A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 4/10.

by Stephen C. Pearson MRACI
[T/A S. C. Pearson Technical Services],
Affiliate of the Royal Society of Chemistry &
Member of the London Mathematical Society,
Dorset, ENGLAND.

Email Address:- scpearson1952@outlook.com.

I. PRELIMINARY REMARKS.

For further details, the reader should accordingly refer to the first page of the author’s previous submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 1/10.”

which has been published under the ‘VIXRA’ Mathematics subheading:- ‘Functions and Analysis’.


For further details, the reader should accordingly refer to the remainder of this submission from Page [2] onwards.
such that the real variable function,

\[ \Theta = \cos^{-1} \left[ \frac{x}{\sqrt{x^2 + y^2 + z^2 + t^2}} \right] \in [0, \pi], \]

\[ v_0 = x + iy + jz + k \bar{t} \in \mathbb{H} - \{0\}, \text{ as required. Q.E.D.} \]

Similarly, in view of Eq. (1-87), we shall likewise define the quaternion analogue of the multi-valued complex function, \( \log(z) \), as follows:

**Definition DI-15.**

Let there exist a multi-valued logarithmic quaternion hypercomplex function, \( \log(z) \), whose domain,

\[ \text{dom}(\log) \subseteq \mathbb{H} - \{0\} \]

Subsequently, this function is denoted by the formula,

\[ \log(z) = \log(z) + \left[ \frac{iy + jz + k \bar{t}}{\sqrt{y^2 + z^2 + t^2}} \right] 2\pi i \ (m \in \mathbb{Z}). \]

Churchill et al. [1] have established the following identities with respect to the multi-valued logarithmic function, \( \log(z) \), namely:

\[ \exp(\log(z)) = e^{\log(z)} = z \quad (1-91); \]
\[ \log(\exp(z)) = \log(\exp(z)) = z + i2\pi n \quad (n \in \mathbb{Z}) \quad (1-92), \]

and thus we are motivated to derive quaternion analogues of these particular results by means of the next two theorems:

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**Theorem TI-21.**

Let there exist a multi-valued logarithmic quaternion hypercomplex function, \( \log(q) \), whose domain,

\[ \text{dom}(\log) \subseteq \mathbb{H} - \{0\}. \]

Subsequently, it may be proven that this function is characterized by the following property, i.e.,

\[ \exp(\log(q)) = q, \quad \forall q = x + iy + jz + k\bar{z} \in \mathbb{H} - \{0\}. \]

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**Proof:**

From Definition DI-15, we recall that the multi-valued logarithmic function,

\[ \log(q) = \log(q) + \left[ \frac{iy + jz + k\bar{z}}{\sqrt{y^2 + z^2 + i^2}} \right] 2\pi \mathbb{N} \]

\[ (n \in \mathbb{Z} \land q = x + iy + jz + k\bar{z} \in \mathbb{H} - \{0\}). \]
Finally, by virtue of Theorems TI-7 & TI-8 and Definition DI-14, we deduce that the function,

\[ \exp(\log(q)) = \exp(\log(q)) + \left[ \frac{iy + jx + k\bar{z}}{\sqrt{y^2 + x^2 + \bar{z}^2}} \right] 2\pi i \]

\[ = \exp(\log(q)) \exp\left( \frac{iy + jx + k\bar{z}}{\sqrt{y^2 + x^2 + \bar{z}^2}} \right) 2\pi i \]

\[ = q \cdot 1 = q, \text{ as required. Q.E.D.} \]

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**Theorem TI-22.**

Let there exist the multi-valued logarithmic quaternion-hypercomplex functions, \( \log(q) \), having both an algebraic structure and associated properties as outlined in the preceding Definitions DI-14 & DI-15 and Theorems TI-20 & TI-21.

Henceforth, it may be shown that the function \( \log(q) \) possesses the additional property:

\[ \log(\exp(q)) = q + \left[ \frac{iy + jx + k\bar{z}}{\sqrt{y^2 + x^2 + \bar{z}^2}} \right] 2\pi i, \quad \forall n \in \mathbb{Z}, \text{ the set of integers.} \]

**Proof:**

From Theorem TI-21, we instantly recall that the exponential and logarithmic functions, \( \exp \) and \( \log \), are correlated by means of the formula:
\[ \exp(\log(q)) = q \]

\[ \therefore \exp(\log(x + iy + j\hat{x} + k\hat{y})) = x + iy + j\hat{x} + k\hat{y} \]

\[ \implies \exp(\log(x + iY + j\hat{x} + k\hat{Y})) = x + iY + j\hat{x} + k\hat{Y}, \]

\[ \forall x, y, \hat{x}, \hat{Y} \in \mathbb{R}. \]

Subsequently, if we define \( X, Y, \hat{x} \text{ and } \hat{Y} \) to be real-valued functions of \( x, y, \hat{x} \text{ and } \hat{y} \) such that

\[ \exp(q) = X + iY + j\hat{x} + k\hat{Y}, \]

it therefore follows that

\[ \exp(\log(\exp(q))) = \exp(q). \]

Let us now define a function,

\[ w = \log(\exp(q)) \]

\[ \implies \exp(w) = \exp(q). \] \hspace{1cm} (ii)

However, from Theorem TI-8, we also perceive that the exponential function,

\[ \exp(q) = \exp(q + \left[\frac{iy + j\hat{x} + k\hat{y}}{\sqrt{\hat{x}^2 + \hat{y}^2 + \hat{Y}^2}}\right]2\pi i) \quad (n \in \mathbb{Z}) \] \hspace{1cm} (iii)
Clearly, Eqs. (i) & (ii) become identical statements by setting

\[ w = q + \left[ \frac{iy + jx + kz}{\sqrt{y^2 + x^2 + z^2}} \right] 2\pi i \]

\[ \implies \log(\exp(w)) = q + \left[ \frac{iy + jx + kz}{\sqrt{y^2 + x^2 + z^2}} \right] 2\pi i , \]

\[ \forall n \in \mathbb{Z} \text{, the set of integers, as required. Q.E.D.} \]

From complex variable analysis, we recall that, for any two non-zero complex numbers, \( z_1 \) and \( z_2 \), there exist the following logarithmic identities:

\[ \log(z_1 z_2) = \log(z_1) + \log(z_2) \quad (1-93); \]

\[ \log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2) \quad (1-94). \]

Hence, the purpose of our next theorem is to show that, subject to certain restrictions, quaternion analogues of Eqs. (1-93) & (1-94) likewise exist.

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**Theorem TI-23.**

Let there exist two quaternion variables,

\[ q_1 = x_1 + iy_1 + jz_1 + k\bar{y}_1, \]
\[ q_2 = x_2 + iy_2 + jz_2 + k\bar{y}_2, \quad \forall q_1, q_2 \in \mathbb{H} - \{0\}. \]
Subsequently, it may be shown that the logarithms of both the product, \( y_1 y_2 \), and also the quotient, \( y_1/y_2 \), are respectively, written as

(a) \( \log(y_1 y_2) = \log(y_1) + \log(y_2) \),

(b) \( \log(y_1/y_2) = \log(y_1) - \log(y_2) \),

wherever \( y_2 = \lambda y_1 ; \hat{y}_2 = \lambda \hat{y}_1 ; \vec{y}_2 = \lambda \vec{y}_1 \), \( \forall \lambda \in \mathbb{R} \).

\[ \ast \ast \ast \]

