Exact Smooth Solution for General Form of Navier-Stokes Equations

Sangwha-Yi

Department of Math , Taejon University 300-716

ABSTRACT

In Wikipedia "Navier-Stokes existence and smoothness", we discover the exact smooth solution for general form of Navier-Stokes Equations by Newton potential and time-function.

PACS Number:04,04.90.+e

Key words:General form of Navier-Stokes Equations;

Exact smooth solutions;

Newton potential;

Time function

e-mail address:sangwha1**@nate.com**

Tel:010-2496-3953

1. Introduction

We discover the exact solution for general form of the Navier-Stokes equation by Newton potential and time function in Wikipedia "Navier-Stokes existence and smoothness".

According Wikipedia's general form of Navier-Stokes Equations(3-dimensional),

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} p + \upsilon \Delta \vec{v} + \vec{f}(\vec{x}, t) \tag{1}$$

$$\vec{\nabla} \cdot \vec{v} = 0 \tag{2}$$

Where $\upsilon > 0$ is the kinematic viscosity, $\vec{f}(\vec{x}, t) = (f_x, f_y, f_z)$ is the external volumetric force,

$\vec{\nabla}$ is the gradient operator and $\Delta = \nabla^2$ is the Laplacian operator,

Coordinate: (x, y, z), Time: t, Pressure: p, Density: ρ

If the velocity $\vec{v} = \vec{v}(\vec{x}, t) = (u, v, w)$, Eq(1),Eq(2) are

$$\frac{\partial u}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})u = -\frac{1}{\rho}\frac{\partial}{\partial x}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u + f_x \tag{3}$$

$$\frac{\partial v}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})v = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})v + f_y \tag{4}$$

$$\frac{\partial w}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})w = -\frac{1}{\rho}\frac{\partial}{\partial z}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})w + f_z \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

2. Exact Solution for General Form of 3-Dimentional Navier-Stokes Equations (Include time) in Wikipedia

For we solve equations, we use Newton potential and time function. If we think the solution of Laplace equation,

$$\vec{v} = \vec{v}(\vec{x}, t) = (u, v, w)$$

$$u = \frac{C_1}{r^3}xf(t) \ , \ v = \frac{C_1}{r^3}yf(t), w = \frac{C_1}{r^3}zf(t), \ r = \sqrt{x^2 + y^2 + z^2} \ , \ r \geq 1 \tag{7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = f(t)C_1[\frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3})]$$

$$= f(t)C_1[(-\frac{3x^2}{r^5} + \frac{1}{r^3}) + (-\frac{3y^2}{r^5} + \frac{1}{r^3}) + (-\frac{3z^2}{r^5} + \frac{1}{r^3})] = 0 \tag{8}$$

Hence, Eq(7) satisfy the continuity equation-Eq(6)

In this case, if we solve left term of Eq(1), the left term of Eq(3) is

$$\frac{\partial u}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})u = \frac{C_1}{r^3}x\dot{f}(t) + f^2(t)\frac{C_1}{r^3}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})(\frac{C_1}{r^3}x)$$

$$= \frac{C_1}{r^3} x\dot{f}(t) + f^2(t)\frac{C_1}{r^3}[x(-3\frac{C_1}{r^5}x^2 + \frac{C_1}{r^3}) + y(-3\frac{C_1}{r^5}xy) + z(-3\frac{C_1}{r^5}xz)]$$

$$= \frac{C_1}{r^3} x\dot{f}(t) + f^2(t)\frac{C_1}{r^3}(-2\frac{C_1}{r^3})x = \frac{C_1}{r^3} x\dot{f}(t) - f^2(t)\frac{2C_1^2}{r^6}x \qquad (9)$$

So, left terms of Eq(4), Eq(5) are

$$\frac{\partial v}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})v = \frac{C_1}{r^3} y\dot{f}(t) - f^2(t)\frac{2C_1^2}{r^6}y \qquad (10)$$

$$\frac{\partial w}{\partial t} + (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})w = \frac{C_1}{r^3} z\dot{f}(t) - f^2(t)\frac{2C_1^2}{r^6}z \qquad (11)$$

In this time, if $f(t)$ is,

$$\frac{1}{[f(t)]^2}\frac{d}{dt}[f(t)] = 1 \;\rightarrow\; f(t) = \frac{1}{C'-t},\;\; C' < 0 \text{ is constant} \qquad (12)$$

Therefore, the left term of Eq(1) is

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} = = \frac{C_1}{r^3}\vec{r}\frac{1}{(C'-t)^2} - \frac{2C_1^2}{r^6}\vec{r}\frac{1}{(C'-t)^2} \qquad (13)$$

In this case, if we solve right term of Eq(1), the right term of Eq(3) is

$$-\frac{1}{\rho_0}\frac{\partial}{\partial x}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u + f_x,\;\; \upsilon > 0,\;\; \rho = \rho_0 \qquad (14)$$

