A demonstration of the Titius–Bode law and the number of Saturn's rings by Newtonian methods using the Kerr-Newman solution of the general relativity theory

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Abstract

The beautiful Titius–Bode law ($\xi = 0.4 + 0.3 \times 2^{\mathrm{n}}$) , discovered 250 years ago, is considered to be a mathematical coincidence rather than an "exact" law, because it has not yet been physically proved. However, considering the disturbance restoration and the stability of the asteroid belt orbit, there must be some underlying logical necessity.

Planetary orbits are often computed by Newtonian mechanics with the kinetic energy and the universal gravitation energy. However, applying the principle of energy-minimum to the Newtonian mechanics leads to the result that the stable orbital radius is only one value, which is totally incompatible with actual phenomena. This discrepancy must result from the shortage of elements which rule over the planetary orbits. Other elements to rule over the planetary orbits are the electric charge energy and the rotation energy, both of which are guided by the Kerr-Newman solution of the general relativity theory discovered in 1965. Here, I mathematically demonstrated the Titius–Bode law, and also calculated the number of Saturn's rings, maximum 35, by applying the principle of energy-minimum of Newtonian methods to the complicated energy equation which adopts mass, electric charge and rotation elements of the central core star such as the Sun.

Keywords

Demonstration; Titius–Bode law; Saturn's rings number; energy stable orbits; Kerr-Newman solution; relativity theory.

1. Introduction

The Titius–Bode law, discovered 250 years ago, is considered to be a mathematical coincidence rather than an "exact" law [1], because it has not yet been physically proved. However, considering the disturbance restoration and the stability of the asteroid belt orbit, there must be some underlying logical necessity. Here, by Newtonian methods using the Kerr-Newman solution of the general relativity theory, I demonstrate the Titius-Bode law and apply its solution method to the calculation of the number of Saturn's rings. This is a mathematical equation calculation. The detailed analysis process is provided in a separate sheet of paper [2].

2. Methods

The outline of the solution method and the key equation numbers in this article are as follows.

1) The equation for energy in the space-time field is obtained from the Kerr-Newman solution, a strict solution of the Einstein equations of the general relativity theory.

$$f_1(\rho, \theta, d\rho/dt, d\theta/dt, d\varphi/dt, \varepsilon) = 0 \qquad \text{(eq. 3)}$$

2) This energy equation is partially differentiated by θ to the minimum energy. The result is $\theta=\pi/2$, and the calculation below proceeds at $\theta=\pi/2$, i.e., on the equatorial plane.

$$f_2(\rho, \pi/2, d\rho/dt, 0, d\varphi/dt, \varepsilon) = 0$$

3) The angular momentum equivalent J is obtained by applying the variational principle to the Kerr-Newman solution to calculate $d\varphi/dt$.

$$\xi(\rho, d\varphi/dt, J) = 0 \qquad \text{(eq. 6)}$$

4) Because an additional radius is $d\rho = 0$ at the aphelion and perihelion distances R, the calculation below proceeds at distance R.

$$f_3(R, \pi/2, 0, 0, d\varphi/dt, \varepsilon) = 0 \qquad \text{(eq. 7)}$$

5) Substituting $d\varphi/dt$ from $\xi = 0$ into $f_3 = 0$ results in a relational expression of the radius, the angular momentum equivalent, and the energy.

$$f_4(R, \pi/2, 0, 0, J, \varepsilon) = 0 \qquad \text{(eq. 9)}$$

6) The orbital distance R is determined by the energy and the angular momentum equivalent, i.e., $R = R(\varepsilon, J)$. R is partially differentiated by ε , that is, f_4 is partially differentiated by ε.

$$g(R, J, \varepsilon, \partial R/\partial \varepsilon) = 0 \qquad \text{(eq. 10)}$$

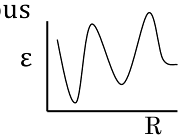
7) Derive the angular momentum equivalent J from $f_4(R, \pi/2, 0, 0, J, \varepsilon) = 0$ and substitute it into $g(R, J, \varepsilon, \partial R/\partial \varepsilon) = 0$. Make this into an important differential equation which is just composed of the radius and the energy to analyze unique characteristics of orbits.

