Exact Smooth Solution for General Form of Navier-Stokes Equations

Sangwha-Yi
Department of Math, Taejon University 300-716

ABSTRACT
In Wikipedia “Navier-Stokes existence and smoothness”, we discover the exact smooth solution for general form of Navier-Stokes Equations by Newton potential and time-function.

PACS Number: 04.04.90.+e
Key words: General form of Navier-Stokes Equations;
  Exact smooth solutions;
  Newton potential;
  Time function

e-mail address: sangwha1@nate.com
Tel: 010-2496-3953
1. Introduction

We discover the exact solution for general form of the Navier-Stokes equation by Newton potential and time function in Wikipedia "Navier-Stokes existence and smoothness".

According Wikipedia’s general form of Navier-Stokes Equations(3-dimensional),

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \vec{v} + \vec{f}(\bar{x},t)
\]

\[
\nabla \cdot \vec{v} = 0
\]

Where \( \nu > 0 \) is the kinematic viscosity, \( \vec{f}(\bar{x},t) = (f_x, f_y, f_z) \) is the external volumetric force,

\( \nabla \) is the gradient operator and \( \Delta = \nabla^2 \) is the Laplacian operator.

Coordinate: \((x, y, z)\). Time: \( t \). Pressure: \( \rho \). Density: \( \rho \).

If the velocity \( \vec{v} = \vec{v}(\bar{x},t) = (u,v,w) \), Eq(1),Eq(2) are

\[
\frac{\partial u}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z})u = -\frac{1}{\rho} \frac{\partial}{\partial x} \rho + \nu \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u + f_x
\]

\[
\frac{\partial v}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z})v = -\frac{1}{\rho} \frac{\partial}{\partial y} \rho + \nu \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v + f_y
\]

\[
\frac{\partial w}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z})w = -\frac{1}{\rho} \frac{\partial}{\partial z} \rho + \nu \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) w + f_z
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

2. Exact Solution for General Form of 3-Dimensional Navier-Stokes Equations (Include time) in Wikipedia

For we solve equations, we use Newton potential and time function. If we think the solution of Laplace equation, if we differentiate the function \( g \) \( k \)-times by the coordinate \( x,y,z \), it is

\[(\partial_x)^k g, (\partial_y)^k g, (\partial_z)^k g.\]

Newton potential \( g = \frac{1}{r} \cdot g^{(k)}_x = (\partial_x)^k g, g^{(k)}_y = (\partial_y)^k g, g^{(k)}_z = (\partial_z)^k g \)

\( \vec{v} = \vec{v}(\bar{x},t) = (u,v,w) = C(t)\vec{v}g^{(k)}_x \) or

\( \vec{v} = \vec{v}(\bar{x},t) = (u,v,w) = C(t)\vec{v}g^{(k)}_y \) or \( \vec{v} = \vec{v}(\bar{x},t) = (u,v,w) = C(t)\vec{v}g^{(k)}_z \)

In this time, we treat only \( \vec{v} = \vec{v}(\bar{x},t) = (u,v,w) = C(t)\vec{v}g^{(k)}_x \)

\[
\frac{\partial}{\partial x} g^{(k)}_x, \frac{\partial}{\partial y} g^{(k)}_x, \frac{\partial}{\partial z} g^{(k)}_x.
\]

(8)
In this case, 

\[ \nabla^{(k)} \cdot \vec{v} g = \nabla^{(k-1)} \nabla \cdot (g_x i + g_y j + g_z k) = \nabla^{(k-1)} \nabla^2 \left( \frac{1}{r} \right) = \nabla^{(k-1)}(0) = 0 \]

Hence, according to Eq(9),

\[ \frac{\partial u}{\partial t} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = C_f(t) \vec{v} \cdot \vec{g}^{(k)} = 0 \]  
(10)

Hence, Eq(8) satisfy the continuity equation-Eq(6)

In this case, if we solve left term of Eq(1), the left term of Eq(3) is

\[ \frac{\partial u}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) u = C_i \partial_x g^{(k)} x f(t) + f^2(t) C_1 \left[ \partial_x g^{(k)} x \partial_x g^{(k)} x + \partial_y g^{(k)} x \partial_y g^{(k)} x + \partial_z g^{(k)} x \partial_z g^{(k)} x \right] \]  
(11)

So, left terms of Eq(4), Eq(5) are

\[ \frac{\partial v}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) v = C_i \partial_x g^{(k)} x f(t) + f^2(t) C_1 \left[ \partial_x g^{(k)} x \partial_x g^{(k)} x + \partial_y g^{(k)} x \partial_y g^{(k)} x + \partial_z g^{(k)} x \partial_z g^{(k)} x \right] \]  
(12)

\[ \frac{\partial w}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) w = C_i \partial_x g^{(k)} x f(t) + f^2(t) C_1 \left[ \partial_x g^{(k)} x \partial_x g^{(k)} x + \partial_y g^{(k)} x \partial_y g^{(k)} x + \partial_z g^{(k)} x \partial_z g^{(k)} x \right] \]  
(13)

