Exact Smooth Solution for General Form of Navier-Stokes Equations

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ABSTRACT

In Wikipedia "Navier-Stokes existence and smoothness", we discover the exact smooth solution for general form of Navier-Stokes Equations by Newton potential and time-function.

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1. Introduction

We discover the exact solution for general form of the Navier-Stokes equation by Newton potential and time function in Wikipedia "Navier-Stokes existence and smoothness".

According Wikipedia's general form of Navier-Stokes Equations(3-dimensional),

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \upsilon\Delta\vec{v} + \vec{f}(\vec{x}, t) \tag{1}$$

$$\vec{\nabla} \cdot \vec{v} = 0 \tag{2}$$

Where $\upsilon > 0$ is the kinematic viscosity, $\vec{f}(\vec{x}, t) = (f_x, f_y, f_z)$ is the external volumetric force,

$\vec{\nabla}$ is the gradient operator and $\Delta = \nabla^2$ is the Laplacian operator,

Coordinate: (x, y, z), Time: t, Pressure: p, Density: ρ

If the velocity $\vec{v} = \vec{v}(\vec{x}, t) = (u, v, w)$, Eq(1),Eq(2) are

$$\frac{\partial u}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u + f_x \tag{3}$$

$$\frac{\partial v}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})v = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})v + f_y \tag{4}$$

$$\frac{\partial w}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})w = -\frac{1}{\rho}\frac{\partial}{\partial z}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})w + f_z \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

2. Exact Solution for General Form of 3-Dimentional Navier-Stokes Equations (Include time) in Wikipedia

For we solve equations, we use Newton potential and time function. If we think the solution of Laplace equation, if we differentiate the function g k-times by the coordinate x,y,z, it is

$(\partial_x)^k g, (\partial_y)^k g, (\partial_z)^k g$.

Newton potential $g = \frac{1}{r}$, $g^{(k)}{}_x = (\partial_x)^k g, g^{(k)}{}_y = (\partial_y)^k g, g^{(k)}{}_z = (\partial_z)^k g$

$$\vec{v} = \vec{v}(\vec{x}, t) = (u, v, w) = C_1 f(t)\vec{\nabla}g^{(k)}{}_x \text{ or}$$

$$\vec{v} = \vec{v}(\vec{x}, t) = (u, v, w) = C_1 f(t)\vec{\nabla}g^{(k)}{}_y \text{ or } \vec{v} = \vec{v}(\vec{x}, t) = (u, v, w) = C_1 f(t)\vec{\nabla}g^{(k)}{}_z \tag{7}$$

In this time, we treat only $\vec{v} = \vec{v}(\vec{x}, t) = (u, v, w) = C_1 f(t)\vec{\nabla}g^{(k)}{}_x$

$$u = C_1 f(t)\partial_x g^{(k)}{}_x \ , \ v = C_1 f(t)\partial_y g^{(k)}{}_x, w = C_1 f(t)\partial_z g^{(k)}{}_x, \tag{8}$$

$$\nabla^{(k)}\cdot\vec{\nabla}g=\nabla^{(k-1)}\vec{\nabla}\cdot(g'_x\,i+g'_y\,j+g'_z\,k)=\nabla^{(k-1)}\nabla^2(\frac{1}{r})=\nabla^{(k-1)}(0)=0$$

$$=\vec{\nabla}\cdot\vec{\nabla}(\nabla^{(k-1)}g)=\vec{\nabla}\cdot\vec{\nabla}(g^{(k-1)}{}_x)=\vec{\nabla}\cdot\vec{g}^{(k)}=0 \qquad (9)$$

Hence, according to Eq(9),

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=C_1 f(t)\vec{\nabla}\cdot\vec{g}^{(k)}=0 \qquad (10)$$

Hence, Eq(8) satisfy the continuity equation-Eq(6)

In this case, if we solve left term of Eq(1), the left term of Eq(3) is

$$\frac{\partial u}{\partial t}+(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})u$$

$$=C_1\partial_x g^{(k)}{}_x\dot{f}(t)+f^2(t)C_1^2[\partial_x g^{(k)}{}_x\partial_x\partial_x g^{(k)}{}_x+\partial_y g^{(k)}{}_x\partial_y\partial_x g^{(k)}{}_x+\partial_z g^{(k)}{}_x\partial_z\partial_x g^{(k)}{}_x]$$

(11)

So, left terms of Eq(4), Eq(5) are

$$\frac{\partial v}{\partial t}+(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})v$$

$$=C_1\partial_y g^{(k)}{}_x\dot{f}(t)+f^2(t)C_1^2[\partial_x g^{(k)}{}_x\partial_x\partial_y g^{(k)}{}_x+\partial_y g^{(k)}{}_y\partial_y\partial_y g^{(k)}{}_x+\partial_z g^{(k)}{}_x\partial_z\partial_y g^{(k)}{}_x]$$