In this time,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial x}(-\frac{3C_1}{r^5}x^2 + \frac{C_1}{r^3}) + \frac{\partial}{\partial y}(-\frac{3C_1}{r^5}xy) + \frac{\partial}{\partial z}(-\frac{3C_1}{r^5}xz)$$

$$= [(15\frac{C_1}{r^7}x^3 - \frac{6C_1}{r^5}x) + (-\frac{3C_1}{r^5}x)] + (\frac{15C_1}{r^7}xy^2 - \frac{3C_1}{r^5}x) + (\frac{15C_1}{r^7}xz^2 - \frac{3C_1}{r^5}x)$$

$$= 15\frac{C_1}{r^5}x - \frac{15C_1}{r^5}x = 0 \qquad (15)$$

So,

$$\nabla^2 v = \nabla^2 w = 0 \qquad (16)$$

Hence, right terms of Eq(3),Eq(4),Eq(5) are

$$-\frac{1}{\rho_0}\frac{\partial}{\partial x}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u + f_x = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + f_x \qquad (17)$$

$$-\frac{1}{\rho_0}\frac{\partial}{\partial y}p + \upsilon(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})v + f_y = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + f_y \qquad (18)$$

$$-\frac{1}{\rho_0}\frac{\partial}{\partial z}p+\upsilon(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})w+f_z=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+f_z \quad (19)$$

Hence, if we solve Eq(1),

$$\frac{C_1}{r^3}x\frac{1}{(C'-t)^2}-\frac{2C_1^2}{r^6}x\frac{1}{(C'-t)^2}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+f_x \quad (20)$$

$$\frac{C_1}{r^3}y\frac{1}{(C'-t)^2}-\frac{2C_1^2}{r^6}y\frac{1}{(C'-t)^2}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+f_y \quad (21)$$

$$\frac{C_1}{r^3}z\frac{1}{(C'-t)^2}-\frac{2C_1^2}{r^6}z\frac{1}{(C'-t)^2}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+f_z \quad (22)$$

In this time, if $\vec{f}(\vec{x},t)=0$ is for smooth function $p(\vec{x},t)$.Hence,

$$p(\vec{x},t)=\rho_0\frac{C_1}{r}\frac{1}{(C'-t)^2}-\frac{1}{2}\rho_0\frac{C_1^2}{r^4}\frac{1}{(C'-t)^2} \quad ,r\geq 1 \quad (23)$$

3. Hypotheses and Growth Conditions(Smoothness)

The initial condition $\vec{V}_0(x)=\frac{C_1}{r^3}\vec{r}\frac{1}{C'}$ is, for every multi-index α and any $K>0$,there exist a constant $C=C(\alpha,K)>0$ such that

$$\left|\partial^\alpha\vec{V}_0(x)\right|=\left|\partial^\alpha(\frac{C_1}{r^3}\vec{r}\frac{1}{C'})\right|\leq\frac{C}{r^K}<\frac{C}{(1+|x|)^K}, \text{ for all } x\in R^3,$$

$$r^K>(1+|x|)^K,\ \ r\geq 1 \quad (24)$$

The external force $\vec{f}(x,t)=0$ is

$$\left|\partial^\alpha\vec{f}(x)\right|=0\leq\frac{C}{(1+|x|+t)^K}, \text{ for all } (x,t)\in R^3\times[0,\infty) \quad (25)$$

More precisely, the following assumptions are made:

1.$\vec{V}(\vec{x},t)=\frac{C_1}{r^3}\vec{r}\frac{1}{(C'-t)}\in[C^\infty(R^3\times[0,\infty))]^3$, $C'<0$, $r\geq 1$ (26)

Hence,

$$p(\vec{x},t)=\rho_0\frac{C_1}{r}\frac{1}{(C'-t)^2}-\frac{1}{2}\rho_0\frac{C_1^2}{r^4}\frac{1}{(C'-t)^2}\in[C^\infty(R^3\times[0,\infty))]^3, C'<0,\ r\geq 1 \quad (27)$$

2.There exists a constant $E \in (0,\infty)$ such that

$$\int_{R^3} \left|\vec{v}(x,t)\right|^2 dx = \int_{R^3} \left|\frac{C_1}{r^3}\frac{\vec{x}}{(C'-t)}\right|^2 dx$$

$$= \int_{R^3} \left| \frac{C_1^2}{r^6}\frac{r^2 dx}{(C'-t)^2} \right| < \int_{R^3} \left| \frac{C_1^2}{r^4}\frac{dr}{(C'-t)^2} \right| \quad , \quad dx < dr$$

$$= \left| -\frac{C_1^2}{3}\frac{1}{r^3}\frac{1}{(C'-t)^2} \right| < \frac{C_1^2}{3} = E \text{, for all } t \geq 0.$$

If $C' < -1$, $r \geq 1$, $r^3(C'-t) > 1$ (28)

*The Millennium Prize Conjectures in the Whole Space

(A) Existence and smoothness of the Navier-Stokes solution in R^3.

Let $\vec{f}(x,t) \equiv 0$. For any intial condition $\vec{v}_0(x)$ satisfying the above hypotheses there exist smooth and globally defined solutions to the Navier-Stokes equations, i.e. there is a velocity vector $\vec{v}(x,t)$ and a pressure $p(x,t)$ satisfying conditions 1 and 2 above.

4. Conclusion

Therefore, the exact smooth solution of Navier-Stokes equations in Wikipedia(3-dimensional) is

Density: $\rho = \rho_0$,

Velocity Components: $\vec{V} = \vec{v}(\vec{x},t) = (u,v,w) = \frac{C_1}{r^3}(x,y,z)\frac{1}{(C'-t)}, C' < -1, r \geq 1$

The external volumetric force: $\vec{f}(\vec{x},t) = 0$

Pressure: $p(\vec{x},t) = \rho_0 \frac{C_1}{r}\frac{1}{(C'-t)^2} - \frac{1}{2}\rho_0 \frac{C_1^2}{r^4}\frac{1}{(C'-t)^2}, C' < -1, r \geq 1$

References

[1]Wikipedia,” Navier-Stokes existence and smoothness”