**PROOF:**

From Definition DI-15 and Theorem TI-20, we recall that the multi-valued logarithmic function,

\[
\log(y) = \log(r) + \left[ \frac{iy + j\hat{x} + k\vec{x}}{\sqrt{y^2 + x^2 + z^2}} \right] 2\pi n + \left[ \frac{iy - j\hat{x} - k\vec{x}}{\sqrt{y^2 + x^2 + z^2}} \right] 2\pi n
\]

\[
= \log(\sqrt{x^2 + y^2 + \hat{x}^2 + \vec{x}^2}) + \left[ \frac{iy + j\hat{x} + k\vec{x}}{\sqrt{y^2 + x^2 + z^2}} \right] \Theta + \left[ \frac{iy - j\hat{x} - k\vec{x}}{\sqrt{y^2 + x^2 + z^2}} \right] 2\pi n
\]

\[
= \log(\sqrt{x^2 + y^2 + \hat{x}^2 + \vec{x}^2}) + \frac{iy + j\hat{x} + k\vec{x}}{\sqrt{y^2 + x^2 + z^2}} (\Theta + 2\pi n)
\]

such that the real variable function,

\[
\Theta = \arccos \left[ \frac{x}{\sqrt{x^2 + y^2 + \hat{x}^2 + \vec{x}^2}} \right] \in [0, \pi], \quad \forall y = x + iy + j\hat{x} + k\vec{x} \in \mathbb{H} - \{0\},
\]
and the integer, $n \in \mathbb{Z}$.

Let there exist two quaternions,

$$q_1 = x_1 + iy_1 + jx_2 + k\tilde{y}_1 \quad \text{and} \quad q_2 = x_2 + iy_2 + j\tilde{x}_2 + k\tilde{y}_2,$$

and, furthermore, by substituting the real and imaginary parts of $q_1$ and $q_2$ into Eq. (i) above, we respectively obtain

$$\log(q_1) = \log(x_1 + iy_1 + jx_2 + k\tilde{y}_1)$$

$$\quad = \log(\sqrt{x_1^2 + y_1^2 + x_2^2 + \tilde{y}_1^2}) + \left[ \frac{iy_1 + jx_2 + k\tilde{y}_1}{\sqrt{x_1^2 + y_1^2 + x_2^2 + \tilde{y}_1^2}} \right] (\Theta_1 + 2n_1\pi) \quad \text{(ii)}$$

such that the real-valued constant,

$$\Theta_1 = \cos^{-1} \left[ \frac{x_1}{\sqrt{x_1^2 + y_1^2 + x_2^2 + \tilde{y}_1^2}} \right] \in [0, \pi],$$

and the integer, $n = n_1 \in \mathbb{Z}$

AND

$$\log(q_2) = \log(x_2 + iy_2 + jx_2 + k\tilde{y}_2)$$

$$\quad = \log(\sqrt{x_2^2 + y_2^2 + x_2^2 + \tilde{y}_2^2}) + \left[ \frac{iy_2 + jx_2 + k\tilde{y}_2}{\sqrt{x_2^2 + y_2^2 + x_2^2 + \tilde{y}_2^2}} \right] (\Theta_2 + 2n_2\pi) \quad \text{(iii)}$$

such that the real-valued constant,

$$\Theta_2 = \cos^{-1} \left[ \frac{x_2}{\sqrt{x_2^2 + y_2^2 + x_2^2 + \tilde{y}_2^2}} \right] \in [0, \pi].$$
\[ \theta_2 = \cos^{-1} \left[ \frac{x_2}{\sqrt{x_2^2 + y_2^2 + z_2^2 + w_2^2}} \right] \in [0, \pi], \]

and the integer, \( m = n_0 \in \mathbb{Z} \).

Since, as stated in the preamble to this proof, we initially set

\[ y_2 = \lambda y_1; \quad z_2 = \lambda z_1; \quad \text{and} \quad w_2 = \lambda w_1; \quad \forall \lambda \in \mathbb{R}, \]

it therefore follows that Eq. (iv) can now be written as

\[ \log(z_2) = \log(x_2 + i\lambda y_1 + j\lambda z_1 + k\lambda w_1) \]

\[ = \log \left( \sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2 z_1^2 + \lambda^2 w_1^2} \right) + \left[ i\frac{\lambda y_1 + j\lambda z_1 + k\lambda w_1}{\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2 z_1^2 + \lambda^2 w_1^2}} \right] (\theta_2 + 2n_2\pi) \]

\[ = \log \left( \sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2 z_1^2 + \lambda^2 w_1^2} \right) + \left( \frac{\lambda}{|\lambda|} \right) \left[ i\frac{\lambda y_1 + j\lambda z_1 + k\lambda w_1}{\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2 z_1^2 + \lambda^2 w_1^2}} \right] (\theta_2 + 2n_2\pi) \quad (iv), \]

such that the real valued constant,

\[ \theta_2 = \cos^{-1} \left[ \frac{x_2}{\sqrt{x_2^2 + \lambda^2 y_1^2 + \lambda^2 z_1^2 + \lambda^2 w_1^2}} \right] \in [0, \pi], \]

and the integer, \( n_2 \in \mathbb{Z} \).

(3) Let us define a quaternion constant, \( w \), whose logarithm,

\[ \log(w) = \log(z_2) + \log(x_2) \quad (A-1), \]
\[ \exp(\log(w)) = \exp(\log(q_1) + \log(q_2)). \]

From Theorem 11-21, however, we recall that

\[ \exp(\log(q)) = q, \quad \forall q = x + iy + jz + k\] and thus, by the 'elementary substitution of variables' process,

\[ \exp(\log(w)) = w \quad \Rightarrow \quad w = \exp(\log(q_1) + \log(q_2)) \quad (A-2). \]

Moreover, let us construct two quaternion hypercomplex functions which are respectively denoted as

\[ \phi_1(q) = \log(q) \quad (A-3); \]

\[ \phi_2(q) = \log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 z^2 + \lambda^2 \tilde{y}^2 + \tilde{z}^2 - \tilde{x}^2}\right) + \]

\[ \left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy + jz + k\tilde{x}}{\sqrt{y^2 + z^2 + \tilde{x}^2}}\right](\Psi + 2n_3\pi) \quad (A-4), \]

such that the real variable function,

\[ \Psi = \text{cos}^{-1}\left[\frac{x + z_2 - \tilde{x}_2}{\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 \tilde{z}^2 + \lambda^2 \tilde{y}^2 + \tilde{z}^2 - \tilde{x}^2}}\right] \in [0, \pi], \]

and the integer, \( n_3 \in \mathbb{Z} \).
In accordance with Theorem T1-7, it subsequently follows that

\[ \exp(\phi_1(y) + \phi_2(y)) = \]

\[ \exp \left[ \log(y) + \log(\sqrt{x^2 + \lambda y^2 + \lambda^2 x^2 + \lambda^2 y^2 + x_2^2 - x_1^2}) + \right. \]

\[ \left. \left( \frac{\lambda}{|\lambda|} \frac{iy + j\hat{e} \cdot \hat{e} \cdot y_3}{\sqrt{y^2 + \hat{e} \cdot \hat{e} \cdot y^2}} \right) (\Psi + 2\pi n) \right] \]

\[ = \exp \left[ \log(\sqrt{x^2 + y^2 + \hat{e} \cdot \hat{e} \cdot y^2}) + \left( \frac{iy + j\hat{e} \cdot \hat{e} \cdot y_3}{\sqrt{y^2 + \hat{e} \cdot \hat{e} \cdot y^2}} \right) (\Theta + 2\pi n) \right. \]

\[ \left. + \log(\sqrt{x^2 + \lambda y^2 + \lambda^2 x^2 + \lambda^2 y^2 + x_2^2 - x_1^2}) + \right. \]

\[ \left. \left( \frac{\lambda}{|\lambda|} \frac{iy + j\hat{e} \cdot \hat{e} \cdot y_3}{\sqrt{y^2 + \hat{e} \cdot \hat{e} \cdot y^2}} \right) (\Psi + 2\pi n) \right] \]

\[ = \exp \left[ \log(\sqrt{x^2 + y^2 + \hat{e} \cdot \hat{e} \cdot y^2}) + \left( \frac{iy + j\hat{e} \cdot \hat{e} \cdot y_3}{\sqrt{y^2 + \hat{e} \cdot \hat{e} \cdot y^2}} \right) (\Theta + 2\pi n) \right] \times \]

\[ \exp \left[ \log(\sqrt{x^2 + \lambda y^2 + \lambda^2 x^2 + \lambda^2 y^2 + x_2^2 - x_1^2}) + \right. \]

\[ \left. \left( \frac{\lambda}{|\lambda|} \frac{iy + j\hat{e} \cdot \hat{e} \cdot y_3}{\sqrt{y^2 + \hat{e} \cdot \hat{e} \cdot y^2}} \right) (\Psi + 2\pi n) \right] \]

(A-5).

