Interval Sieve Algorithm

Creating a Countable Set of Real Numbers from a Closed Interval

I. Abstract

The Interval Sieve Algorithm is a method for generating a list of real numbers on any closed interval of real numbers $[r_i, r_j]$ where $r_i < r_j$. This may seem strange in light of Georg Cantor's 1891 paper wherein he demonstrated a constructive proof that the real numbers are uncountable. Cantor developed a method for showing how a particular objective cannot be accomplished; in this case, establishing a one to one correspondence between the natural numbers and the real numbers, using his diagonal argument.

What Cantor didn't show is that there are no ways of demonstrating a one to one correspondence between the natural numbers and the real numbers. This is important because if one can demonstrate a million ways that something cannot be done, it is only necessary to develop one way that shows how it can be done to trump the million ways that show it cannot.

The interval sieve algorithm partitions a closed interval of real numbers $[r_i, r_j]$ where $r_i < r_j$ to create a complete list, L, of numbers in the interval. We will prove that the list L is complete, and lastly derive the bijective function $f: \mathbb{N} \leftrightarrow [r_1, r_2]$.

II. Definitions

1. The **lower bound** of an interval is the leftmost member of the interval. In the interval $[r_1, r_2]$, r_1 is the lower bound of the interval.

2. The **upper bound** of an interval is the rightmost member of the interval. In the interval $[r_1, r_2]$, r_2 is the upper bound of the interval.

3. Given the set, S = {1, 2, 3}. We define a **closed interval** of the set as $I_S = [1, 3]$ where both upper and lower bounds are included in the interval.

4. Given the set, S = {1, 2, 3}. We define an **open interval** of the set as $I_S = (1, 3)$ where the upper and lower bounds are excluded from the interval.

5. Given the set, S = {1, 2, 3}. We define a **lower open interval** of the set as $I_S = (1, 3]$ where the lower bound is excluded from the interval and the upper bound is included in the interval.

6. Given the set, S = {1, 2, 3}. We define an **upper open interval** of the set as $I_S = [1, 2)$ where the lower bound is included in the interval and the upper bound is excluded from the interval.

7. A **conjoined interval pair** is a pair of intervals where the upper bound of one and the lower bound of the other are the same member. $[r_i, [r_k,] r_j]$ is an example of a conjoined interval pair where r_k is both the upper bound of $[r_i, r_k]$, the lower bound of $[r_k, r_j]$ and $r_i < r_k < r_j$.

8. A **relative bound** is the member that is common to both intervals in a conjoined interval pair. In the conjoined interval pair $[r_1, [r_3,] r_2]$, r_3 is the relative bound in both intervals $[r_1, r_3]$ and $[r_3, r_2]$.

9. An interval of a set may be **partitioned** by creating a conjoined interval pair per definition 7 and then splitting the conjoined interval pair into sub-intervals with the relative bound being the upper bound of one sub-interval and the lower bound of the other sub-interval.

Example:

$S = \{1, 2, 3\}$

$I_S = [1, 3]$ ($I_S \longrightarrow$ the interval I on set S)