$$h(R, \varepsilon, d\varepsilon/dR) = 0 \qquad \text{(eq. 11)}$$

8) Solving the differential equation h results in a complicated set of *arctan*, *log*, power functions and an integration constant K.

$$H(R, \varepsilon, K) = 0 \qquad \text{(eq. 14)} \quad \text{(eq. 15)}$$

9) By using that the minimum energy is $d\varepsilon/dR = 0$ in $h(R, \varepsilon, d\varepsilon/dR) = 0$, following simultaneous equations are obtained and solved.



$$h(r, \varepsilon_{min}, 0) = 0 \quad ① \qquad H(r, \varepsilon_{min}, K) = 0 \quad ② \qquad \text{(eq. 16)}$$

10) Because the integration constant K is common to all orbits, the Titius–Bode law is demonstrated and also the number of Saturn's rings is calculated.

$$I(r, K) = 0 \qquad \text{(eq. 22)} \quad \text{(eq. 23)}$$

2.1. The Energy Equation

2.1.1. Introduction to the Energy Equation

There are two preconditions for the following analysis.

1) The analysis object must be sufficiently far from the center of mass.

2) The rotation speed of the center of mass must not be too fast. The characteristic Boyer-Lindquist coordinates in the Kerr solution are equal to general polar coordinates in the first-order term a/ρ[3].

The strict Boyer-Lindquist metric of the Kerr-Newman geometry [4] is as follows.

$$ds^2 = -\frac{R^2\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{R^2\Delta}dr^2 + \rho^2 d\theta^2 + \frac{R^4\sin^2\theta}{\rho^2}\left(d\phi - \frac{a}{R^2}dt\right)^2$$

At the large radius r, the Boyer-Lindquist metric is as follows.

$$ds^2 \to -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4aM\sin^2\theta}{r}dtd\phi + \left(1 + \frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi\right)$$

Symbols are changed from the Boyer-Lindquist metric to the general polar coordinate.

The Kerr-Newman solution of the general relativity theory is given by (eq.1). In this expression, m, a and e are the mass, rotation and electric charge elements respectively.

$$(1)\quad ds^2 = \left(1 - \frac{2m\rho - e^2}{\rho^2 + a^2\cos^2\theta}\right)(cdt)^2 - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho + e^2}d\rho^2 - (\rho^2 + a^2\cos^2\theta)d\theta^2$$

$$-\left[(\rho^2 + a^2) + \frac{(2m\rho - e^2)a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right]\sin^2\theta\, d\varphi^2 - \frac{2(2m\rho - e^2)a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}cdt \cdot d\varphi$$

Γ is as follows when ds is divided by the time elements ($c\,dt$).

$$\frac{1}{\Gamma^2} = \left(\frac{ds}{cdt}\right)^2$$

The Lorentz transformation factor $\gamma\,(= c\,dt/ds)$ in the Minkowski space-time of the special relativity theory is an important component of the energy $E = M c^2 = M_0\,\gamma\,c^2$. $\Gamma\,(= c\,dt/ds)$ of the Kerr-Newman solution of the general relativity theory is analogous to γ.

On this occasion, by following the principle of minimum energy, the sign of m is changed to $-m$, a is changed to $+a$, and e is changed to $+e$. Therefore, the energy equation is $E = \Gamma(\rho, \theta, \varphi, t, -m, a, e)$.

$$(2)\quad \frac{1}{E^2} = \left(1 + \frac{2m\rho + e^2}{\rho^2 + a^2\cos^2\theta}\right) - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 + 2m\rho + e^2}\left(\frac{d\rho}{cdt}\right)^2 - (\rho^2 + a^2\cos^2\theta)\left(\frac{d\theta}{cdt}\right)^2$$

$$-\left[(\rho^2 + a^2) - \frac{(2m\rho + e^2)a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right]\sin^2\theta\left(\frac{d\varphi}{cdt}\right)^2 + \frac{2(2m\rho + e^2)a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\left(\frac{d\varphi}{cdt}\right)$$

Since E has a decisive massive energy $M_0 c^2$, it is converted into ε $1/E^2 = 1 - 2\varepsilon$ in (eq.3).