Now, by setting

\[ x = x_1; \quad y = y_1; \quad \hat{e} = \hat{e}_1; \quad \hat{y} = \hat{y}_1 \text{ and } n = n_1 \]
and hence substituting these values into Eqs. (A-3), (A-4) & (A-5), we further obtain

\[
\phi_1(y) = \phi_1(x_1 + iy_1 + j\xi_1 + k\eta_1) = \log \left(x_1 + iy_1 + j\xi_1 + k\eta_1\right) = \log (q_1)
\]

\[
\phi_2(y) = \phi_2(x_1 + iy_1 + j\xi_1 + k\eta_1) = \log \left(\sqrt{x_1^2 + \lambda^2 y_1^2 + \lambda^2 \xi_1^2 + \lambda^2 \eta_1^2} \right) + \left(\frac{\lambda}{|1\lambda|} \frac{iy_1 + j\xi_1 + k\eta_1}{\sqrt{y_1^2 + \xi_1^2 + \eta_1^2}}\right) \left(\Theta_2 + 2\pi n_2\right)
\]

\[
= \log (q_2)
\]

\[
\exp(\phi_1(y) + \phi_2(y)) = \exp(\phi_1(x_1 + iy_1 + j\xi_1 + k\eta_1) + \phi_2(x_1 + iy_1 + j\xi_1 + k\eta_1)) = \exp \left(\log \left(\sqrt{x_1^2 + y_1^2 + \xi_1^2 + \eta_1^2}\right) + \left[\frac{iy_1 + j\xi_1 + k\eta_1}{\sqrt{y_1^2 + \xi_1^2 + \eta_1^2}}\right] \left(\Theta_2 + 2\pi n_2\right)\right) \times \exp \left[\log \left(\sqrt{x_1^2 + \lambda^2 y_1^2 + \lambda^2 \xi_1^2 + \lambda^2 \eta_1^2}\right) + \left(\frac{\lambda}{|1\lambda|} \frac{iy_1 + j\xi_1 + k\eta_1}{\sqrt{y_1^2 + \xi_1^2 + \eta_1^2}}\right) \left(\Theta_2 + 2\pi n_2\right)\right]
\]

\[
= \exp(\log(q_1)) \exp(\log(q_2))
\]
\[ \exp(\log(q) + \log(q_2)) = \exp(\log(q)) \exp(\log(q_2)) \quad (A-6), \]

since the real variable functions,

\[ \Theta = \cos^{-1}\left[ \frac{x}{\sqrt{x^2 + y^2 + \frac{\lambda^2}{\lambda_1^2} y^2 + \frac{\lambda^2}{\lambda_1^2} y^2}} \right] = \cos^{-1}\left[ \frac{x_1}{\sqrt{x_1^2 + y_1^2 + \frac{\lambda^2}{\lambda_1^2} y_1^2}} \right] = \Theta_1 \]

AND

\[ \Psi = \cos^{-1}\left[ \frac{x + x_2 - x_1}{\sqrt{x_1^2 + \lambda_2^2 y_2^2 + \lambda_2^2 y_2^2 + \frac{\lambda^2}{\lambda_1^2} y_2^2 + x_2^2 - x_1^2}} \right] \\
= \cos^{-1}\left[ \frac{x_2}{\sqrt{x_2^2 + \lambda_2^2 y_2^2 + \lambda_2^2 y_2^2 + \frac{\lambda^2}{\lambda_1^2} y_2^2}} \right] = \Theta_2, \]

after making the relevant algebraic substitutions into Eqs. (iii) & (iv). Once again, from Theorem T1-21, we deduce that

\[ \exp(\log(q)) = q \quad \Longrightarrow \quad \begin{cases} 
\exp(\log(q_1)) = q_1, \\
\exp(\log(q_2)) = q_2.
\end{cases} \]

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and thus Eq. (A-6) may be written as

\[ \exp(\log(q_1) + \log(q_2)) = q_1 q_2 \quad (A-7). \]

Finally, Eqs. (A-2) & (A-7) can be combined into a single statement, namely—
\( w = q_1 q_2 \),

which is the direct substitution of this value for \( w \) into Eq. (A-1) likewise yields:

\[
\log(w) = \log(q_1) + \log(q_2), \quad \text{as required.} \quad \text{Q.E.D.}
\]

(6) Let us define a quaternion constant, \( W \), whose logarithm,

\[
\log(W) = \log(q_1) - \log(q_2) \quad \text{(B-1),}
\]

\[
\therefore \exp(\log(W)) = \exp(\log(q_1) - \log(q_2)).
\]

From Theorem TI-21, however, we recall that

\[
\exp(\log(q_1)) = q, \quad Vq = x + iy + jz + k \in \mathbb{H} \setminus \{0\},
\]

and thus, by the "dummy substitution of variables" process,

\[
\exp(\log(W)) = W \quad \text{\( \implies \)}
\]

\[
W = \exp(\log(q_1) - \log(q_2)) \quad \text{(B-2).}
\]

Moreover, we recall from part (6) of this proof that the quaternion functions,

\[
\log(q) = \log(q_1)
\]

\[
= \log(\sqrt{x^2 + y^2 + z^2 + w^2}) + \left[ \frac{iy + jz + k}{\sqrt{y^2 + z^2 + w^2}} \right] (\Theta + 2\pi n)
\]
\[
\phi_2(y) = \log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 x^2 + \lambda^2 y^2 + x_2^2 - x_1^2}\right) + \\
\left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy + jz + k\lambda}{\sqrt{z^2 + x^2 + y^2}}\right] (\Psi + 2\pi)
\]

\[
\therefore \phi_2(y) = -\log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 x^2 + \lambda^2 y^2 + x_2^2 - x_1^2}\right) - \\
\left(\frac{\lambda}{|\lambda|}\right)\left[\frac{iy + jz + k\lambda}{\sqrt{z^2 + x^2 + y^2}}\right] (\Psi + 2\pi)
\]

\[
= U_2(x, y, z, \tilde{y}) + \left[\frac{iy + jz + k\lambda}{\sqrt{z^2 + x^2 + y^2}}\right] U_2(x, y, z, \tilde{y})
\]

\[
= \phi_2^*(y) \quad (B-4),
\]

Insofar as the real variable functions,

\[
U_1(x, y, z, \tilde{y}) = \log\left(\sqrt{x^2 + y^2 + \tilde{x}^2 + \tilde{y}^2}\right);
\]

\[
U_1(x, y, z, \tilde{y}) = \Theta + 2\pi i;
\]

\[
U_2(x, y, z, \tilde{y}) = -\log\left(\sqrt{x^2 + \lambda^2 y^2 + \lambda^2 x^2 + \lambda^2 y^2 + x_2^2 - x_1^2}\right);
\]

\[
U_2(x, y, z, \tilde{y}) = -\left(\frac{\lambda}{|\lambda|}\right)(\Psi + 2\pi i);
\]
\[ \Theta = \pi \tan^{-1} \left[ \frac{x}{\sqrt{x^2 + y^2 + z^2 + \lambda^2}} \right] \in [0, \pi); \]

\[ \Psi = \pi \tan^{-1} \left[ \frac{x + z - x_i}{\sqrt{x^2 + x_i^2 + \lambda^2 x_i^2 + \lambda^2 y^2 + x_i^2 - x_i^2}} \right] \in [0, \pi), \]

and the integers, \( n, n_2 \in \mathbb{Z} \).

