Definition of Algebraic Graph and New Perspective on Chromatic Number

Gosu Choi

February 2019

Abstract

This paper is an article on the unusual structure that can be thought on set of simple graph. I discovered various characteristics of graph with such structures and defined algebraic graph as a set of graphs with a new perspective on the chromatic number they have.

1 Introduction

In this article, I will introduce algebraic graph that can be expressed as simple equation, and several theorems can be used in graph coloring theory. Briefly writing, the chromatic number of graph can be expressed as more connection of vertexes, edges.

2 Main text

2.1 Definition of Algebraic Graph

I made several definition to define algebraic graph.

Definition 1. Rearranged Set. $R(G) := \{X : V(G) = V(X), G \cong X\}$

This definition is intuitive to understand. This is set of graph which is isomorphic with G. It is used to express isomorphism of graph more easily.

Definition 2. Congruence. $G_1 \equiv G_2 \Leftrightarrow V(G_1) = V(G_2), E(G_1) = E(G_2)$

This is the definition of graph congruence. I defined it to use congruence of graph clearly.

Definition 3. $G_1 \subseteq G_2 \Leftrightarrow V(G_1) \subseteq V(G_2), E(G_1) \subseteq E(G_2)$

Definition 4. $G_1 \subseteq_E G_2 \Leftrightarrow V(G_1) = V(G_2), E(G_1) \subseteq E(G_2)$

Definition 3 and 4 is about inclusion relationship of graph.

Definition 5. $m \subseteq V, E_m := max\{X \subseteq E : G_0(m, X) \subseteq G(V, E)\} : G_m(m, E_m)$

Definition 6. $A \subseteq_V B \Leftrightarrow \exists_{m \subseteq V(B)} : A = B_m$

And I denote induced subgraph as following expression.

Definition 7. Induced Subgraph. $G_1 \subseteq_{RV} G_2 \Leftrightarrow \exists_{a \in R(G_1), b \in R(G_2)} : a \subseteq_V b$

Definition 8. $G_1 \subseteq_{RE} G_2 \Leftrightarrow \exists_{a \in R(G_1), b \in R(G_2)} : a \subseteq_E b$

Definition 9. $G_1 \subseteq_R G_2 \Leftrightarrow \exists_{G_0 \in R(G_1)} : G_0 \subseteq G_2$

Definition 7, 8 and 9 introduce about inclusion relationship of graph with rearrange set. This is general inclusion relationship that people usually think.

Definition 10. $W(A, R) := \begin{cases} 1, & \exists_{a,b \in A} : aRb \\ 0, & \forall_{a,b \in A} :\sim aRb \end{cases}$

This definition isn't intuitive to understand. But it is very important to read article.

Now I define the algebraic graph with following term.

Definition 11. Let AG is set of algebraic graph, $G \in AG \Leftrightarrow g \nsubseteq_{RV} G$

That G is in algebraic graph is equivalent with that G doesn't have g as induced subgraph. g is following graph, figure 1.

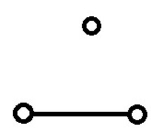

Figure 1: Contradict graph, g

The reason why I define it algebraic graph is that it is only graph set which can be expressed with equation. Now, I will introduce how to express graph to equation. Let us think that vertexes in graph as solution of equation. If edge exists between 2 vertexes, the solution which the vertexes mean is different. Else they are equal. Then the minimum-degree equation which includes all solution in graph is the equation which means that graph, and this is only one.

There is a property in non-algebraic graph, that is non-algebraic graph always need more edge to color how connected vertexes are different, non-connected vertexes are same. This is important property to prove theorem.

Definition 12. If $G \in AG$ and its algebraic expression is $f(x) = 0$, then $G \sim f(x)$

This definition makes the algebraic expression of graph more simple.

2.1.1 Expanded Definition of Perfect Graph with Algebraic Expression of Graph

Every perfect graph is algebraic graph. The reason is that all vertexes in perfect graph are connected because of the definition of perfect graph, so they can expressed as $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$. Using this feature, I expanded definition of perfect graph with algebraic expression of graph.

Definition 13. $G \in K_{ex} \Leftrightarrow G \sim (x-\alpha_1)^m(x-\alpha_2)^m \cdots (x-\alpha_n)^m$

In this case, this graph is called n-degree m-character perfect graph.

2.1.2 Chromatic Number of Algebraic Graph

Chromatic number of algebraic graph is trivially equal with the number of distinct solutions of the equation which expresses the graph. It can be written mathematically as following term, and it's trivial because of the definition of algebraic graph.

Theorem 1. $G \sim f(x) \Rightarrow \chi(G) =$(Number of distinct solutions of $f(x) = 0$)

2.1.3 Property on Chromatic Number of Graph

In general, following proposition can be utilized.

$$G_1 \subseteq_R G_2 \Rightarrow \chi(G_1) \leq \chi(G_2)$$

I use the contrapositive proposition on here. That is written on following theorem. I skipped the proof of this theorem because it's trivial.

Theorem 2. $\chi(A) < \chi(B) \Rightarrow [A \subsetneq_R B$ or $W(\{A,B\}, \subseteq_R) = 0]$

2.1.4 Number of Algebraic Graph and Integer Partitions

I thought about the number of algebraic graph of which vertex number is n. Surprisingly, that is equal with partition number of n.