$$(3)\quad -2\varepsilon = \frac{2m\rho + e^2}{\rho^2 + a^2\cos^2\theta} - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 + 2m\rho + e^2}\left(\frac{d\rho}{cdt}\right)^2 - (\rho^2 + a^2\cos^2\theta)\left(\frac{d\theta}{cdt}\right)^2$$

$$-\left[(\rho^2 + a^2) - \frac{(2m\rho + e^2)a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\right]\sin^2\theta\left(\frac{d\varphi}{cdt}\right)^2 + \frac{2(2m\rho + e^2)a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\left(\frac{d\varphi}{cdt}\right)$$

Partial differentiation is used to minimize the energy $\varepsilon(\rho,\ \theta,\ \varphi,\ t)$ by using $\partial\varepsilon/\partial\theta = 0$.

$$\left\{\begin{aligned} &\frac{(2m\rho + e^2)a^2}{(\rho^2 + a^2\cos^2\theta)^2} + \frac{a^2}{\rho^2 + a^2 + 2m\rho + e^2}\left(\frac{d\rho}{cdt}\right)^2 + a^2\left(\frac{d\theta}{cdt}\right)^2 \\ &-\left[(\rho^2 + a^2) - \frac{(2m\rho + e^2)2a^2\sin^2\theta}{\rho^2 + a^2\cos^2\theta} - \frac{(2m\rho + e^2)a^4\sin^4\theta}{(\rho^2 + a^2\cos^2\theta)^2}\right]\left(\frac{d\varphi}{cdt}\right)^2 \\ &+\left[\frac{2(2m\rho + e^2)a}{\rho^2 + a^2\cos^2\theta} + \frac{2(2m\rho + e^2)a^3\sin^2\theta}{(\rho^2 + a^2\cos^2\theta)^2}\right]\left(\frac{d\varphi}{cdt}\right) \end{aligned}\right\} \cdot \sin 2\theta = 0$$

That is, the energy E and ε are minimized at $\theta = \pi/2$ and the planets gather on the equatorial plane where the energy is stable.

2.1.2. Time component from the variational principle

When the rotation speed of the center of mass is not too fast, the Kerr-Newman solution expanded in the first order of a/ρ takes the form given in (eq.4):

$$(4)\quad \left(\frac{ds}{ds}\right)^2 = 1 = \left(1 - \frac{2m}{\rho} + \frac{e^2}{\rho^2}\right)\left(\frac{cdt}{ds}\right)^2 - \frac{1}{1 - \frac{2m}{\rho} + \frac{e^2}{\rho^2}}\left(\frac{d\rho}{ds}\right)^2 - \rho^2\left(\frac{d\theta}{ds}\right)^2 - \rho^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2$$

$$-\frac{2a}{\rho}\left(2m - \frac{e^2}{\rho}\right)\sin^2\theta\left(\frac{cdt}{ds}\right)\left(\frac{d\varphi}{ds}\right)$$

The Euler–Lagrange equation [5] is adopted by applying the variational principle to the Kerr-Newman solution.

$$\delta\int\left[\left(1-\frac{2m}{\rho}+\frac{e^2}{\rho^2}\right)\left(\frac{cdt}{ds}\right)^2-\frac{1}{1-\frac{2m}{\rho}+\frac{e^2}{\rho^2}}\left(\frac{d\rho}{ds}\right)^2-\rho^2\left\{\left(\frac{d\theta}{ds}\right)^2+\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right\}\right.$$
$$\left.-\frac{2a}{\rho}\left(2m-\frac{e^2}{\rho}\right)\sin^2\theta\left(\frac{cdt}{ds}\right)\left(\frac{d\varphi}{ds}\right)\right]ds=0$$

Eventually, (eq.5) is obtained at the equatorial plane of the rotating center of mass where the energy is stable. Hereafter, I perform the calculation at the equatorial plane ($\theta=\pi/2$) of the rotating center of mass.

