Stochastic Causality and Quantization of Gravity

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ABSTRACT

Causality is the fundamental principle in the Einstein's theory of general relativity. We consider the theory of broken causality by the assumptions of stochastic nature to causal process. We see that the causality breaking of stochastic property brings the general relativity of broken causality, and this is equivalent to the theory of quantum gravity. We investigate some properties of quantum gravity in relation to the holographic principle. We calculate Shannon's entropy. In those investigations, we see the appearance of holographic principle at the first order in expansion of perturbation theory. The result indicates that the theory of stochastic causality, that we established, is the non-perturbative theory of quantum gravity.

1 Introduction

Physics of 20th century has started with two greatest works: quantum theory and general theory of relativity. Einstein established the theory of general relativity as the extension of Newtonian mechanics, which is described as curved space-time.

The 21st century, we are living, is called as an era of information. Invention of such a technology as AI is changing the circumstances of our life. Inspired by the work of deep learning [1], the interaction between theoretical physicists and the researchers of machine learning or information theory seems to be often [2, 3]. The theoretical physical works inspired by machine learning does exist [4, 5, 6, 7].

Causality is the fundamental principle in the Einstein's theory of general relativity. In recent fashion of data science, they are considering the Riemannian manifold of data space. Therefore, what they are treating is merely data, and there is no reason to follow physical causality. Thus, we consider the theory of broken physical causality by the assumptions of stochastic nature to causal process. We will see that the causality breaking of stochastic property brings the general relativity of broken causality, and this is equivalent to the theory of quantum gravity. We investigate some properties in relation to the holographic principle. We see that holographic principle appears at the first order expansion of perturbation theory. This means that the theory of stochastic causality is the non-perturbative theory of quantum gravity.

This paper is organized as follows. In section 2, we review holographic principle and entropy of black hole. Section 3 is devoted for the description of stochastic causality and quantization of gravity. Section 4 is discussions.

2 Holographic Principle and Entropy of Black Hole

The holographic principle is a principle of string theory and quantum gravity. That states that the theory of bulk space can be inferred from a lower dimensional boundary theory. It was proposed first by 't Hooft [8]. Susskind gave an interpretation from string theory [9]. Maldacena proposed the prime theory of holographic principle called as the AdS/CFT correspondence [10].

Holographic principle was inspired by black hole thermodynamics. Black hole thermodynamics states that maximal entropy of black hole is proportional to the area at event horizon. AdS/CFT correspondence gives the interpretation to black hole entropy as a description of a microscopic states at the boundary. This relationship has been derived by calculating the central charge c of Virasoro algebra of boundary CFT, and saw the correspondence between area law of black hole entropy and the entropy calculated from the degree of freedom at the boundary using Cardy's formula [11],

$$S = 2\pi\sqrt{\frac{c\Delta}{6}}, \tag{1}$$

here, Δ is the eigenvalue of L_0. Then, it was applied to de Sitter/CFT correspondence [12], and entanglement entropy [13],

$$S = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right), \tag{2}$$

here, l is the length of subsystem, a is the lattice spacing, and β is related with ultra violet cut off.

The bulk/boundary correspondence is applied in a wide range of fields including condensed matter physics [14] not only in string theory. For the review of AdS/CFT, see ref. [15, 16].

3 Stochastic Causality and Quantization of Gravity

3.1 Stochastic Causality

In the theory of general relativity, they consider such process on light cone as

$$d\mathbf{x} = cdt, \tag{3}$$

here, c is the speed of light, and describe world line of the following form,

$$ds^2 = -c^2dt^2 + d\mathbf{x}^2. \tag{4}$$

We extend this process to include stochastic process,

$$d\mathbf{x} = cdt + \sigma dZ. \tag{5}$$

Here, dZ is a standard Brownian motion and σ is standard deviation. If we write this equation in a world line form, the world line is represented as,

$$ds^2 = -c^2dt^2 + d\mathbf{x}^2 - \sigma^2 dZ^2. \tag{6}$$

Here, σ can be interpreted as a parameter of quantum fluctuation, that represents deviation around classical space-time. In another word, we are assuming the last term in (6) is a perturbation to classical space-time.

3.2 Minkowski Space

Eq. (6) represents the fluctuation around the $(n+1)$-dimensional Minkowski space. We are considering Brownian motion on space-time, therefore the probability density function takes the following normal distribution form. It is given by,

$$f_{\mathbf{x}}(\mathbf{x_0}; c, \sigma, t) = \left(\frac{1}{2\pi\sigma^2 t}\right)^{n/2} \exp\left\{-\frac{(\mathbf{x}-\mathbf{x_0})^2}{2\sigma^2 t}\right\}. \tag{7}$$

The Shannon entropy is obtained as follows.

$$\begin{aligned} H &= -\int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2 t}\right)^{n/2} e^{-\frac{\sum_{i=1}^{n}(x_i - x_{i0})^2}{2\sigma^2 t}} \ln\left\{\left(\frac{1}{2\pi\sigma^2 t}\right)^{n/2} e^{-\frac{\sum_{i=1}^{n}(x_i - x_{i0})^2}{2\sigma^2 t}}\right\} d\mathbf{x} \\ &= \frac{n}{2}\ln(2\pi\sigma^2 t) + \left(\frac{1}{2}\right)^{n} (2\sigma^2 t)^{(n-1)} \end{aligned} \tag{8}$$

By expanding the order of $\sigma^2 t$,

$$H = \frac{n}{2}(\ln(2\pi) - 1) + n\sqrt{\sigma^2 t} + \cdots \quad (\text{for } n \geq 2). \tag{9}$$

The first term is a classical entropy and the second term is the contribution in the first order of perturbative expansion. The entropy will be indifferent to coordinate transformation. Therefore, the entropy of de Sitter space will be the same. The interpretation of Shannon entropy is entropy of space-time fluctuating to external space. The form of eq. (9) reminds us Cardy's formula [11] and the variance is related with the central charge of boundary CFT, but we will consider this relationship in later section.