In accordance with Theorem TI-7, it subsequently follows that the exponential function,

\[ \exp(\phi_1(q) + \phi_2^*(q)) = \exp(\phi_1(q)) \exp(\phi_2^*(q)) \]

\[ \therefore \exp(\phi_1(q) - \phi_2(q)) = \exp(\phi_1(q)) \exp(-\phi_2(q)) \quad (B-5). \]

Furthermore, we deduce from Theorem TI-6 that the formula,

\[ \exp(nq) = (\exp(q))^n, \text{ where } n \in \mathbb{Z} \text{ and } q \in \text{dom} (\exp) \subseteq \mathbb{H}, \]

likewise implies that for any quaternion variable, \( Q^* \in \text{dom} (\exp) \subseteq \mathbb{H} \), the formula,

\[ \exp(nQ^*) = (\exp(Q^*))^n \quad (B-6), \]

is also valid. By setting \( n = -1 \) and \( Q^* = \phi_2(q) \), we can rewrite \( \theta_i \) (B-6) as

\[ \exp(-\phi_2(q)) = (\exp(\phi_2(q)))^{-1} \]
and hence Eq. (B-5) can be rewritten as

$$\exp(\phi_1(q) - \phi_2(q)) = \exp(\phi_1(q)) \exp(\phi_2(q))^{-1}$$  \hspace{1cm} (B-7).

From part (a) of this proof, we recall that for any quaternion constant,

$$q_i = x_i + i y_i + j z_i + k w_i \in \mathbb{H} \setminus \{0\},$$

$$\phi_i(q_i) = \log(q_i) \& \phi_i(q_i) = \log(q_i) \quad (q_i = x_i + i y_i + j z_i + k w_i \in \mathbb{H} \setminus \{0\}),$$

and, after making the appropriate algebraic substitutions into Eq. (B-7), we then obtain

$$\exp(\phi_1(q) - \phi_2(q)) = \exp(\phi_1(q)) \exp(\phi_2(q))^{-1}.$$  \hspace{1cm} (B-8).

Once again, from Theorem TI-21, we deduce that

$$\exp(\log(q_i)) = q_i \quad \rightarrow \quad \left\{ \begin{array}{l}
\exp(\log(q_1)) = q_1; \\
\exp(\log(q_2)) = q_2
\end{array} \right.$$}

whereupon Eq. (B-8) may be written as

$$\exp(\log(q_1)) - \log(q_2) = q_1 q_2^\dagger$$  \hspace{1cm} (B-9).

Clearly, Eqs. (B-2) & (B-9) can be combined into a single statement, namely—
\[ W = g_1 g_2^{-1} , \]

insor a direct substitution of this value for \( W \) into Eq. (B-1) likewise yields

\[ \log(g_1 g_2^{-1}) = \log(g_1) - \log(g_2) \quad \text{(B-10)}. \]

def the quaternion constants,

\[ g_1 = x_1 + i y_1 + j z_1 + k \theta_1 = x_1 + \left( \frac{i y_1 + j z_1 + k \theta_1}{\sqrt{y_1^2 + z_1^2 + \theta_1^2}} \right) \sqrt{y_1^2 + z_1^2 + \theta_1^2} ; \]

\[ g_2 = x_2 + i y_2 + j z_2 + k \theta_2 = x_2 + \left( \frac{i y_2 + j z_2 + k \theta_2}{\sqrt{y_2^2 + z_2^2 + \theta_2^2}} \right) \sqrt{y_2^2 + z_2^2 + \theta_2^2} , \]

then, by virtue of Theorem TI-10, we can set

\[ g_1 = U_1 + Q V_1 \& g_2 = U_2 + Q V_2 \implies g_2^{-1} = g_2 / |g_2|^2 = \frac{U_2 + Q V_2}{|U_2 + Q V_2|^2} , \]

where the real-valued constants,

\[ U_1 = x_1 ; \quad V_1 = \sqrt{y_1^2 + z_1^2 + \theta_1^2} ; \quad U_2 = x_2 ; \quad V_2 = \frac{\lambda \sqrt{y_2^2 + z_2^2 + \theta_2^2}}{\sqrt{y_2^2 + z_2^2 + \theta_2^2}} , \]

and the quaternion constant,

\[ Q = \frac{i y_2 + j z_2 + k \theta_2}{\sqrt{y_2^2 + z_2^2 + \theta_2^2}} \implies Q^2 = -1 , \]
such that we obtain the quotient value,

\[
\frac{q_1}{q_2} = \frac{U_1 + QV_1}{U_2 + QV_2} = \frac{(U_1 + QV_1)(U_2 + QV_2)}{|U_2 + QV_2|^2} = \frac{(U_2 + QV_2)(U_1 + QV_1)}{|U_2 + QV_2|^2}
\]

\[= q_1^{-1}q_2 = q_2^{-1} \quad (B-11).\]

Finally, in view of this particular result, Eq. (B-10) may now be written as

\[
\log \left(\frac{q_1}{q_2}\right) = \log(q_1) - \log(q_2), \quad \text{as required.} \quad \text{Q.E.D.}
\]

We conclude our discussion of the logarithmic quaternion-hypercomplex function with the following remarks:

(a) The results of the preceding Theorems TI-20 & TI-22 are analogous with Eqs. (1-88) & (1-92) insofar as the complex number, \(i\), is replaced by the quaternion variable,

\[
\frac{iz + jk + k\ell}{\sqrt{y^2 + z^2 + y^2}}
\]

and the modulus, \(r = |z| = \sqrt{x^2 + y^2}\), is replaced by the modulus, \(r = \sqrt{z^2 + y^2 + z^2 + y^2}\).
(c) The algebraic properties associated in Theorem TI-23, however, are comparable to but not wholly analogous with Eqs. (1-93) & (1-94) in view of the restrictions placed on the quaternions $q_1$ and $q_2$, via the formulae,

$$\log( q_1 q_2 ) = \log(q_1) + \log(q_2) \quad (1-95)$$

$$\log( q_1 / q_2 ) = \log(q_1) - \log(q_2) \quad (1-96).$$

7. Variables raised to the Power of Fractional Indices; Quaternion Hypercomplex Exponents.

From complex variable analysis, we recall that any non-zero complex number $z$, raised to the power of a fractional index $1/n (n \in \mathbb{N})$, accordingly defines an irrational 'nth root' function $z^{1/n}$, which takes on 'n' distinct values. Indeed, Churchill et al. [1] provide us with the following definitive formula, namely -

$$z^{1/n} = \exp \left( \frac{\log(z)}{n} \right) = \sqrt[n]{r} \exp \left[ i \left( \frac{\theta + 2k\pi}{n} \right) \right]$$

$$\left( r = |z| > 0 ; \ -\pi < \theta < \pi ; \ k \in \{0, 1, 2, \ldots, n-1\} \right) \quad (1-97).$$

Furthermore, it should be noted that this particular multi-valued function also satisfies the property -

$$z^{mn} = (z^m)^n \quad (z \neq 0) \quad (1-98).$$

Hence, in view of Eqs. (1-97) & (1-98), we shall derive the quaternion analogue of $z^{1/n}$ by means of the next definition and theorem -
Definition DI-16.