In this time, if \( f(t) \) is,

\[ \frac{1}{[f(t)]^2} \frac{d}{dt} [f(t)] = 1 \rightarrow f(t) = \frac{1}{C-t} \quad C < 0 \]  
(14)

Therefore, left terms of Eq(3),Eq(4),Eq(5) are

\[ \frac{\partial u}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) u = C_i \partial_x g^{(k)} x \left( \frac{1}{(C-t)^2} \right) + \frac{1}{(C-t)^2} C_1 \left[ \partial_x g^{(k)} x \partial_x g^{(k)} x + \partial_y g^{(k)} x \partial_y g^{(k)} x + \partial_z g^{(k)} x \partial_z g^{(k)} x \right] \]  
(15)

\[ \frac{\partial v}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) v = \frac{1}{[f(t)]^2} \frac{d}{dt} [f(t)] \]
\[
\frac{1}{(C-t)^2} \partial_x \mathbf{g}^{(k)}_x + \partial_y \mathbf{g}^{(k)}_y + \partial_z \mathbf{g}^{(k)}_z 
\]
\[
\mathbf{w}(t) = \frac{1}{(C-t)^2} \mathbf{g}^{(k)}_x + \partial_y \mathbf{g}^{(k)}_y + \partial_z \mathbf{g}^{(k)}_z 
\]
\[
\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{w} = 0
\]
\[
\mathbf{w}(t) = \frac{1}{(C-t)^2} \mathbf{g}^{(k)}_x + \partial_y \mathbf{g}^{(k)}_y + \partial_z \mathbf{g}^{(k)}_z 
\]
\[
\mathbf{w}(t) = \frac{1}{(C-t)^2} \mathbf{g}^{(k)}_x + \partial_y \mathbf{g}^{(k)}_y + \partial_z \mathbf{g}^{(k)}_z 
\]

In this case, if we solve right term of Eq(1), the right term of Eq(3) is
\[
-\frac{1}{\rho_0} \frac{\partial}{\partial t} \mathbf{p} + \mathbf{u} \cdot \nabla \mathbf{p} + \frac{\partial^2}{\partial x^2} \mathbf{u} + \frac{\partial^2}{\partial y^2} \mathbf{u} + \frac{\partial^2}{\partial z^2} \mathbf{u} + f_x, \quad \mathbf{v} > 0, \quad \rho = \rho_0
\]

In this time, according to Eq(9),
\[
\nabla^2 \mathbf{u} = \frac{1}{(C-t)^2} \nabla^2 \partial_x \mathbf{g}^{(k)}_x = \frac{1}{(C-t)^2} \partial_x \nabla^2 \mathbf{g}^{(k)}_x = 0
\]

So,
\[
\nabla^2 \mathbf{v} = \nabla^2 \mathbf{w} = 0
\]

Hence, right terms of Eq(3), Eq(4), Eq(5) are
\[
-\frac{1}{\rho_0} \frac{\partial}{\partial t} \mathbf{p} + \mathbf{u} \cdot \nabla \mathbf{p} + \frac{\partial^2}{\partial x^2} \mathbf{u} + \frac{\partial^2}{\partial y^2} \mathbf{u} + \frac{\partial^2}{\partial z^2} \mathbf{u} = -\frac{1}{\rho_0} \frac{\partial \mathbf{p}}{\partial x} + f_x
\]

In this time, if \( \tilde{f}(\tilde{x}, t) = 0 \) is for smooth function \( \rho(\tilde{x}, t) \). Hence, if we solve Eq(1),
\[
\frac{\rho_0}{(C-t)^2} \partial_x \mathbf{g}^{(k)}_x + \partial_y \mathbf{g}^{(k)}_y + \partial_z \mathbf{g}^{(k)}_z 
\]
\[
\partial_y \mathbf{g}^{(k)}_y + \partial_z \mathbf{g}^{(k)}_z 
\]
\[
-\frac{1}{\rho_0} \frac{\partial \rho}{\partial x}
\]

\[
\partial_y \mathbf{g}^{(k)}_y + \partial_z \mathbf{g}^{(k)}_z 
\]

(24)
\[
\frac{1}{(C-t)^2} \mathcal{C}_1 \left[ \partial_x g^{(k)} \partial_x g^{(k)} + \partial_y g^{(k)} \partial_y g^{(k)} + \partial_z g^{(k)} \partial_z g^{(k)} \right] \\
= - \frac{1}{\rho_0} \frac{\partial \rho}{\partial y} \\
\mathcal{C}_1 \partial_z g^{(k)} \frac{1}{(C-t)^2} \\
+ \frac{1}{(C-t)^2} \mathcal{C}_1 \left[ \partial_x g^{(k)} \partial_x g^{(k)} + \partial_y g^{(k)} \partial_y g^{(k)} + \partial_z g^{(k)} \partial_z g^{(k)} \right] \\
= - \frac{1}{\rho_0} \frac{\partial \rho}{\partial z}
\]