Partition I_S as follows -
$I_S = [1, 3]$

$= [1, [2], 3]$

$= [1, 2], [2, 3]$

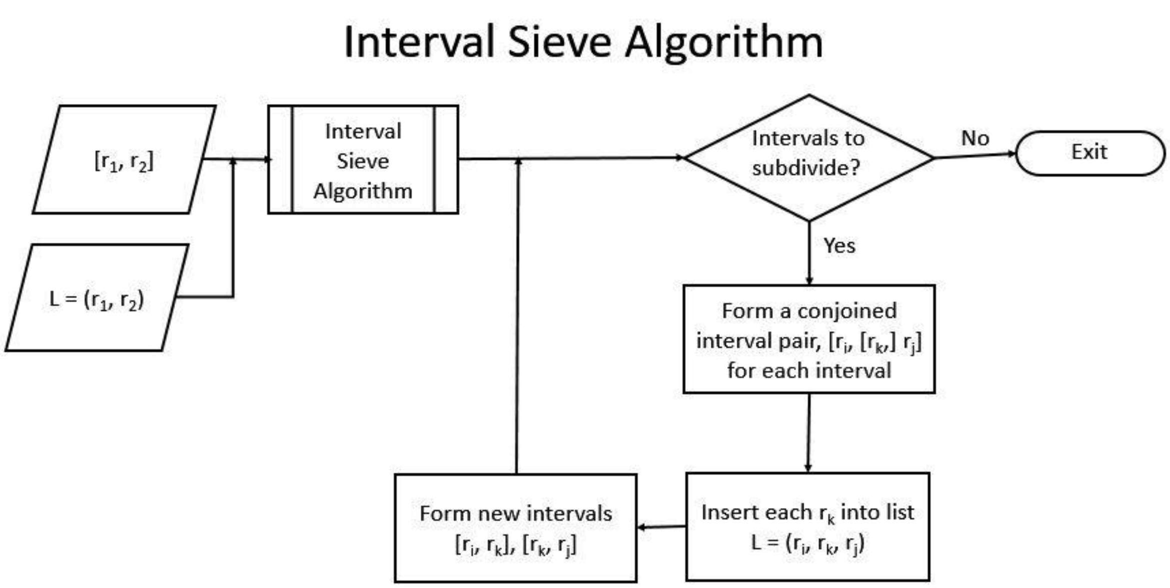
10. When no sub-intervals can be further subdivided then the interval is called **fully partitioned**.

11. The **immediate predecessor** of a number λ is a number β such that there exists no number δ where $\beta < \delta < \lambda$.

12. The **immediate successor** of a number λ is a number β such that there exists no number δ where $\lambda < \delta < \beta$.

13. For any 2 real numbers λ and β in $[r_1, r_2]$, we can always find another real number, δ, such that if $\lambda > \beta$ then $\beta < \delta < \lambda$ and if $\lambda < \beta$ then $\lambda < \delta < \beta$. Therefore from definitions 11 and 12 we know that there are no immediate predecessors or successors of any of the elements of $[r_1, r_2]$; that is, $[r_1, r_2]$ is a **continuum**.

III. The Interval Sieve Algorithm



Procedure:

0. We begin the procedure given the interval $[r_1, r_2]$ where $r_1 < r_2$ and r_1, r_2 are real numbers and the list $L = (r_1, r_2)$.

1. If there are intervals that can be sub-divided next step else stop..

2. Sub-divide each interval $[r_i, r_j]$ by selecting a number r_k such that $r_i < r_k < r_j$ to get a conjoined interval pair $[r_i, [r_k,] r_j]$

3. Insert the relative bound number, r_k, into the list L to get $L = (r_i, r_k, r_j)$.

4. Form new sub-intervals $[r_i, r_k], [r_k, r_j]$.

4. Return to step 1.

IV. Partitioning Intervals Using the Algorithm

We will use the partitioning of a set interval according to the algorithm as a method for creating a list of the elements of the interval. Partitioning allows us to construct a list whereby all the elements of the interval will be included in the list.

Below are two examples of partitioning set intervals.

Example 1 -

Let:

$S = \{1, 2, 3, 4, 5\}$

$I_S = [1, 5]$

First we note that in defining the interval [1, 5] we have specified the first two elements of our list L. That is L = (1, 5). To continue populating L we will partition [1, 5], adding each relative bound to L as it is created, and continue the process until no intervals are left that can be partitioned.

$I_S = [1, 5]$	L = (1, 5)
= [1, [3], 5]	L = (1, 3, 5)
= [1, 3], [3, 5]	
= [1, [2], 3], [3, [4], 5]	L = (1, 2, 3, 4, 5)
= [1, 2], [2, 3], [3, 4], [4, 5]	

Since no interval in I_S can be further subdivided the interval is fully partitioned and L is complete.

Example 2 -

Let:

$S = \{1, 2, 3, \ldots \omega\}$ where ω is the first infinite ordinal number.

$I_S = [1, \omega)$

Partition the interval $[1, \omega)$ to create a list L of natural numbers.

$I_S = [1, \omega)$	$L = (1)$
$= [1, [2], \omega)$	$L = (1, 2)$
$= [1, 2], [2, \omega)$	
$= [1, 2], [2, [3], \omega)$	$L = (1, 2, 3)$
$= [1, 2], [2, 3], [3, \omega)$	
…	…

Taking the procedure to its limit will create a list of all the natural numbers, $L = (1, 2, 3, \ldots)$. Since I_S is defined as an upper open interval, ω is not included in L.

