The Geometric Phase as Analog of Fractional Exponential Function

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Abstract

The notion of geometric phase arises in connection with parallel transport in differential geometry and the formulation of gauge transformation in field theory. Here we show that the geometric phase is locally equivalent to the action of fractional exponential, which is applicable to manifolds having minimal fractal topology or for modeling complex phenomena using fractional calculus.

Key words: Geometric phase, gauge transformation, metric connection, fractional exponential, fractional calculus, minimal fractal manifold.

1. Introduction

It was recently shown that, under general boundary conditions, non-equilibrium Renormalization Group flows are prone to converge to strange attractors [8, 9]. Since multifractals are the natural language for the characterization of strange attractors, the dynamic content of effective field theories may be best understood using the tools of multifractal analysis. It has been known for some time that fractional calculus is an attractive framework for the description of dynamics on fractal or multifractal sets. The goal of this work is to suggest that several foundational concepts of Quantum Field Theory (QFT) and the Standard Model – gauge symmetry, Lie groups, spin-statistics theorem – arise from the asymptotic behavior of the fractional exponential, a concept derived in connection with fractional calculus [5]. Moreover, given the geometric similarity between
the gauge field strength and the curvature tensor, the same formalism may be applied to
the low-energy limit of General Relativity (GR).

2. Berry phase in quantum physics

A quantum system adiabatically transported around a closed path \( C \) in the space of
external parameters acquires a non-vanishing phase (\textit{Berry phase}, BP in short). Since BP
is exclusively path-dependent, it provides key insights into the geometric structure of
quantum mechanics and QFT. The BP concept is closely tied to \textit{holonomy}, that is, the
extent to which some of variables change as other variables or parameters defining a
system return to their initial values \([1, 2]\).

Consider a quantum system described by the time-independent Hamiltonian \( H(t) \), whose
associated eigenstate is \( |\psi(t)\rangle \) and which is embedded in a slowly changing environment.
After a periodic evolution of the environmental parameters \( (t \rightarrow t + T) \), the eigenstate
returns to itself, apart from a phase angle,

\[
|\psi(t)\rangle = e^{i\alpha}|\psi(0)\rangle
\]  

If \( \omega \) denotes the eigenvalue of \( |\psi(t)\rangle \), a generalization of the phase angle \( \alpha = \omega T \) in units
of \( \hbar = 1 \) is given by the \textit{dynamical phase}”

\[
\gamma_d = \int_0^T \omega(t) \, dt = \int_0^T \langle \psi(t) | H(t) | \psi(t) \rangle \, dt
\]  

Berry has shown that there is a time-independent (but contour dependent) supplemental
\textit{geometric phase}” entering the phase angle, namely,
\[ \alpha = \gamma_d + \gamma(C) \]  

where

\[ \gamma(C) = \int_C \langle \psi | i \nabla \psi \rangle \, dr \]  

The dynamical phase $\gamma_d$ encodes information about the duration associated with the cyclic evolution of the complex vector $|\psi(t)\rangle$. By contrast, (4) encodes information about the geometry of the environment where the transport takes place.

### 3. The geometry of gauge and gravitational fields

The gauge field concept may be built from a straightforward geometric interpretation [3, 4]. Consider the parallel transport of a complex vector $|\psi\rangle$ round a closed rectangular loop. The difference between the value of $|\psi\rangle$ at the starting point ($|\psi\rangle_0$) and at the end point $|\psi\rangle_0 \rightarrow |\psi\rangle_f$ is given by

\[ \Delta \psi = \psi_f - \psi_0 = -ig \Delta S^{\mu \nu} F_{\mu \nu} \psi \]  

in which $\Delta S^{\mu \nu}$ denotes the area enclosed by the rectangle and the strength of the gauge field is

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \]  

Echoing the formation of the Berry phase, the effect of parallel transport is to induce a non-vanishing rotation of $|\psi\rangle$ in internal space proportional to the strength of the gauge field.
Likewise, the curvature tensor of GR may be motivated through similar arguments. Taking a vector $V^\kappa$ on a round trip by parallel transport, the difference between the initial and final components of the vector amounts to

$$\Delta V^\kappa = \frac{1}{2} R^\kappa_{\lambda\mu\nu} V^\lambda \Delta S^{\mu\nu}$$

(7)

This equation faithfully replicates (5) and signals the presence of a gravitational field, via the curvature tensor $R^\kappa_{\lambda\mu\nu}$. The geometric analogy between gauge theory and General Relativity is captured in the table below.

<table>
<thead>
<tr>
<th><strong>Gauge Theory</strong></th>
<th><strong>General Relativity</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauge transformation</td>
<td>Coordinate transformation</td>
</tr>
<tr>
<td>Gauge group</td>
<td>Group of coordinate transformations</td>
</tr>
<tr>
<td>Gauge potential $A_\mu$</td>
<td>Connection coefficient $\Gamma^\kappa_{\mu\nu}$</td>
</tr>
<tr>
<td>Field strength $F_{\mu\nu}$</td>
<td>Curvature tensor $R^\kappa_{\lambda\mu\nu}$</td>
</tr>
</tbody>
</table>

Comparison between the geometry of gauge and gravitational fields.

### 4. Fractional exponential function

For a function $u \in \Phi(\mathbb{R})$, the fractional Fourier transform of order $0 < \beta \leq 1$ is defined as [5]

$$\hat{u}_\beta(\omega) = \int_{-\infty}^{\infty} u(t) e^{i\beta(\omega, t)} dt, \quad \omega \in \mathbb{R}$$

(8)

where the kernel functions are
\[ e_\beta(\omega,t) = \begin{cases} \exp(-i |\omega|^{\beta/\nu} t), \omega \leq 0 \\ \exp(i |\omega|^{\beta/\nu} t), \omega \geq 0 \end{cases} \] (9)

When \( \beta = 1 - \varepsilon \), \( \varepsilon \ll 1 \) the frequency entering (6) takes the form

\[ |\omega|^{\beta/\nu} = |\omega|^{(1-\varepsilon)/\nu} \approx |\omega|^{(1+\varepsilon)} \] (10)

leading to a non-vanishing correction to the conventional phase angle given by

\[ \alpha_\varepsilon(\varepsilon) = |\omega|^\varepsilon t = e^{\varepsilon \ln |\omega|} t \approx (1 + \varepsilon \ln |\omega|) t \] (11)

The adiabatic condition \( \omega \ll 1 \) yields an undefined phase (11), which signals unbounded phase fluctuations on all scales and the onset of critical behavior.

...text to follow...

5. Further implications of fractional exponential in field theory

...text to follow...

Significant consequences for Beyond the Standard Model (BSM) physics and ultraviolet completion programs.

- \textit{Lie groups and Lie algebra} [6]

Linear representation of a group of elements \( a(\theta) \)

\[ a \mapsto D_\theta(a) \] (12)
\[ D_{R}(a(\theta)) = \exp(i \theta \alpha T_{R}^{\alpha}) \] (13)

- **Spin-statistics theorem** [7]

As the wavefunction of a system of \( n \) identical particles stays invariant in modulus, the interchange of quantum numbers between the \( i \)-th and the \( j \)-th particle picks up a non-vanishing phase according to

\[ \psi(q_i, q_j) = \exp(2\pi i \sigma) \psi(q_j, q_i) \] (14)

The parameter \( \sigma \) is defined only modulo integers and fixes the statistics of particles indexed by \( i, j \).

...text to follow...

**6. Conclusions and outlook.**

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**References**

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