(25)

Hence,

\[
\rho(x, t) = -\rho_0 \mathcal{C}_1 g^{(k)} \frac{1}{(C-t)^2} - \frac{1}{2} \rho_0 \mathcal{C}_1^2 \left[ (\partial_x g^{(k)})^2 + (\partial_y g^{(k)})^2 + (\partial_z g^{(k)})^2 \right] \frac{1}{(C-t)^2}
\]

(26)

3. Hypotheses and Growth Conditions (Smoothness)

The initial condition \( \mathcal{V}_0(x) = \mathcal{C}_1 (\partial_x g^{(k)} x, \partial_y g^{(k)} x, \partial_z g^{(k)} x) \frac{1}{C} = \mathcal{C}_2 \mathcal{V} g^{(k)} x \) is, for every multi-index \( \alpha \) and any \( K > 0 \), there exist a constant \( C = C(\alpha, K) > 0 \) such that

\[
|\partial^\alpha \mathcal{V}_0(x)| = |\mathcal{C}_2 \partial^\alpha \partial_x (\mathcal{V} g^{(k)} x)| < \frac{C}{(1 + |x|^K)}, \text{ for all } x \in \mathbb{R}^3,
\]

(28)

The external force \( \tilde{f}(x, t) = 0 \) is

\[
|\partial^\alpha \tilde{f}(x)| = 0 \leq \frac{C}{(1 + |x| + t)^K}, \text{ for all } (x, t) \in \mathbb{R}^3 \times [0, \infty)
\]

(29)

More precisely, the following assumptions are made:

1. \( \mathcal{V}(x, t) = \mathcal{C}_1 (\partial_x g^{(k)} x, \partial_y g^{(k)} x, \partial_z g^{(k)} x) \frac{1}{(C-t)^2} \in [C^\infty(\mathbb{R}^3 \times [0, \infty))]^3, \ C < 0\).

(30)

Hence,

\[
\rho(x, t) = -\rho_0 \mathcal{C}_1 g^{(k)} \frac{1}{(C-t)^2} - \frac{1}{2} \rho_0 \mathcal{C}_1^2 \left[ (\partial_x g^{(k)})^2 + (\partial_y g^{(k)})^2 + (\partial_z g^{(k)})^2 \right] \frac{1}{(C-t)^2}
\]

\( \in [C^\infty(\mathbb{R}^3 \times [0, \infty))]^3, C < 0 \)

(31)

2. There exists a constant \( \mathcal{E} \in (0, \infty) \) such that
\[ \bar{v}(x,t) = C(x) (e_x g_{(k)}^x, e_y g_{(k)}^y, e_z g_{(k)}^z) \frac{1}{(C-t)} \]

\[ \int_{\mathbb{R}^3} |\bar{v}(x,t)|^2 \, dx = \int_{\mathbb{R}^3} C_1^2 \frac{1}{(C-t)^2} \left[ (e_x g_{(k)}^x)^2 + (e_y g_{(k)}^y)^2 + (e_z g_{(k)}^z)^2 \right] \, dx \]

\[ < E \text{, for all } t \geq 0 \text{ if } C < 0 \]  (32)

*The Millennium Prize Conjectures in the Whole Space*

(A) Existence and smoothness of the Navier-Stokes solution in \( \mathbb{R}^3 \).

Let \( \vec{f}(x,t) \equiv 0 \). For any initial condition \( \bar{v}_0(x) \) satisfying the above hypotheses there exist smooth and globally defined solutions to the Navier-Stokes equations, i.e. there is a velocity vector \( \bar{v}(x,t) \) and a pressure \( \rho(x,t) \) satisfying conditions 1 and 2 above.

4. Conclusion

Therefore, the exact smooth solution of Navier-Stokes equations in Wikipedia(3-dimensional) is

Density: \( \rho = \rho_0 \),

Velocity Components: \( \bar{v}(x,t) = C_1 (e_x g_{(k)}^x, e_y g_{(k)}^y, e_z g_{(k)}^z) \frac{1}{(C-t)} \cdot C < 0 \)

The external volumetric force: \( \vec{f}(x,t) = 0 \)

Pressure:

\[ \rho(x,t) = -\rho_0 C_1 (e_x g_{(k)}^x) - \rho_0 C_1^2 \left[ (e_x g_{(k)}^x)^2 + (e_y g_{(k)}^y)^2 + (e_z g_{(k)}^z)^2 \right] \frac{1}{(C-t)^2} \cdot C < 0 \]

References