Let there exist a quaternion variable,

\[ q = x + iy + jz + k \in \mathbb{H} - \{0\}. \]

The \( n \)th root of \( q \) is subsequently denoted as \( q^{1/n} \) and thus satisfies the property:

\[ (q^{1/n})^n = q, \quad \forall n \in \mathbb{N}, \text{ the set of natural numbers}. \]

Theorem TI-24.

Let there exist a quaternion variable,

\[ q = x + iy + jz + k \in \mathbb{H} - \{0\}. \]

In the circumstances, it may be shown that the \( n \)th root quaternion hypercomplex function,

\[ q^{1/n} = \sqrt[n]{\cos \left( \frac{\Theta + 2K\pi}{n} \right) + \frac{\left[ iy + jz + k \right]}{\sqrt{y^2 + z^2 + k^2}}} \text{sin} \left( \frac{\Theta + 2K\pi}{n} \right) \]

where the real variable functions,

\[ \Theta = \arccos \left[ \frac{x}{\sqrt{x^2 + y^2 + z^2 + k^2}} \right] \in [0, \pi] \quad \text{and} \quad r = \sqrt{x^2 + y^2 + z^2 + k^2} \]
and the integers,

\[ k \in \{0, 1, \ldots, (n-1)\} \quad \text{and} \quad n \in \mathbb{N}, \text{ the set of natural numbers}. \]

**PROOF:**

From Definition DI-16, we initially recall that the \( n \)th root of a quaternion number is defined by the equation,

\[ (q^{1/n})^n = q, \quad \forall n \in \mathbb{N}, \text{ the set of natural numbers}. \]

Furthermore, by virtue of the preceding Definitions DI-14 & DI-15 and Theorems TI-20 & TI-21, we deduce that

\[ (q^{1/n})^n = q = \exp(\log(q)) \]

\[ = e^{\left[ \log(\sqrt{x^2 + y^2 + z^2 + w^2}) + \left[ \frac{i x + j y + k z}{-\sqrt{x^2 + y^2 + z^2 + w^2}} \right] (2\pi n + \Theta) \right]}, \]

\[ \forall Z \in \{0, \pm 1, \pm 2, \ldots, \pm \infty\}, \]

such that the real variable function,

\[ \Theta = \cos^{-1} \left[ \frac{x}{\sqrt{x^2 + y^2 + z^2 + w^2}} \right] \in [0, \pi]. \]
Now let the $\sqrt[n]{\text{quaternion}}$ hyperspace complex function,

$$q^{\frac{1}{n}} = e^{(U + [\frac{\frac{i}{\sqrt{y^2 + z^2 + y^2}} + \frac{j}{\sqrt{1 + z^2 + y^2}} + \frac{k}{\sqrt{1 + z^2 + y^2}} + \frac{l}{\sqrt{1 + z^2 + y^2}}]V)},$$

such that the undefined variables, $U, V \in \mathbb{R}$.

Subsequently, in the light of Theorem TI-6, we perceive, after making the appropriate algebraic substitution, that

$$\left(e^{(U + [\frac{i}{\sqrt{y^2 + z^2 + y^2}} + \frac{j}{\sqrt{1 + z^2 + y^2}} + \frac{k}{\sqrt{1 + z^2 + y^2}} + \frac{l}{\sqrt{1 + z^2 + y^2}}]V)}\right)^n = e^{n(U + [\frac{i}{\sqrt{y^2 + z^2 + y^2}} + \frac{j}{\sqrt{1 + z^2 + y^2}} + \frac{k}{\sqrt{1 + z^2 + y^2}} + \frac{l}{\sqrt{1 + z^2 + y^2}}]V)} =
\left(nU + [\frac{i}{\sqrt{y^2 + z^2 + y^2}} + \frac{j}{\sqrt{1 + z^2 + y^2}} + \frac{k}{\sqrt{1 + z^2 + y^2}} + \frac{l}{\sqrt{1 + z^2 + y^2}}]nV\right) =
\left[n\log(\sqrt{x^2 + y^2 + z^2 + y^2}) + \frac{1}{\sqrt{y^2 + z^2 + y^2}}(2\pi + \theta)\right].$$

\[\rightarrow nU = n\log(\sqrt{x^2 + y^2 + z^2 + y^2}) \& nV = 2\pi + \theta\]

\[\therefore U = \frac{1}{n}\log(\sqrt{x^2 + y^2 + z^2 + y^2}) = \frac{1}{n}\log(\tau) = \log(\sqrt{\tau}),\]

and similarly,

\[V = \frac{2\pi + \theta}{n}.\]

Henceforth, we may write the $\sqrt[n]{\text{quaternion}}$ function, $q^{\frac{1}{n}}$, as

$$q^{\frac{1}{n}} = e^{(U + [\frac{i}{\sqrt{y^2 + z^2 + y^2}} + \frac{j}{\sqrt{1 + z^2 + y^2}} + \frac{k}{\sqrt{1 + z^2 + y^2}} + \frac{l}{\sqrt{1 + z^2 + y^2}}]V)}$$

$$= e^{U \left[\frac{i}{\sqrt{y^2 + z^2 + y^2}} + \frac{j}{\sqrt{1 + z^2 + y^2}} + \frac{k}{\sqrt{1 + z^2 + y^2}} + \frac{l}{\sqrt{1 + z^2 + y^2}}\right]V}$$

$$= e^{\log(\sqrt{\tau}) \left[\frac{i}{\sqrt{y^2 + z^2 + y^2}} + \frac{j}{\sqrt{1 + z^2 + y^2}} + \frac{k}{\sqrt{1 + z^2 + y^2}} + \frac{l}{\sqrt{1 + z^2 + y^2}}\right] \left(\frac{2\pi + \theta}{n}\right)}.$$
\[
\sqrt[n]{\cos \left( \frac{2\pi n + \Theta}{n} \right) + \left( \frac{ix + jz + \frac{iy_j}{n}}{\sqrt{x^2 + y^2 + z^2 + y_j^2}} \right) \sin \left( \frac{2\pi n + \Theta}{n} \right)}
\]

By virtue of Theorem TI-3, and whilst we had originally postulated that the integer, \( Z \in \{0, \pm 1, \pm 2, \ldots, \pm \infty\} \), it is also evident from real variable analysis that the periodicity of the trigonometric functions, \( \cos \left( \frac{2\pi n + \Theta}{n} \right) \) and \( \sin \left( \frac{2\pi n + \Theta}{n} \right) \), will always ensure that the \( n \)th root function, \( \sqrt[n]{\cdot} \), can only take on \( n \) distinct values whenever \( Z = 0, 1, \ldots, n-1 \).