(12)

$$\frac{\partial w}{\partial t}+(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})w$$

$$=C_1\partial_z g^{(k)}{}_x\dot{f}(t)+f^2(t)C_1^2[\partial_x g^{(k)}{}_x\partial_x\partial_z g^{(k)}{}_x+\partial_y g^{(k)}{}_x\partial_y\partial_z g^{(k)}{}_x+\partial_z g^{(k)}{}_x\partial_z\partial_z g^{(k)}{}_x]$$

(13)

In this time, if $f(t)$ is,

$$\frac{1}{[f(t)]^2}\frac{d}{dt}[f(t)]=1\ \rightarrow\ f(t)=\frac{1}{C'-t},\quad C'<0 \text{ is constant} \qquad (14)$$

Therefore, left terms of Eq(3),Eq(4),Eq(5) are

$$\frac{\partial u}{\partial t}+(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})u$$

$$=C_1\partial_x g^{(k)}{}_x\frac{1}{(C'-t)^2}$$

$$+\frac{1}{(C'-t)^2}C_1^2[\partial_x g^{(k)}{}_x\partial_x\partial_x g^{(k)}{}_x+\partial_y g^{(k)}{}_x\partial_y\partial_x g^{(k)}{}_x+\partial_z g^{(k)}{}_x\partial_z\partial_x g^{(k)}{}_x] \qquad (15)$$

$$\frac{\partial v}{\partial t}+(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z})v$$

$$= C_1 \partial_y g^{(k)}{}_x \frac{1}{(C'-t)^2}$$

$$+ \frac{1}{(C'-t)^2} C_1^2 [\partial_x g^{(k)}{}_x \partial_x \partial_y g^{(k)}{}_x + \partial_y g^{(k)}{}_y \partial_y \partial_y g^{(k)}{}_x + \partial_z g^{(k)}{}_x \partial_z \partial_y g^{(k)}{}_x] \quad (16)$$

$$\frac{\partial w}{\partial t} + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}) w$$

$$= C_1 \partial_z g^{(k)}{}_x \frac{1}{(C'-t)^2}$$

$$+ \frac{1}{(C'-t)^2} C_1^2 [\partial_x g^{(k)}{}_x \partial_x \partial_z g^{(k)}{}_x + \partial_y g^{(k)}{}_x \partial_y \partial_z g^{(k)}{}_x + \partial_z g^{(k)}{}_x \partial_z \partial_z g^{(k)}{}_x] \quad (17)$$

In this case, if we solve right term of Eq(1), the right term of Eq(3) is

$$-\frac{1}{\rho_0} \frac{\partial}{\partial x} p + \upsilon (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) u + f_x, \ \upsilon > 0, \ \rho = \rho_0 \quad (18)$$

In this time, according to Eq(9),

$$\nabla^2 u = C_1 \frac{1}{(C'-t)} \nabla^2 \partial_x g^{(k)}{}_x = C_1 \frac{1}{(C'-t)} \partial_x \nabla^2 g^{(k)}{}_x = 0 \quad (19)$$

So,

$$\nabla^2 v = \nabla^2 w = 0 \quad (20)$$

Hence, right terms of Eq(3),Eq(4),Eq(5) are

$$-\frac{1}{\rho_0} \frac{\partial}{\partial x} p + \upsilon (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) u + f_x = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + f_x \quad (21)$$

$$-\frac{1}{\rho_0} \frac{\partial}{\partial y} p + \upsilon (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) v + f_y = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + f_y \quad (22)$$

$$-\frac{1}{\rho_0} \frac{\partial}{\partial z} p + \upsilon (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) w + f_z = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + f_z \quad (23)$$

In this time, if $\vec{f}(\vec{x}, t) = 0$ is for smooth function $p(\vec{x}, t)$.Hence, if we solve Eq(1),

$$C_1 \partial_x g^{(k)}{}_x \frac{1}{(C'-t)^2}$$

$$+ \frac{1}{(C'-t)^2} C_1^2 [\partial_x g^{(k)}{}_x \partial_x \partial_x g^{(k)}{}_x + \partial_y g^{(k)}{}_x \partial_y \partial_x g^{(k)}{}_x + \partial_z g^{(k)}{}_x \partial_z \partial_x g^{(k)}{}_x]$$

$$= -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad (24)$$

$$C_1 \partial_x g^{(k)}{}_x \frac{1}{(C'-t)^2}$$

$$+\frac{1}{(C'-t)^2}C_1^2[\partial_x g^{(k)}{}_x\partial_x\partial_y g^{(k)}{}_x+\partial_y g^{(k)}{}_y\partial_y\partial_y g^{(k)}{}_x+\partial_z g^{(k)}{}_x\partial_z\partial_y g^{(k)}{}_x]$$

