\[
\lim_{n \to \infty} \ln(n) + \sum_{k=1}^{n} \frac{1}{k} = 0.5772156649...
\]

I know I can make this into two limits:

\[
\lim_{x \to \infty} \ln(x) + \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k}
\]

I know \(\ln x\) can be defined as a limit a rearranged version of the one on Wikipedia under properties to avoid a 0/0, or undefined.

The this limit is equal to natural log \(\lim_{n \to 0} \frac{x^n - 1}{n}\) which can be rearranged \(\lim_{n \to 0} (x^n - 1) \frac{1}{n}\) using the distributive property this can be made into \(\lim_{n \to 0} \frac{1}{n} x^n - \frac{1}{n}\) finally, this can be put as an infinite limit by inverting every n in the limit \(\lim_{n \to \infty} n x^\frac{1}{n} - n\)

Now I can replace \(\ln(x)\) to make this double limit

\[
\lim_{x \to \infty} (\lim_{n \to \infty} n x^\frac{1}{n} - n)
\]

Using simplification because all variable in the function are approaching infinity, I can now put this as one limit

\[
\lim_{n \to \infty} n x^\frac{1}{n} - n
\]

Breaking this down into several limits using order of operation

\[
\lim_{n \to \infty} n \quad \lim_{n \to \infty} x^\frac{1}{n} - \lim_{n \to \infty} n
\]

I know \(\lim_{n \to \infty} n^\frac{1}{n}\) approaches to 1 by using a calculator.

The limit \(\lim_{n \to \infty} n\) can be solved by direct substitution just making it \(\infty\). Wolfram Alpha proof here.

That leaves me with

\[\infty \cdot 1 - \infty\]

Which is 0

Thus I can now replace the \(\lim_{x \to \infty} \ln(x)\) to be 0 this leaves me with the sum and limit

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k}
\]

Which can just be put as

\[\sum_{k=1}^{\infty} \frac{1}{k} = 0.5772156649...\]

Anywho I should be off stuff like this I’m meh at especially late at night I don’t expect to be perfect