Divergence measure of belief function

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The work is partially supported by National Natural Science Foundation of China (Grant Nos. 61573290, 61503237).

ABSTRACT Dempster Shafer evidence theory (D-S theory) is widely applied to information fusion for its efficiency in dealing with uncertain information. Furthermore, it is important to measure the divergent or conflicting degree among pieces of information for information preprocessing in case for the unreliable results which come from the combination of conflicting bodies of evidence using Dempster's combination rules. However, how to measure the divergence of different evidence is still an open issue. In this paper, a new divergence measure of belief function based on Deng entropy is proposed in order to measure the divergence of different belief function. The divergence measure is the generalization of Kullback-Leibler divergence for probability since when the basic probability assignment (BPA) is degenerated as probability, divergence measure is equal to Kullback-Leibler divergence. Numerical examples are used to illustrate the effectiveness of the proposed divergence measure.

INDEX TERMS Likelihood function, Soft likelihood function, OWA operator, POWA operator, Geometric mean

I. INTRODUCTION

Dempster-Shafer evidence theory [1], [2] is widely used in real applications like information fusion [3], decision making [4], target recognition [5] and so on for its efficiency in handling the combining uncertain information. However, it counter-intuitive results [6], [7] may come from the combination of conflicting different evidence.

To address this problem, one widely accepted solution is to revise the Dempster’s combination rules and thus reallocate the conflict. For example, Deng [8] proposed to averaged weight different evidences based on Jousselme distance [9] to measure the divergent degree among the evidences. In this paper, a new divergence measure called Song-Deng divergence (D-S divergence) is proposed based on Deng entropy [10] in this paper. The divergence measure is the generalization of Kullback-Leibler divergence [11], [12] for probability since when the basic probability assignment (BPA) is degenerated as probability, divergence measure is equal to Kullback-Leibler divergence. The new divergence measure could allocate masses not only to the propositions consisting of single objects, but also to the unions of such objects so it has a better performance in modelling both of the uncertainty and impression.

The remainder of this paper is constituted as follows, Section 2 introduces some necessary preliminaries of D-S evidence theory, and Deng entropy. In Section 3, a new divergence measure for belief function is proposed. There is an example for the new method and some properties of it are given in Section 4. Finally, the conclusion is given in Section 5.

II. PRELIMINARIES

In this section, some basic preliminaries on Dempster-Shafer evidence theory [1], [2], belief function and Deng entropy are introduced.

A. DEMPSTER-SHAFER EVIDENCE THEORY

Dempster-Shafer evidence theory (D-S theory) [1], [2] is firstly proposed by Dempster and then developed by Shafer. Dempster-Shafer evidence theory extends the elementary event space in probability theory to its power set and the basic probability assignment (BPA) is constructed in the frame of discernment. The basic definitions about D-S theory is shown as follows:

1) Frame of discernment

Definition II.1. D-S theory supposes the definition of a set of elementary hypotheses called the frame of discernment.

\[ \theta = \{ H_1, H_2, ..., H_N \} \]  

\( \theta \) is a set of mutually exclusively exhaustive events, and \( 2^\theta \) is the power set of \( \theta \).
2) Mass function

In the frame of discernment, the mass function \( m \) is defined as follows.

**Definition II.2.** D-S theory supposes the definition of a set of elementary hypotheses called the frame of discernment.

\[
m = 2^\theta \rightarrow [0, 1]
\]  

And there are some conditions for mass function to satisfy.

\[
m(\phi) = 0
\]

\[
\sum_{A \in \mathcal{A}} m(A) = 1
\]

In D-S theory, a mass function is also called a basic probability assignment (BPA).

**B. DENG ENTROPY**

Deng entropy [10] is a uncertainty measure proposed by Deng to measure the uncertainty of a BPA. Deng entropy is the generalization of Shannon entropy and the value of its is equal to Shannon entropy [13] when BPAs degenerate to probabilities. Deng entropy is defined as follows.

**Definition II.3.** For \( m \) is a mass function defined on the frame of discernment \( X \).

\[
E_d(m) = - \sum_{A \in X} m(A) \frac{m(A)}{2|A| - 1}
\]

**III. A NEW DIVERGENCE MEASURE OF BELIEF FUNCTION**

A new divergence measure of belief function is proposed in this section. Considering the relationship between BPA and belief function, in this section, the main goal is to measure the divergence based on BPAs. The new divergence measure is defined as follows.

**Definition III.1.** D-S theory supposes the definition of a set of elementary hypotheses called the frame of discernment.

\[
Div(m_1, m_2) = \frac{D_d(m_1 \| m_2) + D_d(m_2 \| m_1)}{2}
\]

In which the function \( D_d(m_1 || m_2) \) is defined as follows.

**Definition III.2.** D-S theory supposes the definition of a set of elementary hypotheses called the frame of discernment.

\[
D_d(m_1 \| m_2) = \sum_i \frac{1}{2|F_i| - 1} m_1(F_i) \log \left( \frac{m_1(F_i)}{m_2(F_i)} \right)
\]

The divergence measure is the generalization of Kullback-Leibler divergence for probability since when the basic probability assignment (BPA) is degenerated as probability, divergence measure is equal to Kullback-Leibler divergence. The new divergence measure could allocate masses not only to the propositions consisting of single objects, but also to the unions of such objects so it has a better performance in modelling both of the uncertainty and impression.

**A. AN EXAMPLE OF THE NEW DIVERGENCE MEASURE**

There is an example to show the properties of S-D divergence.

1) The application statement

Assume there are several sources(N) of evidence from sensors. The BPAs given by the sensors are shown in Table.1. \( i \) and \( j \) are the numbers under the amount of all sensors and \( i \) is smaller than \( j \).

2) The properties of new divergence measure

As the the amount of masses in the last BPA increases, the value of S-D divergence varies as the following figure shows. It shows that value of D-S divergence decreases as the number of focal elements increases. It shows that value of

![FIGURE 1: The value of S-D divergence for different sizes of masses](image)

D-S divergence decreases as the number of focal elements increases. When the number of focal elements increases, it means that the sensor’s reliability for any target decreases while the uncertainty of the all target increases. Moreover, the value of D-S divergence is positive for different sizes of masses.

**IV. CONCLUSION**

In order to address the problem of measuring the divergent degree among different evidence, a new divergence measure of BPA is proposed in this paper. The divergence measure is the generalization of Kullback-Leibler divergence for probability since when the basic probability assignment (BPA) is degenerated as probability, S-D divergence is equal to Kullback-Leibler divergence when all the belief are assigned to single elements. Making use of Deng entropy, the new divergence measure can greatly measure the divergent degree among evidences.

**CONFLICT OF INTERESTS**

The authors declare that there is no conflict of interests regarding the publication of this paper.
$BPA \times _1 \times _2 \times _3 \times _4 \times _5 \times _6 \times _i, \ldots, _j$

<table>
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<th>$BPA$</th>
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<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
<th>$X_i, \ldots, X_j$</th>
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TABLE 1: The BPAs for $N$ sensors

REFERENCES


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