Interval Sieve Algorithm

Creating a Countable Set of Real Numbers from a Closed Interval

I. Abstract

The Interval Sieve Algorithm is a method for generating a list of real numbers on any closed interval of real numbers $[r_i, r_j]$ where $r_i < r_j$. The purpose of this paper is to:

1. Present the algorithm and demonstrate the construction of the list L,
2. Prove that the list L is complete,
3. Derive the bijective function $f: \mathbb{N} \leftrightarrow [r_1, r_2]$.

II. Given

1. The closed interval $[r_1, r_2]$ where $r_1 < r_2$ and r_1, r_2 are real numbers

2. The list $L = \{r_1, r_2\}$

III. Definitions

1. The **lower bound** of a closed interval is the smaller of the two numbers comprising the interval. In the interval $[r_1, r_2]$ where $r_1 < r_2$, r_1 is the lower bound of the interval.

2. The **upper bound** of a closed interval is the larger of the two numbers comprising the interval. In the interval $[r_1, r_2]$, where $r_1 < r_2$, r_2 is the upper bound of the interval.

4. A **conjoined interval pair** is a pair of closed intervals where the upper bound of one and the lower bound of the other are the same number. $[r_i, [r_k,] r_j]$ is an example of a conjoined interval pair where r_k is both the upper bound of $[r_i, r_k]$ and the lower bound of $[r_k, r_j]$.

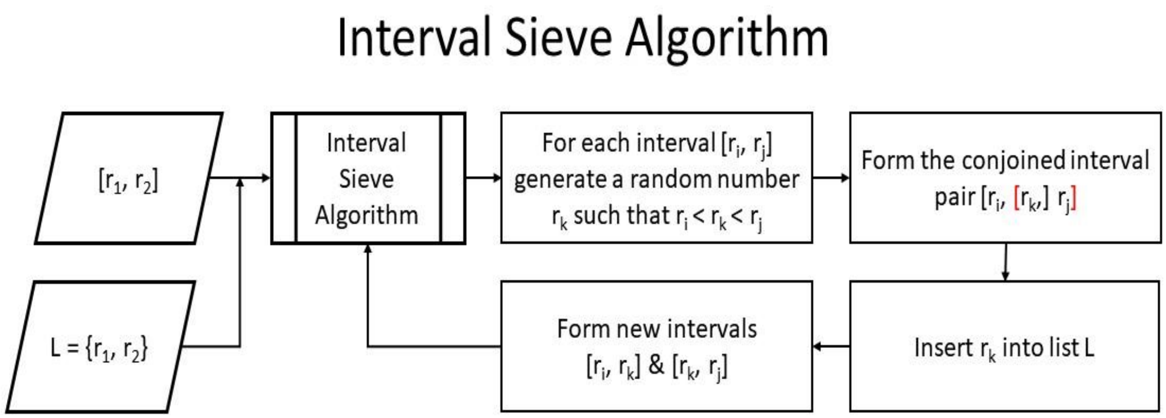
5. A **relative bound** is the number that is common to both intervals in a conjoined interval pair. In the conjoined interval pair $[r_1, [r_3,] r_2]$, where $r_1 < r_3 < r_2$, r_3 is a relative bound in both intervals $[r_1, r_3]$ and $[r_3, r_2]$.

6. The **immediate predecessor** of a number λ is a number β such that there exists no number δ where $\beta < \delta < \lambda$.

7. The **immediate successor** of a number λ is a number β such that there exists no number δ where $\lambda < \delta < \beta$.

For any 2 real numbers λ and β in $[r_1, r_2]$, we can always find another real number, δ, such that if $\lambda > \beta$ then $\beta < \delta < \lambda$ and if $\lambda < \beta$ then $\lambda < \delta < \beta$. Therefore from definitions 6 and 7 we know that there are no immediate predecessors or successors of any of the elements of $[r_1, r_2]$.

IV. Interval Sieve Algorithm - Flowchart



V. Interval Sieve Algorithm – Procedural Narrative

The algorithm acts like a sieve in that each iteration of its steps produces a finer degree of sub-intervals within $[r_1, r_2]$.

Central to the algorithm is a random number generator that will take as input the lower and upper bounds of a closed interval, $[r_i, r_j]$ and output a number r_k of the form $d_n \ldots d_1.d_{-1}d_{-2}\ldots$ such that $r_i < r_k < r_j$. Because there are no overlapping intervals, the random number generator will always produce numbers that have not already been generated. The algorithm will complete its work in the limit and the result will be a list of numbers, $L = \{r \mid r \in [r_1, r_2]\}$, that will later be shown to be a complete list of numbers in the interval $[r_1, r_2]$. Alternatively we can show the completed L as $L = \{r_1 \ldots r_3 \ldots r_2\}$. r_1 has no immediate successors, r_3 has no immediate predecessors and no immediate successors and r_2 has no immediate predecessors.

Procedure:

0. We begin the procedure given the interval $[r_1, r_2]$ where $r_1 < r_2$ and r_1, r_2 are real numbers and the list $L = \{r_1, r_2\}$.

1. Sub-divide each interval $[r_i, r_j]$ by generating a random real number r_k such that $r_i < r_k < r_j$ to get a conjoined interval pair: $[r_i, [r_k,] r_j]$.