V. Creating L over $[r_1, r_2]$ where r_1, r_2 are Real Numbers

Let:

$S = \{\mathbb{R}\}$

$I_S = [r_1, r_2], r_1 < r_2$

Partition I_S to create the list L of real numbers between r_1 and r_2.

$I_S = [r_1, r_2]$	$L = (r_1, r_2)$
$= [r_1, [r_3], r_2]$	$L = (r_1, r_3, r_2)$
$= [r_1, r_3], [r_3, r_2]$	
$=[r_1, [r_4], r_3], [r_3, [r_5], r_2]$	$L = (r_1, r_4, r_3, r_5, r_2)$
$= [r_1, r_4], [r_4, r_3], [r_3, r_5], [r_5, r_2]$	
$= [r_1, [r_6], r_4], [r_4, [r_7], r_3], [r_3, [r_8], r_5], [r_5, [r_9], r_2]$	$L = (r_1, r_6, r_4, r_7, r_3, r_8, r_5, r_9, r_2)$
$= [r_1, r_6], [r_6, r_4], [r_4, r_7], [r_7, r_3], [r_3, r_8], [r_8, r_5], [r_5, r_9], [r_9, r_2]$	
…	…

At the limit of the process L will appear as follows: $L = (r_1,\ldots r_6,\ldots r_4,\ldots r_7,\ldots r_3,\ldots r_8,\ldots r_5,\ldots r_9,\ldots r_2)$.

By definition 13 there are no immediate predecessors or successors in the set of real numbers. It follows that the partitioning of sub-intervals of real numbers can go on indefinitely. Also, except for r_1 and r_2, every number in the original interval must, at some point during the process, become a relative bound and only then added to L. And because no number will be a relative bound more than once, there will be no duplicates in L.

As can be seen in the examples above, each relative bound becomes the lower bound of one sub-interval and the upper bound of another sub-interval. This means that every number in the each sub-interval will be approached from both below and above its value and the interval lengths will become infinitesimally small.

Example 3, using numbers:

Let:

$S_\infty = \{\mathbb{R}\}$

$I_S = [1, 4]$

Partition I_S to create the list L of real numbers between 1 and 4.

$I_S = [1, 4]$	$L = (1, 4)$
$= [1, [\pi], 4]$	$L = (1, \pi, 4)$
$= [1, \pi], [\pi, 4]$	
$= [1, [e], \pi], [\pi, [3.2], 4]$	$L = (1, e, \pi, 3.2, 4)$
$= [1, e], [e, \pi], [\pi, 3.2], [3.2, 4]$	
$= [1, [\sqrt{2}], e], [e, [3], \pi], [\pi, [3.15], 3.2], [3.2, [3.3], 4]$	$L = (1, \sqrt{2}, e, 3, \pi, 3.15, 3.2, 3.3, 4)$
$= [1, \sqrt{2}], [\sqrt{2}, e], [e, 3], [3, \pi], [\pi, 3.15], [3.15, 3.2], [3.2, 3.3], [3.3, 4]$	

…

At the limit of the process $L = (1,\ldots\sqrt{2},\ldots e,\ldots 3,\ldots \pi,\ldots 3.15,\ldots 3.2,\ldots 3.3,\ldots 4)$.

VI. Proving the List L is Complete

The question remains as to whether or not the list L will contain all real numbers in $[r_1, r_2]$. We will prove that: All the real numbers in $[r_1, r_2]$ are contained in the list L.
Proof: Assume that there exists a number X such that $r_1 < X < r_2$ and that $X \notin L$.

1. Since X is an element of $[r_1, r_2]$ then it must be an element of a sub-interval $[r_i, r_j]$ contained in $[r_1, r_2]$.

2. If X is an element of a sub-interval of $[r_1, r_2]$ then at some finite point before the limit it will become a relative bound of a conjoined interval pair $[r_i, [X], r_j]$.