(5)
$$\begin{cases}\dfrac{d}{ds}\left[\left(1-\dfrac{2m}{\rho}+\dfrac{e^2}{\rho^2}\right)\left(\dfrac{cdt}{ds}\right)-\dfrac{a}{\rho}\left(2m-\dfrac{e^2}{\rho}\right)\left(\dfrac{d\varphi}{ds}\right)\right]=0 & \text{time component}\\ \dfrac{d}{ds}\left[\rho^2\left(\dfrac{d\varphi}{ds}\right)+\dfrac{a}{\rho}\left(2m-\dfrac{e^2}{\rho}\right)\left(\dfrac{cdt}{ds}\right)\right]=0 & \varphi\text{ component}\end{cases}$$

The two equations in (eq.5) are integrated over ds. $d\varphi/dt$ (eq. 6) with an integration variable J is obtained by using the resulting pair of simultaneous equations,

(6) $$\frac{d\varphi}{dt}=\frac{\left(\frac{d\varphi}{ds}\right)}{\left(\frac{dt}{ds}\right)}=\frac{J\left(\rho-2m+\frac{e^2}{\rho}\right)+a\left(\frac{e^2}{\rho}-2m\right)}{\rho^3+Ja\left(2m-\frac{e^2}{\rho}\right)}\cdot c$$

J : the angular momentum equivalent
(a kind of Carter constant in relativity theory)

The distance variables are defined as follows:

ρ : An arbitrary orbital distance in two- or three-dimensional coordinates.

R : The aphelion and perihelion distances at the equatorial plane of the rotating center of mass.

r: The aphelion and perihelion distances, both of which are energetically stable at the equatorial plane.

2.1.3. Introduction of the angular momentum equivalent

Because additional ρ at the aphelion and perihelion distances is $d\rho=0$, the energy equation is given by (eq.7).

(7) $$0=2\varepsilon+\frac{2m}{R}+\frac{e^2}{R^2}-R^2\left(\frac{d\varphi}{cdt}\right)^2+\frac{4a}{R}\left(m+\frac{e^2}{2R}\right)\left(\frac{d\varphi}{cdt}\right)$$

$d\varphi/cdt$ (eq. 6) is composed of the angular momentum equivalent and is substituted into (eq.7). J is obtained as in (eq.8) by adopting the secondary order R.

(8) $$J=\frac{4am+R\delta\sqrt{R(2\varepsilon R+2m+C)}}{R^2(R-2m+C)-a(2m-C)\delta\sqrt{R(2\varepsilon R+2m+C)}}R^2$$

Here, $\delta=\pm1$ and $C=e^2/R$. δ is related to the orbital rotation direction.

2.2. The Space Fantasy Differential Equation

2.2.1. Introduction of the Space Fantasy differential equation

It leads not to the numerical analysis but to the analytical unique characteristics. The relation of R, ε, and J are given as (eq.9) at the aphelion and perihelion distances R by changing the angular momentum equivalent J (eq. 8).

(eq.9) is far more complicated than the Kepler-Newton equation $2\varepsilon R^2+2mR-J^2=0$

$$(9)\quad 0 = 2\varepsilon + \frac{2m}{R} + \frac{e^2}{R^2} - R^2\left[\frac{J\left(R - 2m + \frac{e^2}{R}\right) + a\left(\frac{e^2}{R} - 2m\right)}{R^3 + Ja\left(2m - \frac{e^2}{R}\right)}\right]^2$$

$$+\frac{4a}{R}\left(m + \frac{e^2}{2R}\right)\left[\frac{J\left(R - 2m + \frac{e^2}{R}\right) + a\left(\frac{e^2}{R} - 2m\right)}{R^3 + Ja\left(2m - \frac{e^2}{R}\right)}\right]$$

Since the orbital distance R is determined by the energy ε and the angular momentum equivalent J, $R = R(\varepsilon, J)$. A new differential equation is given as (eq.10) by partially differentiating R by ε, then substituting J into this, and adopting the reciprocal of $\partial R/\partial\varepsilon$.

$$(10)\quad \frac{\partial\varepsilon}{\partial R}[R^3 + Ja(2m - C)]^2$$

$$= \frac{(m + C)[R^3 + Ja(2m - C)]^2}{R^2} + \frac{[J(R - 2m + C) - 2am + aC]\,[J(R - 2m + C) + 3aC]\cdot R}{1}$$

$$+\frac{2R^2[J(R - 2m + C) - 4am]\,[J^2a(m - C) - JR^2(R - 3m + 2C) + aR^2(3m - 2C)]}{R^3 + Ja(2m - C)}$$

Here, by substituting J of (eq. 8) into (eq.10), the second order R is obtained.