In the circumstances, we now let the integer, \( K \in \{0, 1, \ldots, (n-1)\} \), act as a dummy substitute for the integer, \( \frac{Z}{(n-1)} \), and hence we conclude that the \( n \)th root function,

\[
\sqrt[n]{\cos \left( \frac{\Theta + 2K\pi}{n} \right) + \left( \frac{ix + jz + \frac{iy_j}{n}}{\sqrt{x^2 + y^2 + z^2 + y_j^2}} \right) \sin \left( \frac{\Theta + 2K\pi}{n} \right)}
\]

where the real variable functions,

\[\Theta = \cos^{-1} \left[ \frac{x}{\sqrt{x^2 + y^2 + z^2 + y_j^2}} \right] \in [0, \pi] \hspace{1em} \rho = \sqrt{x^2 + y^2 + z^2 + y_j^2} > 0,\]

and the integer,

\( K \in \{0, 1, \ldots, (n-1)\}, \hspace{1em} \forall n \in \mathbb{N}, \) the set of natural numbers,

as required. Q.E.D.
Churchill et al. [1] have defined the multi-valued complex exponent functions, $z^z$ and $c^z$, in terms of the following formulas:

$$z^z = \exp(c \log(z)) \quad (z \neq 0) \quad (1-99);$$
$$c^z = \exp(z \log(c)) \quad (c \neq 0) \quad (1-100),$$

where $c$ is some arbitrary complex constant. Needless to say, the purpose of our next definition is to enumerate the quaternion analogues of Eqs. (1-99) & (1-100):

**Definition DI-17.**

Let there exist two quaternions, $q'$ and $e'$, where $e'$ is some arbitrary constant. We accordingly define the existence of two quaternion hypercomplex exponent functions, $q'^{e'}$ and $e'^q$, such that

$$q'^{e'} = \begin{cases} 
\exp(c \log(q')) & (q \in \mathbb{H} \setminus \{0\} \& c \in \mathbb{H}) \\
\exp(c \log(q)) & \end{cases}$$

AND

$$e'^q = \begin{cases} 
\exp(q \log(e')) & (q \in \mathbb{H} \& c \in \mathbb{H} \setminus \{0\}) \\
\exp(c \log(q)) & \end{cases}.$$
We conclude our discussion of variables raised to the power of fractional indices and quaternion hypercomplex exponents with the following remarks:

(a) The algebraic property enunciated in Theorem TI-24 is analogous with Eq. (1-97) insofar as the complex number, \( i \), is replaced by the quaternion variable,

\[
\frac{x + iy + iz}{\sqrt{x^2 + y^2 + z^2}}
\]

and the modulus, \( r = |z| = \sqrt{x^2 + y^2} \), is replaced by the modulus, \( r = |z| = \sqrt{x^2 + y^2 + z^2} \).

(b) The algebraic properties enunciated in Definition DI-17, however, are comparable to but not wholly analogous with Eqs. (1-99) & (1-100) since the quaternion products, \( e \log(z) \), \( \log(q) e \), \( q \log(e) \) and \( \log(q) e \), are generally non-commutative, in other words:

\[
e \log(q) \neq \log(q) e \quad \& \quad q \log(e) \neq \log(e) q.
\]

3. The Inverse Trigonometric Functions.

From complex variable analysis, we recall that the inverse trigonometric functions, \( \sin^{-1}(z) \), \( \cos^{-1}(z) \) and \( \tan^{-1}(z) \), are defined in the following manner:

(c) the inverse sine function,

\[
\omega = \sin^{-1}(z) \quad \Rightarrow \quad z = \sin(\omega) \quad (1-101);
\]
(a) the inverse cosine function,

\[ w = \cos^{-1}(z) \quad \Rightarrow \quad z = \cos(w) \quad (1-102); \]

(b) the inverse tangent function,

\[ w = \tan^{-1}(z) \quad \Rightarrow \quad z = \tan(w) \quad (1-103). \]

Moreover, Churchill et al. [1] provide us with three supplementary formulas for \( \sin^{-1}(z) \), \( \cos^{-1}(z) \) and \( \tan^{-1}(z) \), namely:

\[ \sin^{-1}(z) = -i \log [i z + (1 - z^2)^{1/2}] \quad (1-104); \]

\[ \cos^{-1}(z) = -i \log [z + i(1 - z^2)^{1/2}] \quad (1-105); \]

\[ \tan^{-1}(z) = \frac{i}{2} \log \left( \frac{i + z}{i - z} \right) \quad (1-106). \]

Hence, the purpose of our next definition and theorem is to derive the quaternion analogues of Eqs. (1-101) \( \Rightarrow \) (1-106).

**Definition DI-18.**

Let there exist three inverse trigonometric quaternion hypercomplex functions, \( \sin^{-1}(q) \), \( \cos^{-1}(q) \) and \( \tan^{-1}(q) \), whereupon \( q = x + iy + jz + kj \epsilon \mathbb{H} \). In the circumstances, we postulate that:

(c) the inverse sine function,
\( w = \sin^{-1}(q) \implies q = \sin(w), \)

(6) the inverse cosine function,

\( w = \cos^{-1}(q) \implies q = \cos(w), \)

(7) the inverse tangent function,

\( w = \tan^{-1}(q) \implies q = \tan(w). \)

Theorem TI-25.

Let there exist the inverse trigonometric functions, \( \sin^{-1}(q), \cos^{-1}(q), \) and \( \tan^{-1}(q), \) as previously defined. Subsequently, we may prove that the following formulas, namely:

(a) \( \sin^{-1}(q) = -Q \log \left[ Qq + (1-q^2)^{1/2} \right], \)

(b) \( \cos^{-1}(q) = -Q \log \left[ q + Q(1-q^2)^{1/2} \right], \)

(c) \( \tan^{-1}(q) = \frac{1}{2} Q \log \left( \frac{Q+q}{Q-q} \right), \)

are likewise valid, insofar as the auxiliary function.
\[ Q = \frac{ix + jy + kz}{\sqrt{x^2 + y^2 + z^2}}. \]

**PROOF:**

From Definitions DI-10, DI-11 and DI-18, we initially recall that

(i) the inverse sine function,

\[ w = \sin^{-1}(q) \rightarrow q = \sin(w) = \frac{\exp(Q^* w) - \exp(-Q^* w)}{2Q^*}, \]

(ii) the inverse cosine function,

\[ w = \cos^{-1}(q) \rightarrow q = \cos(w) = \frac{\exp(Q^* w) + \exp(-Q^* w)}{2}, \]

(iii) the inverse tangent function,

\[ w = \tan^{-1}(q) \rightarrow q = \tan(w) = \frac{\sin(w)}{\cos(w)} \]

\[ = \frac{\exp(Q^* w) - \exp(-Q^* w)}{2Q^*}, \]

\[ = \frac{\exp(Q^* w) + \exp(-Q^* w)}{2}, \]

such that the auxiliary function,
\[ Q^* = \frac{ix_v + iy_v + h_2}{\sqrt{x_v^2 + y_v^2 + z_v^2}}, \quad \text{where} \quad w = u + iv + jw + kw. \]

We will now examine each of these inverse trigonometric functions separately.

