+1) \times A_1/B_1 \times A_2/B_2 \dots A_y(p_y+1)/(B_y p_y) = 1$$

$$p(p_y+1)A_1 A_2 \dots A_y = p_y(p+1)B_1 B_2 \dots B_y$$

When $p_y > p$, it becomes contradiction since right side does not include p as a factor.

When $p_y = p$,

$$A_1 A_2 \dots A_y = B_1 B_2 \dots B_y$$

is established. It is contradiction. By this operation, $R=1$ holds at one point. However when $a=1$ and $b=1$, if $R=1$ is established $p=1$ holds. Therefore, it is inappropriate when $n=1$, since $R=1$ is not satisfied at points other than $p=1$.

From the above Ⅰ, Ⅱ, Ⅲ, Ⅳ, there are no odd perfect numbers.

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5. References

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