Through all these extensive calculation processes, the relation between ε and R is summarized as (eq.11).

$$(11)\quad \boldsymbol{\frac{d\varepsilon}{dR}R^4(R^2 - 4mR + 2CR + 4m^2)}$$

$$\boldsymbol{= mR^2(-R^2 + 8mR - 4CR - 12m^2) + \varepsilon\cdot 2R^3(-R^2 + 6mR - 4CR - 8m^2)}$$

$$\boldsymbol{+2am(2R^2 + 2mR - CR - 12m^2)\delta\sqrt{R(2\varepsilon R + 2m + C)}}$$

$$\boldsymbol{+\varepsilon\cdot 4aR(3mR - 2CR - 6m^2 + 7Cm)\delta\sqrt{R(2\varepsilon R + 2m + C)}} \qquad C = e^2/R \quad (R\ \ \text{2ry order})$$

I call this second order equation (eq. 11) "the Space Fantasy (SF) differential equation".

The change of variables is performed to solve the SF differential equation for S. The result is (eq.12).

$$S = R\sqrt{R(2\varepsilon R + 2m + C)}$$

$$(12)\quad \boldsymbol{\frac{dS}{dR} = \frac{2e^2(e^2 + 2m^2)}{SR} + \frac{4a\delta m + S}{R} + \frac{6a\delta m S^2}{R^5}} \qquad (R\ \ \text{0 order})$$

The form of the differential equation in (eq.12) is more complicated than the Riccati's differential equation, which never has an exact general solution [6]. Since $6a\delta m S^2/R^5$ is much smaller than S/R and $4a\delta m/R$, it can be treated as a constant θ. Also, (eq.12) can be reduced to the problem of an approximate differential equation, and it is given as (eq.13).

$$\frac{dS}{dR} = \frac{1}{S}\left[\frac{2E^4}{R} + \frac{4a\delta m S}{R}\left(1 + \frac{6S^2}{4R^4}\right) + \frac{S^2}{R}\right] \qquad E^4 = e^2(e^2 + 2m^2)$$

$$\fallingdotseq \frac{1}{S}\left[\frac{2E^4}{R} + \frac{4a\delta m S}{R}(1 + \theta) + \frac{S^2}{R}\right] \qquad \theta = \frac{3S_0^2}{2R_0^4} \quad (S_0^2, R_0^4 \text{ are centroids } S^2/3\ , R^4/5)$$

$$(13)\quad \frac{SdS}{S^2 + 4a\delta m S(1 + \theta) + 2E^4} = \frac{dR}{R}$$

Solving (eq.13) by the quadrature formulae [7] leads to (eq.14) and (eq.15).

In the case that the discriminant is $\Delta = E^4 - 2a^2m^2(1+\theta)^2 > 0$:

$$\frac{1}{2}\log[S^2 + 4a\delta m(1+\theta)S + 2E^4] - \frac{4a\delta m(1+\theta)}{2\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\arctan\left(\frac{2S + 4a\delta m(1+\theta)}{2\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\right)$$

$$= \log R + K$$

$$(14) \quad K = \frac{S^2 + 4a\delta m(1+\theta)S + 2E^4}{R^2} \cdot \mathrm{EXP}\left[\frac{-4a\delta m(1+\theta)}{\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\arctan\left(\frac{S + 2a\delta m(1+\theta)}{\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\right)\right]$$

In the case that the discriminant is $\Delta = E^4 - 2a^2m^2(1+\theta)^2 < 0$:

$$\log[S^2 + 4a\delta mS(1+\theta) + 2E^4]$$
$$- \frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}} \cdot \log\left[\frac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right] = 2\log R + K$$

$$(15) \quad K = \log\left[\frac{\frac{S^2 + 4a\delta mS(1+\theta) + 2E^4}{R^2}}{\left[\frac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]^{\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}}}\right]$$

2.2.2. Conditions of the energy minimum orbit

Since the minimum energy is $d\varepsilon/dR = 0$ in the SF differential equation (eq. 11), it is a cubic equation in ε.