(a) Let us define a variable, \( W_f \), such that

\[ W_f = w = \sin^{-1}(q) \quad \Rightarrow \quad q = \sin(w) = \sin(W_f); \]

\[ w = W_f = u + iv + jw + kw = U_1 + \left( \frac{ix + iy + h_2}{\sqrt{x^2 + y^2 + z^2}} \right) V_1 \]

\[ u_1 = U_1, \quad v_1 = \frac{y V_1}{\sqrt{y^2 + x^2 + z^2}}, \quad u_2 = \frac{\hat{x} V_1}{\sqrt{y^2 + x^2 + z^2}} \quad \text{and} \quad v_2 = \frac{\hat{y} V_1}{\sqrt{y^2 + x^2 + z^2}}; \]

where the variables, \( U_1, V_1 \in \mathbb{R}. \)

Hence, it follows that the auxiliary function,

\[ Q^* = \frac{ix_v + iy_v + h_2}{\sqrt{x_v^2 + y_v^2 + z_v^2}} \]

\[ = \frac{V_1}{IV_1} \left( \frac{ix + iy + h_2}{\sqrt{x^2 + y^2 + z^2}} \right) \]
\[ Q = \frac{i y + j x + k z}{\sqrt{x^2 + y^2 + z^2}}, \quad \text{where} \quad I^* = \frac{V_1}{|V_1|} = \pm 1 \implies I^{*2} = 1. \]

Subsequently, we may write

\[ q = \sin(W_q) = \frac{\exp(I^*QW_1) - \exp(-I^*QW_1)}{2I^*Q}. \]

\[ = -\frac{1}{2} I^*Q \left( \exp(I^*QW_1) - \exp(-I^*QW_1) \right) \]

\[ \therefore \exp(I^*QW_1) - \exp(-I^*QW_1) = 2I^*Qq = 2qI^*Q, \]

**by virtue of Theorem 11-10, such that**

\[ (\exp(I^*QW_1))^2 - \exp(-I^*QW_1) \exp(I^*QW_1) = 2I^*Qq \exp(I^*QW_1) \]

\[ \therefore (\exp(I^*QW_1))^2 - \exp(-I^*QW_1 + I^*QW_1) = 2I^*Qq \exp(I^*QW_1) \]

\[ \therefore (\exp(I^*QW_1))^2 - 2I^*Qq \exp(I^*QW_1) - \exp(0) = 0 \]

\[ \therefore (\exp(I^*QW_1))^2 - 2I^*Qq \exp(I^*QW_1) - 1 = 0 \]

\[ \therefore (\exp(I^*QW_1))^2 - 2I^*Qq \exp(I^*QW_1) = 1 \]

\[ \therefore (\exp(I^*QW_1))^2 - 2I^*Qq \exp(I^*QW_1) + I^{*2}Q^2q^2 = 1 + I^{*2}Q^2q^2 \]
\[
\begin{aligned}
\text{exp}(W_3) = I^*Q_3 \quad (N.B. \quad I^*Q^2 = -1) \\
\text{exp}(I^*Q_3) - I^*Q_3 = (1 - q^2)^{1/2} \\
\text{exp}(I^*Q_3) = I^*Q_3 + (1 - q^2)^{1/2} \quad \rightarrow \\
I^*QW_3 = \log[I^*Q_3 + (1 - q^2)^{1/2}] \\
-I^*Q^2W_3 = -I^*Q \log[I^*Q_3 + (1 - q^2)^{1/2}] \\
W_3 = -I^*Q \log[I^*Q_3 + (1 - q^2)^{1/2}].
\end{aligned}
\]

However, since we had originally postulated that the inverse function,

\[
W_3 = w = \sin^{-1}(q) \quad \rightarrow \quad q = \sin(w) = \sin(W_3),
\]

we therefore obtain the result,

\[
\sin^{-1}(q) = -I^*Q \log[I^*Q_3 + (1 - q^2)^{1/2}]. \quad (A-1)
\]

Let us now define another variable, \(W_2\), such that

\[
\begin{aligned}
\text{so} \quad W_2 = w = \sin^{-1}(q) \quad \rightarrow \quad q = \sin(w) = \sin(W_2); \\
\text{or} \quad W_2 = w_2 + iw_1 + jw_2 + kw_2 = U_2 + \left(\frac{iw_1 + jw_2 + kw_2}{\sqrt{y^2 + z^2 + w^2}}\right)k_2 \quad \rightarrow \\
U_2 = U_2, \quad v_1 = \frac{yV_3}{\sqrt{y^2 + z^2 + w^2}}, \quad v_2 = \frac{zV_3}{\sqrt{y^2 + z^2 + w^2}} \quad \text{and} \quad v_3 = \frac{wV_3}{\sqrt{y^2 + z^2 + w^2}}; \\
\text{with the variables,} \quad U_2 \in \mathbb{R} \quad \text{and} \quad V_3 \in [0, \infty).
\end{aligned}
\]
Hence it follows that the auxiliary function,

\[ Q^* = \frac{\text{i}y V_1 + \text{j}z V_3 + \text{k}^2 V_2}{\sqrt{y^2 + z^2 + \text{j}^2}} \]

\[ = \frac{V_2}{IV_2} \left( \frac{\text{i}y + \text{j}z + \text{k}^2}{\sqrt{y^2 + z^2 + \text{j}^2}} \right) \]

\[ = Q, \]

where \( Q = \frac{\text{i}y + \text{j}z + \text{k}^2}{\sqrt{y^2 + z^2 + \text{j}^2}} \) and \( V_2/IV_2 = 1, \)

since, as previously stated, \( V_2 = [0, \infty) \implies V_2 = 1V_2. \)

Subsequently, we may write

\[ q = \sin(W_2) = \frac{\exp(QW_2) - \exp(-QW_2)}{2Q} \]

\[ = -\frac{1}{2}Q (\exp(QW_2) - \exp(-QW_2)) \]

\[ \therefore \exp(QW_2) - \exp(-QW_2) = 2Qq = 2q Q, \]

by virtue of Theorem TI-10, such that

\[ (\exp(QW_2))^2 - \exp(-QW_2) \exp(QW_2) = 2Qq \exp(QW_2) \]
\[
\begin{align*}
\therefore (\exp(QW_2))^2 - \exp(-QW_2 + QW_2) &= 2Q_e \exp(QW_2) \\
\therefore (\exp(QW_2))^2 - 2Q_e \exp(QW_2) - \exp(0) &= 0
\end{align*}
\]

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\[
\begin{align*}
\therefore (\exp(QW_2))^2 - 2Q_e \exp(QW_2) - 1 &= 0 \\
\therefore (\exp(QW_2))^2 - 2Q_e \exp(QW_2) &= 1 \\
\therefore (\exp(QW_2))^2 - 2Q_e \exp(QW_2) + Q_e^2 &= 1 + Q_e^2 \\
\therefore (\exp(QW_2) - Q_e)^2 &= 1 - Q_e^2 \\
\therefore \exp(QW_2) - Q_e &= (1 - Q_e)^{1/2} \\
\therefore \exp(QW_2) &= Q_e + (1 - Q_e)^{1/2} \quad \rightarrow
\end{align*}
\]

\[
QW_2 = \log [Q_e + (1 - Q_e)^{1/2}]
\]

\[
\therefore -Q^2W_2 = -Q \log [Q_e + (1 - Q_e)^{1/2}]
\]

\[
W_2 = -Q \log [Q_e + (1 - Q_e)^{1/2}].
\]

However, since we had originally postulated that the inverse function,

\[
W_2 = \omega = \sin^{-1}(q) \quad \rightarrow \quad q = \sin(\omega) = \sin(W_2),
\]

we therefore obtain the result,
\[
\sin^{-1}(y) = -Q \log \left[ Qy + (1-y^2)^{1/2} \right]. \quad (A-2)
\]

We now wish to prove that
\[
\sin^{-1}(y) = -1*Q \log \left[ 1*Qy + (1-y^2)^{1/2} \right] = -Q \log \left[ Qy + (1-y^2)^{1/2} \right],
\]

where \( I^* = \pm 1 \). To do this, let us consider the separate cases where \( I^* = 1 \) and \( I^* = -1 \). Firstly, by putting \( I^* = 1 \), we observe that
\[
-1*Q \log \left[ 1*Qy + (1-y^2)^{1/2} \right] = -Q \log \left[ Qy + (1-y^2)^{1/2} \right].
\]