The following argument justifies statement 2:
Let
$S = \{\mathbb{R}\}$
$I_S = [r_1, r_2]$, $r_1 < r_2$
$r_1 < X < r_2$
Partition I_S at $r_1 < r_3 < X$
$[r_1, \ldots [r_3], \ldots X, \ldots r_2]$
Form new intervals
$[r_1, r_3], [r_3, \ldots X, \ldots r_2]$

We know that $|r_3 - r_2| < |r_1 - r_2|$ therefore the interval $[r_3, \ldots X, \ldots r_2]$ starts to narrow in on X.
Continue partitioning (we will ignore $[r_1, r_3]$ since we're only interested in the interval containing X).
Partition $[r_3, \ldots X, \ldots r_2]$ at $r_3 < r_4$, $r_4 > X$
$[r_3, \ldots X, \ldots [r_4], \ldots r_2]$
Form new intervals
$[r_3, \ldots X, \ldots r_4]$, $[r_4, r_2]$
Now, $|r_3 - r_4| < |r_3 - r_2|$ and we continue to close in on X.
Partition $[r_3, \ldots X, \ldots r_4]$ at $r_3 < r_5 < X$
$[r_3, \ldots [r_5], \ldots X, \ldots r_4]$
Form new intervals
$[r_3, r_5]$, $[r_5, \ldots X, \ldots r_4]$
Now, $|r_5 - r_4| < |r_3 - r_4|$ and we draw still closer to X.
As the process continues, $|r_i - r_j|$ of the interval containing X gets closer and closer to 0 and before the transition to the limit, X must be identified as a relative bound, that is $\lim_{|r_i - r_j| \to 0} |r_i - r_j| = X$

3. Once X becomes a relative bound of the conjoined interval pair, $[r_i, [X], r_j]$ it will be inserted into L.

4. Since at the limit of the process, X must be an element of L then the original assertion that $X \notin L$ leads to a contradiction and must be false.

5. We can then assert that at the limit, L will be complete and this ends the proof.

VII. Derivation of $f: \mathbb{N} \leftrightarrow [r_1, r_2]$

We have constructed the list L from $[r_1, r_2]$ and have shown that the list is complete, containing all the real numbers in $[r_1, r_2]$.

We will now demonstrate that there exists a bijective function from $\mathbb{N}$ to $[r_1, r_2]$, $f: \mathbb{N} \leftrightarrow [r_1, r_2]$.

We have used the Interval Sieve Algorithm to create: $L = (r_1, \ldots r_6, \ldots r_4, \ldots r_7, \ldots r_3, \ldots r_8, \ldots r_5, \ldots r_9, \ldots r_2)$

and have proved L is complete. It is readily apparent that for every r in the list there is an associated natural number subscript. Since L is complete, containing all numbers in $[r_1, r_2]$ and each number in $[r_1, r_2]$ is associated with a single unique natural number we can assert that $f: \mathbb{N} \leftrightarrow [r_1, r_2]$ exists.

The existence of $f: \mathbb{N} \leftrightarrow [r_1, r_2]$ confirms a one to one correspondence between the natural numbers and any closed interval of real numbers.

VIII. Final Thoughts

1. The upper and lower bounds of $[r_1, r_2]$ are limits that are approached from above and below respectively. Since they have neither direct successors or direct predecessors they can be reached only at the limit.

2. All other numbers generated by repeatedly partitioning $[r_1, r_2]$ become limits in their own right and are approached both from above and below. Since they have neither direct successors or direct predecessors they can be reached only at the limit.

3. The partitioning of $[r_1, r_2]$ is akin to finding the area under a curve of a graphed continuous function. It's not an exact analog of integration; but the process of creating sub-intervals within $[r_1, r_2]$ such that $|r_1 - r_2|$ becomes infinitesimally small bears a resemblance to creating infinitesimally narrow rectangles and summing their areas in order to determine the area under the curve.

4. The fact that we have shown the existence of $f: \mathbb{N} \leftrightarrow [r_1, r_2]$ implies that Cantor's continuum hypothesis is true for closed intervals of real numbers. Interestingly, it's not that there are no infinite sets with cardinality between $\aleph_0$ and $\aleph_1$, rather $\mathbb{N}$ and $[r_1, r_2]$ turn out to be the same size.