$$\begin{aligned}
0 = {} & \varepsilon^3 \cdot 32a^2r^3(3mr - 2Cr - 6m^2 + 7Cm)^2 \\
& + \varepsilon^2 \cdot r^2 \begin{bmatrix} 16a^2(3mr - 2Cr - 6m^2 + 7Cm)^2(2m + C) \\ +32a^2m(2r^2 + 2mr - Cr - 12m^2)(3mr - 2Cr - 6m^2 + 7Cm) \\ -4r^3(-r^2 + 6mr - 4Cr - 8m^2)^2 \end{bmatrix} \\
& + \varepsilon \cdot 4mr \begin{bmatrix} 2a^2m(2r^2 + 2mr - Cr - 12m^2)^2 \\ +4a^2(2r^2 + 2mr - Cr - 12m^2)(3mr - 2Cr - 6m^2 + 7Cm)(2m + C) \\ -r^3(-r^2 + 8mr - 4Cr - 12m^2)(-r^2 + 6mr - 4Cr - 8m^2) \end{bmatrix} \\
& + m^2[4a^2(2r^2 + 2mr - Cr - 12m^2)^2(2m + C) - r^3(-r^2 + 8mr - 4Cr - 12m^2)^2]
\end{aligned}$$

Solve this cubic equation. A solution ε_{min} (eq. 16) very close to 0 is adopted in accordance with the principle of the energy minimum.

$$(16) \quad \varepsilon_{min} = \frac{-m}{4r} \cdot \frac{r^3(r^2 - 8mr + 4Cr + 12m^2)^2 - 4a^2(2m + C)(2r^2 + 2mr - Cr - 12m^2)^2}{\begin{bmatrix} r^3(r^2 - 8mr + 4Cr + 12m^2)(r^2 - 6mr + 4Cr + 8m^2) \\ -4a^2(2m + C)(2r^2 + 2mr - Cr - 12m^2)(3mr - 2Cr - 6m^2 + 7Cm) \\ -2a^2m(2r^2 + 2mr - Cr - 12m^2)^2 \end{bmatrix}}$$

$$\fallingdotseq \frac{-m}{4r} \qquad (r \ 0 \text{ order})$$

ε_{min} (eq. 16) is substituted into the change of variables $S = r\sqrt{r(2\varepsilon r + 2m + C)}$ (eq. 12).

$$S^2 = \frac{-mr^4}{2} \cdot \frac{r^4(r^2 - 8mr + 4e^2 + 12m^2)^2 - 4a^2(2mr + e^2)(2r^2 + 2mr - e^2 - 12m^2)^2}{\begin{bmatrix} r^5(r^2 - 8mr + 4e^2 + 12m^2)(r^2 - 6mr + 4e^2 + 8m^2) \\ -4a^2(2mr + e^2)(2r^2 + 2mr - e^2 - 12m^2)(3mr^2 - 2e^2r - 6m^2r + 7me^2) \\ -2a^2mr^2(2r^2 + 2mr - e^2 - 12m^2)^2 \end{bmatrix}}$$
$$+ r^2(2mr + e^2)$$

$$= \frac{r^4 \times [r^8 \text{ polynomial}] + r^2(2mr + e^2) \times [r^9 \text{ polynomial}]}{[r^9 \text{polynomial}]} = \frac{r^2 \times P}{Q}$$

$$\fallingdotseq \frac{3m}{2} r^3 \qquad (r \ \ 0 \text{ order})$$

Here, P and Q are given by (eq.17) and (eq.18).

(17) $$P = -mr^2/2\,[\,r^4(r^2 - 8mr + 4e^2 + 12m^2)^2 - 4a^2(2mr + e^2)(2r^2 + 2mr - e^2 - 12m^2)^2\,] + (2mr + e^2) \times Q \qquad [r^{10} \text{ polynomial}]$$

(18) $$Q = r^5(r^2 - 8mr + 4e^2 + 12m^2)(r^2 - 6mr + 4e^2 + 8m^2) - 4a^2(2mr + e^2)(2r^2 + 2mr - e^2 - 12m^2)(3mr^2 - 2e^2r - 6m^2r + 7me^2) - 2a^2mr^2(2r^2 + 2mr - e^2 - 12m^2)^2 \qquad [r^9 \text{ polynomial}]$$

And for θ,

$$\theta = \frac{3S_0^2}{2R_0^4} = \frac{5S^2}{2r^4} = \frac{5P}{2Qr^2} \fallingdotseq \frac{15m}{4r} \qquad (r \text{ at } 0 \text{ order})$$

