Two VŁ4 theorems not in S5

Abstract: We evaluate two equations not found as theorems in S5; both are theorems in VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪; - Not Or; & And, ∧, ∩; \ Not And;
> Imply, greater than, →, ↦, ≻, ⊃, ⊢, ⊨, ↠; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞;
= Equivalent, ≡, :=, ⟺, ↔, ≜; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, ∅, Null, ⊥, zero;
(%z<#z) C as contingency, Δ, ordinal 1; (%z>#z) N as non-contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B).

From: Font, J.M.; Hájek, P. (2000). On Łukasiewicz's four-valued modal logic.
citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.25.8024&rep=rep1&type=pdf

Remark 0: The modal truth table of the authors, while mistaken and corrected below, does not affect the logical results which follow:

	given as ◊	given as □	should be ◊	should be □	
11	11	10	11	01	11
10	11	10	11	01	10
01	01	00	10	00	01
00	01	00	10	00	00

Theorem 2 proof: It is straightforward to show that

$$\Diamond\varphi\equiv(\Diamond\top\&\ \varphi) \quad (2.1.1)$$

%p=(%(p=p)&p) ; NTNT NTNT NTNT NTNT (2.1.2)

$$\text{and } \Box\varphi\equiv(\Diamond\top\rightarrow\varphi) \quad (2.2.1)$$

#p=(%(p=p)>p) ; TNTN TNTN TNTN TNTN (2.2.2)

hold in *L*.

Remark 2: Eqs. 2.1.2 and 2.2.2 are *not* tautologous, meaning Theorem 2 is refuted.

Certainly *L* is a rather unusual class of Kripke models; e.g.

[unnumbered equations on page 6]

$\Diamond\varphi\rightarrow\varphi$ (6.1.1)

%p>p ; NTNT NTNT NTNT NTNT (6.1.2)

and $\varphi\rightarrow\Box\varphi$ (6.2.1)

p>#p ; TNTN TNTN TNTN TNTN (6.2.2)

hold in *L,*

while $\varphi\rightarrow\Diamond\varphi$ (6.3.1)

p>%p ; TTTT TTTT TTTT TTTT (6.3.2)

and $\Box\varphi\rightarrow\varphi$ (6.4.1)

#p>p ; TTTT TTTT TTTT TTTT (6.4.2)

don't.

Remark 6: Eqs. 6.1.2 and 6.2.2 are *not* tautologous and do not hold in *L* while6.3.2 and 6.4.2 are tautologous and hold in *L*. These results are the opposite of what the authors claim.

[I]t is clear that the resulting formulas [unnumbered equations on page 16]

LET p, q, r, s: x, a, b, c

$(\Box(\forall x)(b(x)\rightarrow a(x))\ \&\ \ (\forall x)(c(x)\rightarrow b(x)))\rightarrow\Box(\forall x)(c(x)\rightarrow a(x))$ (16.1.1)

(#((r&#p)>(q&#p))& ((s&#p)>(r&#p)))>
#((s&#p)>(q&#p)) ; TTTT TTTT TTTT TTTT (16.1.2)

$(\ \ (\forall x)(b(x)\rightarrow a(x))\&\Box(\forall x)(c(x)\rightarrow b(x)))\rightarrow\Box(\forall x)(c(x)\rightarrow a(x))$ (16.2.1)

(((r&#p)>(q&#p))&#((s&#p)>(r&#p)))>
#((s&#p)>(q&#p)) ; TTTT TTTT TTTT TTTT (16.2.2)

are not theorems of any of the normal modal logics, as they are not theorems of predicate S5.

Remark 16: Eqs. 16.1.2 and 16.2.2 are theorems of the normal modal logic VŁ4, the opposite result of what the authors claim.

Results for eight Eqs. are the opposite of what the authors claim, thereby refuting their assertions.