Theorem 3. $AG_n := \{G \in AG : |V(G)| = n\} \Rightarrow |AG_n| = p_n$

Proof. Let $G \in AG$ and vertex number is n, and let $G \sim f(x)$. And let $\{\alpha_n\}$ is solution of $f(x) = 0$, let a_i is number of same solution with α_i. Then the problem changes to find $|\{\{a_1, a_2, a_3, \cdots a_n\} : a_1 + a_2 + a_3 + \cdots + a_n = n$ and $a_i \in \mathbb{N}_0\}| = p_n$ $\square$

2.2 Algebraic Lower, Upper Graph

Algebraic lower, upper graph can defined by these several definitions.

Definition 14. $Max_G S := \{G \in S : \forall_{K \in S} : [K \subseteq_E G$ or $W(\{G,K\}, \subseteq_E) = 0]\}$

Definition 15. $min_G S := \{G \in S : \forall_{K \in S} : [G \subseteq_E K$ or $W(\{G,K\}, \subseteq_E) = 0]\}$

Definition 16. $f_l(G) := Max_G\{X : X \subseteq_E G, X \in AG\}$

Definition 17. $f_u(G) := min_G\{X : G \subseteq_E X \in AG\}$

Definition 15, 16 is algebraic lower graph and upper graph, respectively. Trivially, the algebraic upper graph and lower graph exist on any graph, because of existence of perfect graph and edgeless graph.

2.2.1 Chromatic Number of Algebraic Upper Graph

The $f_u(G)$ usually doesn't have only one graph. But their chromatic numbers are same each other. It is introduced in next theorem.

Lemma 4. $\forall_{A,B\in AG_n} : \chi(A) < \chi(B) \Rightarrow W(\{A, B\}, \subseteq_E) = 1$

This is trivial for definition of algebraic graph.

Lemma 5. $\forall_{A,B\in f_u(G)} : W(\{A, B\}, \subseteq_E) = 0$

This is trivial for definition of algebraic upper graph.

Theorem 6. $\forall_{A,B\in f_u(G)} : \chi(A) = \chi(B)$

Proof. If $\chi(A) \neq \chi(B) \Rightarrow W(\{A, B\}, \subseteq_E) = 1$ by lemma 4. But by lemma 5, $\forall_{A,B\in f_u(G)} : W(\{A, B\}, \subseteq_E) = 0$. It is contradiction. So theorem 6 can be proved by these inferences. □

I expanded this theorem to theorem 7. It explains the chromatic number of general graph.

Theorem 7. $\forall_{A\in f_u(G)} : \chi(G) = \chi(A)$

Proof. It is trivial when $G \in AG$. By theorem 6, to prove this proposition, $\forall_{A\in f_u(G)} : \chi(G) = \chi(A)$ is equivalent with $\exists_{A\in f_u(G)} : \chi(G) = \chi(A)$. Therefore, the proposition $G \notin AG \Rightarrow \exists_{K:G\subsetneq_E K} : \chi(G) = \chi(K)$ Because non algebraic graph always need more edge to color connected vertexes are different, non-connected vertexes are same. Trivially, It can be substituted with $G \notin AG \Rightarrow \exists_{K:G\subsetneq_E K\in f_u(G)} : \chi(G) = \chi(K)$. It proves theorem 7. □

Following terms are lemmas to prove theorem 10.

Lemma 8. $\exists_{a\in f_u(G)} : K_n \subseteq_V a \Rightarrow \chi(a) \geq n$

Lemma 9. $\exists_{G_0\in AG} : \chi(G_0) \geq n \Rightarrow K_n \subseteq_V G_0$

Lemma 8, 9 are trivial because of definition of algebraic graph so I don't exhibit proof of it.

Theorem 10. $\exists_{a\in f_u(G)} : K_n \subseteq_V a \Leftrightarrow \forall_{a\in f_u(G)} : K_n \subseteq_V a$

Proof. $\chi(a) \geq n$ because $\exists_{a\in f_u(G)} : K_n \subseteq_V a$. And by theorem 7, $\chi(a) = \chi(G)$ so that $\forall_{a\in f_u(G)} : \chi(G) = \chi(a) \geq n$. So Applying lemma 9, $\forall_{a\in f_u(G)} : K_n \subseteq_V a$. It proves theorem 10. □

3 Result

In this article, I proved theorem 1, 2, 3, 6, 7, 10. All of these results imply the necessity and importance of algebraic graph in the mathematical section of graph theory. The main theorems are summarized as follows.

Theorem. $G \sim f(x) \Rightarrow \chi(G) =$(Number of distinct solutions of $f(x) = 0$)

Theorem. $\chi(A) < \chi(B) \Rightarrow [A \subsetneq B$ or $W(\{A, B\}, \subseteq) = 0]$

Theorem. $AG_n := \{G \in AG : |V(G)| = n\} \Rightarrow |AG_n| = p_n$

Theorem. $\forall_{A,B \in f_u(G)} : \chi(A) = \chi(B)$

Theorem. $\forall_{A \in f_u(G)} : \chi(G) = \chi(A)$

Theorem. $\exists_{a \in f_u(G)} : K_n \subseteq_V a \Leftrightarrow \forall_{a \in f_u(G)} : K_n \subseteq_V a$

4 Discussion

Theorem 7 can change that how we think about chromatic number of graph. In the past, researchers considered that chromatic number is property of graph which can not find with relation of edges. But it turned out to be wrong in this article. And definition of algebraic graph and algebraic upper, lower graph are the main points of this article so it should be observed carefully. Concept of algebraic graph can be applied in abstract algebra, and algebraic lower graph can be applied in researching of stability. So I am in progress of follow-up researching about this algebraic lower graph.

5 Reference

[1] Kuratowski, Kazimierz (1930), ”Sur le problème des courbes gauches en topologie”, Fund. Math. 15: 271–283.
[2] Reinhard Diestel (2017), ”Graph theory”, Springer.