2.3. The Titius–Bode Law

In the case that the discriminant is $\Delta = E^4 - 2a^2m^2(1+\theta)^2 > 0$ of the SF differential equation, the function $f(\theta)$ is given in (eq.14) and is subjected to a Maclaurin series expansion. Terms above θ^2 are neglected. The result is given in (eq.19).

$$f(\theta) = \frac{S^2 + 4a\delta m(1+\theta)S + 2E^4}{R^2} \text{ EXP}\left[\frac{-4a\delta m(1+\theta)}{\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}} \arctan\left(\frac{S + 2a\delta m(1+\theta)}{\sqrt{2E^4 - 4a^2m^2(1+\theta)^2}}\right)\right] - K = 0$$

$$f(\theta) = f(0) + \frac{1}{1!} \cdot \frac{\partial f(0)}{\partial \theta}\theta + \frac{1}{2!} \cdot \frac{\partial^2 f(0)}{(\partial \theta)^2}\theta^2 + \cdots = 0$$

(19) $$f(\theta) = \frac{3mr}{2} \text{ EXP}\left[\frac{-4a\delta m}{\sqrt{2E^4 - 4a^2m^2}} \arctan\left(\frac{r\sqrt{3mr}}{2\sqrt{E^4 - 2a^2m^2}}\right)\right] \times \left[1 - \frac{30a\delta m^2E^4}{r[2E^4 - 4a^2m^2]^{\frac{3}{2}}} \times \arctan\left(\frac{r\sqrt{3mr}}{2\sqrt{E^4 - 2a^2m^2}}\right)\right] - K = 0$$

Since r is very large, it is given as $\arctan\left(\frac{r\sqrt{3mr}}{2\sqrt{E^4 - 2a^2m^2}}\right) = \pi/2 + \pi N$. This is substituted into (eq.19).

$$K = \frac{3\text{mr}}{2} \text{ EXP}\left[\frac{-2a\delta m\pi(1+2N)}{\sqrt{2E^4 - 4a^2m^2}}\right] \cdot \left[1 - \frac{30a\delta m^2E^4}{r[2E^4 - 4a^2m^2]^{\frac{3}{2}}} \cdot \frac{\pi(1+2N)}{2}\right]$$

Since the integration constant K is common to all planets that orbit the center of mass, the base planet and the distance ratio to the base planet can be set as follows. r_1, N_1, $N - N_1 = n - 1$, and $\xi = r/r_1$. The result is given in (eq.20).

(20) $$n - 1 = \frac{\sqrt{2E^4 - 4a^2m^2}}{4a\delta m\pi} \cdot \log\left[\frac{\xi - \dfrac{15a\delta m^2E^4\pi(2N_1 + 2n - 1)}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}}{1 - \dfrac{15a\delta m^2E^4\pi(2N_1 + 1)}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}}\right]$$

On the other hand, the Titius-Bode law is changed into (eq.21).

$$\xi_{Earth} = 0.4 + 0.3 \times 2^n = 0.4 + 0.6 \times 2^{n-1} \qquad (\xi_{Earth} : \text{the Earth basis } \xi)$$

(21) $$n - 1 = \frac{1}{\log 2} \cdot \log \frac{\xi_{Earth} - 0.4}{1 - 0.4}$$

The Titius-Bode law (eq. 21) is remarkably similar to the solution (eq. 20) of the approximate SF differential equation. If the two coefficients are the same, the two equations are almost equal.
(The Earth is the base planet, n=1.)

$$\frac{1}{\log 2} = \frac{\sqrt{2E^4 - 4a^2m^2}}{4a\delta m\pi} \qquad 0.4 = \frac{15a\delta m^2E^4\pi(2N_1+1)}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}$$

Since $r_1 = 1.5 \times 10^8 km$ for the Earth, and, m=1.476km and a=0.32km [8] for the Sun, it is calculated that $e = 2.1km$, and $N_1 = 1.5 \times 10^7$. The 2n on the right side of (eq.20) is neglected because of the very large N_1. Thus,

$$(22) \qquad \boldsymbol{\xi = \left[1 - \frac{30a\delta m^2E^4\pi N_1}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}\right] \cdot \mathrm{EXP}\left[\frac{4am\pi(n-1)}{\sqrt{2E^4 - 4a^2m^2}}\right] + \frac{30a\delta m^2E^4\pi N_1}{r_1[2E^4 - 4a^2m^2]^{\frac{3}{2}}}}$$

$$\xi_{Earth} = \boldsymbol{(1-0.4)\cdot 2^{n-1} + 0.4}$$

$\delta = \pm 1$ is related to the orbital rotation direction.
(Eq.22) is now exactly equal to (eq.21). The Titius-Bode law has therefore been demonstrated.