Secondly, by putting \( I^* = -1 \), we likewise note that
\[
-1*Q \log \left[ 1*Qy + (1-y^2)^{1/2} \right] = Q \log \left[ -Qy + (1-y^2)^{1/2} \right]
\]
\[
= -Q \left( -\log \left[ -Qy + (1-y^2)^{1/2} \right] \right)
\]
\[
= -Q \log \left[ \frac{1}{-Qy + (1-y^2)^{1/2}} \right] = -Q \log \left[ \frac{1}{(1-y^2)^{1/2} - Qy} \right],
\]

by virtue of Theorem TI-6, since it is evident that the exponential function,
\[
\exp \left( -\log \left[ -Qy + (1-y^2)^{1/2} \right] \right) = \exp \left( \log \left[ -Qy + (1-y^2)^{1/2} \right] \right)^{-1}
\]
\[
= \left[ -Qy + (1-y^2)^{1/2} \right]^{-1}
\]
\[
\frac{1}{-Q q + (1 - q^2)^{1/2}}
\]

\[
\implies -\log[-Q q + (1 - q^2)^{1/2}] = \log\left[\frac{1}{-Q q + (1 - q^2)^{1/2}}\right].
\]

Furthermore, let the dual-valued function,

\[(1 - q^2)^{1/2} = X + QY \quad (X, Y \in \mathbb{R}) \quad (A-3).\]

Subsequently, the square of this function,

\[1 - q^2 = (X + QY)^2\]

\[\therefore 1 - q^2 = x^2 + 2xyQ - y^2\]

\[\therefore 1 - (x + Q\sqrt{y^2 + x^2 + \hat{z}^2})^2 = x^2 + 2xyQ - y^2\]

\[\therefore 1 - (x^2 + 2x\sqrt{y^2 + x^2 + \hat{z}^2}Q - (y^2 + z^2 + \hat{z}^2)) = x^2 - y^2 + 2xyQ\]

\\
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\\
\[\therefore 1 - x^2 - 2x \sqrt{y^2 + x^2 + \hat{z}^2}Q + y^2 + z^2 + \hat{z}^2 = x^2 - y^2 + 2xyQ\]

\[\therefore 1 - x^2 + y^2 + z^2 + \hat{z}^2 = 2x \sqrt{y^2 + x^2 + \hat{z}^2}Q = x^2 - y^2 + 2xyQ.\]

By equating the corresponding real and imaginary parts, we thus obtain the pair of simultaneous equations:

\[x^2 - y^2 = 1 - x^2 + y^2 + z^2 + \hat{z}^2 \quad \text{AND}\]

\[x^2 + y^2 + z^2 + \hat{z}^2 = 2x \sqrt{y^2 + x^2 + \hat{z}^2}Q = x^2 - y^2 + 2xyQ.\]
\[ XY = -x \sqrt{y^2 + z^2 + \frac{1}{y^2}}. \]

The solution of these equations for \( X \) and \( Y \) leads us to conclude that Eq. (A-3) is a valid statement. In other words, \( (1 - z^2)^{1/2} Q_y \) will form a commutative product with \( Q_y \), in other words:

\[
Q_y (1 - z^2)^{1/2} = (1 - z^2)^{1/2} Q_y. \quad \text{[N.B.]} \quad Q_y = Q(x + Q \sqrt{y^2 + z^2 + \frac{1}{y^2}}) = -\sqrt{y^2 + z^2 + \frac{1}{y^2}} + Qx
\]

In the circumstances, we further deduce that

\[
-\log \left[ I * Q_y + (1 - z^2)^{1/2} \right] = -Q \log \left[ \frac{1}{(1 - z^2)^{1/2} - Qy} \right]
\]

\[
= -Q \log \left[ \frac{(1 - z^2)^{1/2} + Qy}{((1 - z^2)^{1/2} - Qy)((1 - z^2)^{1/2} + Qy)} \right]
\]

\[
= -Q \log \left[ \frac{Qy + (1 - z^2)^{1/2}}{1 - z^2 + (1 - z^2)^{1/2} Qy - Qy(1 - z^2)^{1/2} - Q^2 z^2} \right]
\]

\[
= -Q \log \left[ \frac{Qy + (1 - z^2)^{1/2}}{1 - z^2 + (1 - z^2)^{1/2} Qy - (1 - z^2)^{1/2} Qy + z^2} \right]
\]

\[
= -Q \log \left[ \frac{Qy + (1 - z^2)^{1/2}}{1} \right]
\]

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\[= -Q \log [Qy + (1 - z^2)^{1/2}] .\]
Clearly, we have proven that the function,

\[-I^* \log \left[ I^* \rho_q + (1-q^2)^{1/2} \right] = -\log \left[ \rho_q + (1-q^2)^{1/2} \right],\]

where \( I^* = \pm 1 \), and, in the light of Eqs. (A-1) and (A-2), it actually follows that the inverse sine function,

\[\sin^{-1}(q) = -I^* \log \left[ I^* \rho_q + (1-q^2)^{1/2} \right] = -\log \left[ \rho_q + (1-q^2)^{1/2} \right],\]

as required. Q.E.D.

(6) From Theorem T1-15, we recall that the trigonometric identity,

\[\cos(q) = \sin \left( \frac{\pi}{2} - q \right),\]

is valid for any \( q \in \mathbb{H}_1 \). Hence, it analogously follows that, for any \( w \in \mathbb{H}_1 \), the cosine function,

\[\cos(w) = \sin \left( \frac{\pi}{2} - w \right).\]

Moreover, we likewise note from Definition D1-18 that the inverse cosine function,

\[w = \cos^{-1}(q) \implies q = \cos(w),\]

\[\therefore \ q = \sin \left( \frac{\pi}{2} - w \right),\]

\[\therefore \ \frac{\pi}{2} - w = \sin^{-1}(q),\]

\[\therefore \ w - \frac{\pi}{2} = -\sin^{-1}(q),\]

\[\therefore \ w = \frac{\pi}{2} - \sin^{-1}(q).\]
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\]

\[ \frac{\pi}{2} - \left(-Q \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] \right) \]
\[ = \frac{\pi}{2} + Q \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] \]
\[ = -Q^2 \pi + Q \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] \]
\[ = -Q \left( Q \frac{\pi}{2} - \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] \right) \]
\[ = -Q \left( Q \left( 2n + \frac{1}{2} \right) \pi - \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] - Q \left( 2n \pi \right) \right) \]
\[ = -Q \left( \log \left( Q \right) - \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] - Q (2n \pi) \right) \]
\[ = -Q \left( \log \left( Q \right) - \left( \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] + Q (2n \pi) \right) \right) \]
\[ = -Q \left( \log \left( Q \right) - \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] \right) \]
\[ = -Q \log \left[ \frac{Q}{Qx + (1 - x^2)^{\frac{1}{2}}} \right], \]

by virtue of Theorem 11-23, since the dual-valued function,
\[ Qx + (1 - x^2)^{\frac{1}{2}} = x - \sqrt{y^2 + x^2 + \frac{\pi^2}{4}} + Q(x + y), \]
as previously stated in the proof of part (5) of this theorem [viz. Eq.(3)],
and hence the logarithmic function,
\[ \log \left[ Qx + (1 - x^2)^{\frac{1}{2}} \right] = \log \left[ x - \sqrt{y^2 + x^2 + \frac{\pi^2}{4}} + Q(x + y) \right]. \]
\[ \log \left[ \frac{1}{(x + y)} \left( \cos^{-1} \left( \frac{x - \sqrt{x^2 + z^2}}{\sqrt{(x - \sqrt{x^2 + z^2})^2 + (x + y)^2}} \right) + 2\pi \right) \right] \quad (\text{NeZ}), \]

as a direct consequence of Theorem TI-20 and Definition DI-15. In view of
To be continued via the author's next submission, namely-

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PART 5/10.”