Finite faithful G-sets are asymptotically free

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Let G be a finite nontrivial group, let X be a finite faithful G-set, let P^iX be the i-th power set of X, let $n(i)$ be the number of points of P^iX, let $m(i)$ be the number of points of P^iX with non-trivial stabilizer, let k be the number of prime order subgroups of G, and set $E(j) := 2^j$ for any integer j. We prove $\frac{n(i)}{m(i)} \geq \frac{1}{k}E(\frac{n(i-1)}{4})$ for $i \geq 2$.

Let G be a finite nontrivial group, X a finite faithful G-set, write PX for the power set of X, define the sets P^iX for $i \in \mathbb{N}$ by $P^0X = X$ and $P^{i+1}X = PP^iX$, let $n(i)$ be the number of points in P^iX, let $M(i)$ be the set of all $\xi \in P^iX$ such that there is a $g \in G$ satisfying $g \neq 1$ and $g\xi = \xi$, let $m(i)$ be the number of points of $M(i)$, and set $E(j) := 2^j$ for any integer j.

We expect the sequence

$$\frac{n(1)}{m(1)}, \frac{n(2)}{m(2)}, \dots$$

to grow explosively. Let us illustrate this with the toy example where G and X have cardinality two. We get

$$\frac{n(1)}{m(1)} = \frac{4}{2} = 2, \quad \frac{n(2)}{m(2)} = \frac{16}{8} = 2, \quad \frac{n(3)}{m(3)} = \frac{E(16)}{E(12)} = 16\,,$$

$$\frac{n(4)}{m(4)} = \frac{E(E(16))}{E(E(15)+E(11))} = E(15\,E(11)),$$

and $\frac{n(4)}{m(4)}$ turns out to have 9248 digits.

The purpose of this text is to try to shed some light on this phenomenon.

Theorem. *Let G be a finite nontrivial group and X a finite faithful G-set, and i an integer ≥ 2. Then, in the above setting, we have*

$$\frac{n(i)}{m(i)} \geq \frac{1}{k}E\left(\frac{n(i-1)}{4}\right) = \frac{1}{k}\,E\Big(E\big(n(i-2)-2\big)\Big),$$

where k is the number of prime order subgroups of G.

Proposition 1. *Let G be a group acting on a set Y with finitely many orbits. Then G has precisely 2^r fixed points in PY, where r is the number of G-orbits in Y.*

Proof. The fixed points in PY are the invariant subsets of Y, which are unions of G-orbits. □

Recall that $n(i)$ is the cardinality of $P^i X$, so that we have $n(i+1) = E(n(i))$ for all i.

We start by assuming that the order of G is a prime number p; in particular $m(i)$ is the number of fixed points of G in $P^i X$.

Proposition 2. *We have, in the above setting,*

$$\frac{n(i)}{m(i)} \geq E\left(\frac{n(i-1)}{4}\right)$$

for all $i \geq 2$.

Proof. Let $r(i)$ be the number of G-orbits in $P^i X$, and $f(i)$ the number of *free* G-orbits (that is, of orbits with p points) in $P^i X$.

Let i be a nonnegative integer. We have

$$\begin{cases} n(i) &= & pf(i) &+ & m(i) \\ r(i) &= & f(i) &+ & m(i). \end{cases}$$

Let us express $f(i)$ and $r(i)$ in term of $n(i)$ and $m(i)$:

$$f(i) = \frac{n(i) - m(i)}{p},$$

$$r(i) = \frac{n(i) + (p-1)\, m(i)}{p}.$$

In view of the equalities

$$n(i+1) = E(n(i)), \quad m(i+1) = E(r(i))$$

(Proposition 1), we get

$$m(i+1) = E\left(\frac{n(i) + (p-1)\, m(i)}{p}\right),$$

$$\frac{n(i+1)}{m(i+1)} = E\left(\frac{p-1}{p}\, \big(n(i) - m(i)\big)\right) = E\big((p-1)f(i)\big) \geq 2,$$

the inequality $f(i) \geq 1$ following from our faithfulness assumption. This is easily seen to imply

$$n(i) - m(i) \geq \frac{n(i)}{2} \;\forall\; i \geq 1,$$

so that we get for $i \geq 2$

$$\frac{n(i)}{m(i)} = E\left(\frac{p-1}{p}\,\big(n(i-1) - m(i-1)\big)\right)$$

$$\geq E\left(\frac{p-1}{2p}\;n(i-1)\right) \geq E\left(\frac{n(i-1)}{4}\right).$$

□

Proof of the Theorem. We know that the Theorem is true of G has prime order. Let us show that the Theorem hold for any finite group G.

Let $\mathcal{H}$ be the set of prime order subgroups of G. For each subgroup H of G let $M(i,H)$ be the set of all $\xi \in P^iX$ such that there is a $h \in H$ satisfying $h \neq 1$ and $h\xi = \xi$, and write $m(i,H)$ for the cardinality of $M(i,H)$. We get $M(i,G) = M(i)$,

$$M(i) = \bigcup_{H \in \mathcal{H}} M(i,H), \quad m(i) \leq \sum_{H \in \mathcal{H}} m(i,H),$$

and, for $i \geq 2$,

$$\frac{m(i)}{n(i)} \leq \sum_{H \in \mathcal{H}} E\left(\frac{n(i-1)}{4}\right)^{-1} = kE\left(\frac{n(i-1)}{4}\right)^{-1},$$

where k is the cardinality of $\mathcal{H}$. □

Tex file available at

https://tinyurl.com/y3ls6ql8 and https://tinyurl.com/y2cewbgq

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