2.4. The Saturn's Rings

Since the autorotation of the Saturn is fast, the discriminant is $\Delta = E^4 - 2a^2m^2(1+\theta)^2 > 0$. The solution (eq. 15) of the SF differential equation is as follows.

$$K = \log\left[\frac{\frac{S^2 + 4a\delta mS(1+\theta) + 2E^4}{R^2}}{\left[\frac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]^{\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}}}\right]$$

Since the power number $\left[\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]$ is nearly 1, the denominator is expressed as $(1-\lambda)$. λ is extremely small, but not zero. The solution of the SF differential equation is (eq.23).

$$1 - \lambda = \left[\frac{S + 2a\delta m(1+\theta) - \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}{S + 2a\delta m(1+\theta) + \sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}\right]^{\frac{2a\delta m(1+\theta)}{\sqrt{4a^2m^2(1+\theta)^2 - 2E^4}}}$$

$$(23) \qquad K = \frac{S^2 + 4a\delta mS(1+\theta) + 2E^4}{r^2} \cdot \frac{1}{(1-\lambda)}$$

The integration constant K is common to all the rings that belong to the Saturn. For the base ring, the variables are r_1 and $F=K$, and the polynomial of S is (eq.24).

$$(24) \qquad S^4 - 2S^2[F(1-\lambda)r^2 - 2E^4 + 8a^2m^2(1+\theta)^2] + [F(1-\lambda)r^2 - 2E^4]^2 = 0$$

P (eq. 17) and Q (eq. 18) are substituted into (eq.24) to give S and θ. Finally, the polynomial of r is (eq.25).

$$(25) \qquad \boldsymbol{Qr^2\,(Pr^2 - Q\,[F(1-\lambda)r^2 - 2E^4\,])^2 - 4a^2m^2P\,(2Qr^2 + 5P)^2 = 0}$$

The degree of (eq.25) is the highest at the first term P^2Qr^6, and is r to the power of 35 〔10×2+9+6〕. That is, (eq.25) is a polynomial of r^{35} with high degree coefficient $\boldsymbol{\lambda}$. Thus, planets with rings such as the Saturn have a maximum of 35 rings. The real number of rings decreases because of roots of complex number, minus roots, equal roots, four micro roots and the swelling of the center core. It is expected to observe and determine the rotation element a and the electric charge element e .

3. Discussion

The Titius-Bode law, discovered 250 years ago, is considered to be a mathematical coincidence rather than an "exact" law. However, I successfully proved the law and showed here that the Saturn can physically have a maximum of 35 rings. 250 years of astronomical mystery is now solved not by the computer analysis but by the theoretical analysis.

The Kerr-Newman solution of the Einstein's equation is considered as follows. The no-hair theorem postulates that all black hole solutions of the Einstein-Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only three externally observable classical parameters: mass, electric charge, and angular momentum. [9], [10]

In this manner, since this theory is based on the steady state of Kerr-Newman solution in the mature galaxies, it cannot be applied to the galaxies which are still young, unstable and transitional. Three important equations can be summarized as follows.

(eq.11) is a fundamental differential equation based on the steady state, and it can be applied to the Solar system, other planets and rings in the galaxies. There must be many solutions of (eq.11).

(eq.22) is one of the approximate solutions of (eq.11). Since this is energetically stable, it is applicable to the Solar system planets and many of the around 4000 extrasolar planets in the galaxies. However, it is not applicable to still young, unstable and transitional planets like comets.

(eq.25) is one of the approximate solutions of (eq.11). This is also energetically stable and applicable to Saturn's rings and some other extrasolar planets' rings.

This theory is applied to planets which belong to the center of mass in galaxies, but not available to the bulge space near to the center of mass.

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Author Contributions

F. I. developed the theory and wrote the manuscript.

Competing Interests

The author declares no competing interests including financial and non-financial interests.

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