$$=-\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{25}$$

$$C_1\partial_z g^{(k)}{}_x\frac{1}{(C'-t)^2}$$

$$+\frac{1}{(C'-t)^2}C_1^2[\partial_x g^{(k)}{}_x\partial_x\partial_z g^{(k)}{}_x+\partial_y g^{(k)}{}_x\partial_y\partial_z g^{(k)}{}_x+\partial_z g^{(k)}{}_x\partial_z\partial_z g^{(k)}{}_x]$$

$$=-\frac{1}{\rho_0}\frac{\partial p}{\partial z} \tag{26}$$

Hence,

$$p(\vec{x},t)=-\rho_0 C_1 g^{(k)}{}_x\frac{1}{(C'-t)^2}-\frac{1}{2}\rho_0 C_1^2[(\partial_x g^{(k)}{}_x)^2+(\partial_y g^{(k)}{}_x)^2+(\partial_z g^{(k)}{}_x)^2]\frac{1}{(C'-t)^2} \tag{27}$$

3. Hypotheses and Growth Conditions(Smoothness)

The initial condition $\vec{V}_0(x)=C_1(\partial_x g^{(k)}{}_x,\partial_y g^{(k)}{}_x,\partial_z g^{(k)}{}_x)\frac{1}{C'}=C_2\vec{\nabla}g^{(k)}{}_x$ is, for every multi-index α and any $K>0$,there exist a constant $C=C(\alpha,K)>0$ such that

$$\left|\partial^\alpha\vec{V}_0(x)\right|=\left|C_2\partial^{\alpha_1\cdots\alpha_i}\partial_x(\vec{\nabla}g^{(k)}{}_x)\right|<\frac{C}{(1+|x|)^K},\text{ for all } x\in R^3,$$

$$\partial^\alpha=\partial^{\alpha_1\alpha_2\cdots}\partial^{\alpha_n}, \tag{28}$$

The external force $\vec{f}(x,t)=0$ is

$$\left|\partial^\alpha\vec{f}(x)\right|=0\le\frac{C}{(1+|x|+t)^K},\text{ for all } (x,t)\in R^3\times[0,\infty) \tag{29}$$

More precisely, the following assumptions are made:

1. $\vec{v}(x,t)=C_1(\partial_x g^{(k)}{}_x,\partial_y g^{(k)}{}_x,\partial_z g^{(k)}{}_x)\frac{1}{(C'-t)}\in[C^\infty(R^3\times[0,\infty))]^3$,

$$C'<0, \tag{30}$$

Hence,

$$p(\vec{x},t)=-\rho_0 C_1 g^{(k)}{}_x\frac{1}{(C'-t)^2}-\frac{1}{2}\rho_0 C_1^2[(\partial_x g^{(k)}{}_x)^2+(\partial_y g^{(k)}{}_x)^2+(\partial_z g^{(k)}{}_x)^2]\frac{1}{(C'-t)^2}$$

$$\in[C^\infty(R^3\times[0,\infty))]^3, C'<0 \tag{31}$$

2. There exists a constant $E\in(0,\infty)$ such that

$$\vec{V}(x,t) = C_1(\partial_x g^{(k)}{}_x, \partial_y g^{(k)}{}_x, \partial_z g^{(k)}{}_x)\frac{1}{(C'-t)}$$

$$\int_{R^3} |\vec{V}(x,t)|^2 dx = \int_{R^3} C_1^2 \frac{1}{(C'-t)^2}[(\partial_x g^{(k)}{}_x)^2 + (\partial_y g^{(k)}{}_x)^2 + (\partial_z g^{(k)}{}_x)^2]dx$$

$< E$, for all $t \geq 0$. if $C' < 0$ (32)

*The Millennium Prize Conjectures in the Whole Space

(A) Existence and smoothness of the Navier-Stokes solution in R^3.

Let $\vec{f}(x,t) \equiv 0$. For any intial condition $\vec{V}_0(x)$ satisfying the above hypotheses there exist smooth and globally defined solutions to the Navier-Stokes equations, i.e. there is a velocity vector $\vec{V}(x,t)$ and a pressure $p(x,t)$ satisfying conditions 1 and 2 above.

4. Conclusion

Therefore, the exact smooth solution of Navier-Stokes equations in Wikipedia(3-dimensional) is

Density: $\rho = \rho_0$,

Velocity Components: $\vec{V}(x,t) = C_1(\partial_x g^{(k)}{}_x, \partial_y g^{(k)}{}_x, \partial_z g^{(k)}{}_x)\frac{1}{(C'-t)}, C' < 0$

The external volumetric force: $\vec{f}(\vec{x},t) = 0$

Pressure:

$$p(\vec{x},t) = -\rho_0 C_1 g^{(k)}{}_x \frac{1}{(C'-t)^2} - \frac{1}{2}\rho_0 C_1^2[(\partial_x g^{(k)}{}_x)^2 + (\partial_y g^{(k)}{}_x)^2 + (\partial_z g^{(k)}{}_x)^2]\frac{1}{(C'-t)^2}$$

$$C' < 0$$

References

[1]Wikipedia,” Navier-Stokes existence and smoothness”