2. Insert the relative bound number, r_k, into the list L to get L = $\{r_i, r_k, r_j\}$

3. Form 2 new sub-intervals $[r_i, r_k]$ $[r_k, r_j]$

4. Return to step 1.

VI. Interval Sieve Algorithm – Graphical Depiction of Output

The algorithm produces the following results:

Interval Sieve Algorithm

[r_1																r_2]
[r_1								[r_3]								r_2]
[r_1				[r_4]				[r_3]				[r_5]				r_2]
[r_1		[r_6]		[r_4]		[r_7]		[r_3]		[r_8]		[r_5]		[r_9]		r_2]
[r_1	[r_{10}]	[r_6]	[r_{11}]	[r_4]	[r_{12}]	[r_7]	[r_{13}]	[r_3]	[r_{14}]	[r_8]	[r_{15}]	[r_5]	[r_{16}]	[r_9]	[r_{17}]	r_2]

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Intervals Generated by the Algorithm

$[r_1, r_2]$

$[r_1, r_3][r_3, r_2]$

$[r_1, r_4][r_4, r_3][r_3, r_5][r_5, r_2]$

$[r_1, r_6][r_6, r_4][r_4, r_7][r_7, r_3][r_3, r_8][r_8, r_5][r_5, r_9][r_9, r_2]$

$[r_1, r_{10}][r_{10}, r_6][r_6, r_{11}][r_{11}, r_4][r_4, r_{12}][r_{12}, r_7][r_7, r_{13}][r_{13}, r_3][r_3, r_{14}][r_{14}, r_8][r_8, r_{15}][r_{15}, r_5][r_5, r_{16}][r_{16}, r_9][r_9, r_2]$

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List of Real Numbers Generated by the Algothrim

$L = \{r_1, r_2\}$

$L = \{r_1, r_3, r_2\}$

$L = \{r_1, r_4, r_3, r_5, r_2\}$

$L = \{r_1, r_6, r_4, r_7, r_3, r_8, r_5, r_9, r_2\}$

$L = \{r_1, r_{10}, r_6, r_{11}, r_4, r_{12}, r_7, r_{13}, r_3, r_{14}, r_8, r_{15}, r_5, r_{16}, r_9, r_{17}, r_2\}$

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Beginning with one interval, growth of the number of sub-intervals created is exponential and after the fourth iteration we have a total of 16 sub-intervals. If n is the number of iterations and I is the number of sub-intervals, we have $I = 2^n$ and if L_n is the number of list elements then $L_n = 2^n + 1$.

Numbers are inserted into L when they are produced by the random number generator as relative bounds of conjoined interval pairs. Each iteration of the algorithm bifurcates all sub-intervals at r_k. Once r_k is inserted into L two new sub-intervals are formed with r_k becoming the upper bound of one sub-interval and the lower bound of the other sub-interval, when the process begins again.

This ensures that the random number generator will never produce a duplicate number within the bounds of $[r_1, r_2]$ since every r_k generated must satisfy the requirement that $r_i < r_k < r_j$.

As the process continues the sub-intervals grow smaller and numbers are added to L. At any finite point L contains a finite number of values but in the end we have $L = \{r \mid r \in [r_1, r_2]\}$ which we will prove contains all numbers in $[r_1, r_2]$. The transition occurs in the limit.

VII. Proving the List L is Complete

The question remains as to whether or not the list L will contain all real numbers in $[r_1, r_2]$. We will prove that: **All the real numbers in $[r_1, r_2]$ are contained in the list L**.

Proof: Assume that there exists a number X such that $r_1 < X < r_2$ and that $X \notin L$.

1. Since X is an element of $[r_1, r_2]$ then it must be an element of a sub-interval $[r_i, r_j]$ contained in $[r_1, r_2]$.

2. If X is an element of a sub-interval of $[r_1, r_2]$ then at some finite point before the limit it will be produced by the random number generator as a relative bound of a conjoined interval pair [r_i, [X,] r_j].

3. Once X becomes a relative bound of the conjoined interval pair, [r_i, [X,] r_j] it will be inserted into L.

4. Since X must be an element of L then the original assertion that $X \notin L$ leads to a contradiction and must be false.

5. We can then assert that at the limit, L will be complete and this ends the proof.

VIII. Derivation of $f: \mathbb{N} \leftrightarrow [r_1, r_2]$

We have constructed the list L from $[r_1, r_2]$ and have shown that the list is complete, containing all the real numbers in $[r_1, r_2]$.

We will now demonstrate that there exists a bijective function from $\mathbb{N}$ to $[r_1, r_2]$,

$$f: \mathbb{N} \leftrightarrow [r_1, r_2].$$

We can substitute the digits $d_n \ldots d_1.d_{-1}d_{-2}\ldots$ for the condensed, r_i, representation of the numbers in the list L, drop the r from their condensed forms leaving the subscript, change = to ↔ and go from

Elements of L= $\{r \mid r \in [r_1, r_2]\}$		$f: \mathbb{N} \leftrightarrow [r_1, r_2]$
$r_1 = d_n \ldots d_1.d_{-1}d_{-2}\ldots$		$1 \leftrightarrow d_n \ldots d_1.d_{-1}d_{-2}\ldots$
…	to	…
$r_3 = d_n \ldots d_1.d_{-1}d_{-2}\ldots$		$3 \leftrightarrow d_n \ldots d_1.d_{-1}d_{-2}\ldots$
…		…
$r_2 = d_n \ldots d_1.d_{-1}d_{-2}\ldots$		$2 \leftrightarrow d_n \ldots d_1.d_{-1}d_{-2}\ldots$

Since the set comprised of the elements of L is complete and for every element of the set there is a corresponding natural number we can conclude that $f: \mathbb{N} \leftrightarrow [r_1, r_2]$ exists.

The existence of $f: \mathbb{N} \leftrightarrow [r_1, r_2]$ confirms a one to one correspondence between the natural numbers and any closed interval of real numbers.