Refutation of the Banach-Tarski paradox

Abstract: We evaluate the crucial claim of the proof in Step 3, as a fleshed out detail. It is not tautologous, nor is it contradictory. This means the claim is a non tautologous fragment of the universal logic VL4 and constitutes the briefest known refutation of the Banach-Tarski paradox.

We assume the method and apparatus of Meth8/VL4 with Tautology as the designated proof value, F as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

Let:  ¬ Not, ¬;  + Or, ∨, ∪;  - Not Or;  & And, ∧, ∩;  \ Not And;
> Imply, greater than, →, ⇒, ⊃, ⊢, ⊩;
< Not Imply, less than, ∈, ⊂, ⊆, ⊊, ⊆;
= Equivalent, ≡, ≐, ↔, ≡, ≣, ≃, ≉, ≈;
% possibility, for one or some, ∃, ◊, M;  # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, T, ordinal 3;  (z@z) F as contradiction, ∅, Null, ⊥, zero;
(%z<#z) C as contingency, ∆, ordinal 1;  (%z>#z) N as non-contingency, ∇, ordinal 2;
¬( y < x) ( x ≤ y), ( x  y);   (A=B) (A~B).
Note: For clarity we usually distribute quantifiers on each variable as designated.

From: en.wikipedia.org/wiki/Banach–Tarski_paradox

Some details, fleshed out ... [for Step 3 of 4]
What remains to be shown is the Claim: S^2 − D is equidecomposable with S^2.
Proof. Let λ be some line through the origin that does not intersect any point in D. This is possible since D is countable. Let J be the set of angles, α, such that for some natural number n, and some P in D, r(nα)P is also in D, where r(nα) is a rotation about λ of nα. Then J is countable. So there exists an angle θ not in J. Let ρ be the rotation about λ by θ. Then ρ acts on S^2 with no fixed points in D, i.e., ρ^p(D) is disjoint from D, and for natural n<m, ρ^n(D) is disjoint from ρ^m(D). Let E be the disjoint of ρ^n(D) over n = 0, 1, 2, ... Then

S^2 = E ∪ (S^2 − E) ~ ρ(E) ∪ (S^2 − E) = (E − D) ∪ (S^2 − E) = S^2 − D,

(3.1)

LET  p, q, r, s:  E, D, ρ, S^2

(s=((p+(s-p))= ( (r&p)+(s-p))))=(( (p-q)+(s-p))=(s-q))  ;

FFTT FTTT FTTF FTF FTTT

(3.2)

where ~ denotes "is equidecomposable to".

Remark 3.2: We write "~" as "equivalent to". Eq. 3.2 as rendered is not tautologous. Because it is the crucial claim of the proof, the result is that the Banach-Tarski paradox is also not contradictory, and hence a nontautologous fragment of the universal logic VL4.