3.3 Anti-de Sitter Space

If we use the convention of eq. (6), incorporating 3-dimensional anti-de Sitter space with stochastic fluctuation gives world line of the following form,

$$ds^2 = \frac{-dt^2 + dy^2 + dx^2}{y^2} - \sigma^2 dZ^2. \tag{10}$$

As in the same way of the last section, we consider a probability density function. Probability density function is obtained by solving the diffusion equation on hyperbolic space in 3-dimension [17],

$$f(\rho, \rho_0, ; \sigma, t) = \frac{1}{2\pi\sqrt{4\pi\sigma^2 t}} \mathrm{e}^{-\sigma^2 t} \frac{\sinh(\frac{\rho_0 \rho}{2\sigma^2 t})}{\sinh \rho_0 \sinh \rho} \exp\left(-\frac{\rho^2 + \rho_0^2}{4\sigma^2 t}\right). \tag{11}$$

Here, ρ is the coordinate in radial direction (radial rapidity in [17]), and ρ_0 is the position at which boundary condition is given. The Shannon entropy is

$$H = -\int_0^\infty f(\rho, \rho_0, ; \sigma, t) \ln f(\rho, \rho_0, ; \sigma, t) d\rho. \tag{12}$$

By expanding the order of $\sigma^2 t$,

$$\begin{aligned} H \quad \simeq \quad & \frac{1}{4\pi} \frac{\sinh \frac{\rho_0 \rho}{2\sigma^2 t}}{\rho_0 \rho} \left(\frac{3}{2} + \ln(2\pi) + \frac{1}{2} \ln(4\pi) - \ln\left(\frac{\sinh \frac{\rho_0 \rho}{2\sigma^2 t}}{\rho_0 \rho} \right) \right) \\ & + \frac{1}{4\pi} \frac{\sinh \frac{\rho_0 \rho}{2\sigma^2 t}}{\rho_0 \rho} \left(-\ln(2\pi) - \frac{1}{2} \ln(4\pi) + \ln \frac{\sinh \frac{\rho_0 \rho}{2\sigma^2 t}}{\rho_0 \rho} \right) \sigma^2 t \\ & + \cdots \end{aligned} \tag{13}$$

Here, we are assuming that $\sigma^2 t$ is very small, $\sigma^2 t \sim \rho_0 \rho$, and ρ is finite. Also, we assumed $\rho_0 \to 0$. $\sigma^2 t$ is playing a role of cutoff of quantum fluctuation.

By scaling of $\sigma^2 \to a\sigma^2$, the entropy (13) is renormalizable at the first order of perturbative expansion, i.e., constant terms inside parenthesis in the second line of eq. (13) vanish. In this case, by setting $a = (16\pi^3)^{-1}$, constant terms vanish.

4 Discussions

4.1 Relation to Boundary CFT

Eqs. (9) and (13) invoke the relationship between variance σ^2 and central charge. If there's such a relationship, Cardy's formula [11, 18] in de Sitter/CFT correspondence [12] and in entanglement entropy [13] holds at the first order of perturbative expansion. This means that black hole thermodynamics and the holographic principle are phenomena of the first order of perturbative expansion of quantum gravity.

However, what we did is the quantization with stochastic causality, there is no reasoning to CFT, therefore the relationship between variance and central charge is obscure. In SLE/CFT correspondence [19], the variance is related with the central charge of boundary CFT, but it does not satisfies the standard connection between SLE parameter and central charge.

4.2 Relation to Noncommutative Geometry

Non-commutative geometry has gained attraction in the context of string theory [20]. In ref. [21], the variance σ^2 of eq. (5) is related with Θ_{ij} of commutation relation of position,

$$[x_i, x_j] = \Theta_{ij}. \tag{14}$$

In ref. [22], it is related as,

$$[x, dx] = J\sigma^2. \tag{15}$$

Here, J is time shift operator. So, our formulation will be related to non-commutativity of space.

4.3 Comments on Dimensionality

In former sections, we investigated with the standard Brownian motion. The dimension of standard Brownian motion is $dZ^2 = \Delta t$, and the dimension of σ^2 is t^{-1} to let $\sigma^2 t$ be dimensionless. Therefore, the last terms in eq. (6) and (10) are dimensionless. This means that the quantum fluctuation, currently considering, is dimensionless effect.

One of the future study is to introduce the fractional Brownian motion. By Ito's work [23], the integral of Brownian motion was related with the difference of time. The fractional Brownian motion is an extension of this relationship.

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