Refutation of the axiomatic theory of betweenness

Abstract: We evaluate sixteen equations as axioms and conclusions, for nine as *not* tautologous. This refutes the axiomatic theory of betweenness.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪; - Not Or; & And, ∧, ∩; \ Not And;
> Imply, greater than, →, ↦, ≻, ⊃, ⊢, ⊨, ↠; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞;
= Equivalent, ≡, :=, ⟺, ↔, ≜; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, ∅, Null, ⊥, zero;
(%z<#z) C as contingency, Δ, ordinal 1; (%z>#z) N as non-contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆y); (A=B) (A~B).

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4. Axioms
For any given set V , our axiomatic system consists of the following axioms in the language Φ(V):

1. Trivial Path: ¬(A|B|C) if A∩C≠∅ (4.1.1)

LET p, q, r: A, B, C
((p&r)@(p@p))>~((p<q)<r) ; TTTT TTTT TTTT TTTT (4.1.2)

2. Empty Set: ∅|B|C (4.2.1)

LET q, r: B, C
(p@p)<(q<r) ; **FFFF FFFF FFFF FFFF** (4.2.2)

3. Aggregation: A_1|B|C→(A_2|B|C→A_1,A_2|B|C) (4.3.1)

LET p, q, r, s: A_1, A_2, B, C
((p<r)<s)>(((q<r)<s)>((p&q)<(r<s))) ; TTTT TTTT TTTT TTTT (4.3.2)

4. Symmetry: A|B|C→C|B|A (4.4.1)

LET p, q, r: A, B, C
((p<q)<r)>((r<q)<p) ; T**F**TT TTTT T**F**TT TTTT (4.4.2)

5. Left Monotonicity: A_1,A_2|B|C→A_1|B|C (4.5.1)

LET p, q, r, s: A_1, A_2, B, C
((p&q)<(r<s))>(p<(r<s)) ; TTTT TTTT TTTT TTTT (4.5.2)

6. Central Monotonicity: $A|B_1|C \rightarrow A|B_1,B_2|C$ (4.6.1)

LET p, q, r, s: A, B1, B2, C
(p<(q<s))>((p<(q&r))<s) ; TTTT TTTT T**F**T**F** T**F**T**F** (4.6.2)

7. Insertion: $A|B_1,I,B_2|C \rightarrow (A|I,C|B_1 \rightarrow (B_2|A,I|C \rightarrow A|I|C))$ (4.7.1)

LET p, q, r, s, t, u: A, I, B, C, B_1, B_2
(p<((t&(q&u))<s))>((p<((q&s)<t))>((u<((p&q)<s))>(p<~(s>q)))) ;
TTTT TTTT TTTT TTTT(2),
TTTT TTTT T**F**TT T**F**TT(2) (4.7.2)

8. Transitivity: $\neg (A|B|d) \rightarrow (\neg(d|B|C) \rightarrow \neg(A|B|C))$, where $d \notin B$ (4.8.1)

LET p, q, r, s: A, B, C, d
(s<q)>(~((p<q)<s)>(~(s<(q<r))>~((p<q)<r))) ;
TTTT TTTT TTTT TTTT (4.8.2)

In the above axioms by A; B we denote the union of sets A and B. Note that we represent union by comma only inside [the] betweenness predicate. In all other setting[s], to avoid confusion, we use standard notations A [∩] B.

3. Conclusion

With minimal modifications to the proofs given in this article, one can show the following logical system completely axiomatizes the non-strict betweenness relation:

1. Trivial Path: $\neg(A|B|C)$ if $(A \cap C) \backslash B \neq \varnothing$ (7.1.1)

LET p, q, r: A, B, C
(((p&r)\q)@(p@p))>~((p<q)<r) ; T**F**TT TTTT T**F**TT TTTT (7.1.2)

Remark 7.1: Eq. 7.1.2 differs from 4.1.2.

2. Empty Set: $\varnothing|B|C$ (7.2.1)

LET q, r: B, C
(p@p)<(q<r) ; **FFFF FFFF FFFF FFFF** (7.2.2)

Remark 7.2: Eq. 7.2.2 is the same as 4.2.2.

3. Reflexivity $A|B|C$, where $A \subseteq B$ (7.3.1)

LET p, q, r: A, B, C
~(q<p)>((p<q)<r) ; TTTT TTTT TTTT TTTT (7.3.2)

Remark 7.3: Eq. 7.3.2 is not included in the list of 4. Axioms.

4. Aggregation: $A_1|B|C\rightarrow(A_2|B|C\rightarrow A_1,A_2|B|C)$ (7.4.1)

LET p, q, r, s: A_1, A_2, B, C
((p<r)<s)>(((q<r)<s)>((p&q)<(r<s))) ; TTTT TTTT TTTT TTTT (7.4.2)

Remark 7.4: Eq. 7.4.2 is renamed for 4.3.2.

5. Symmetry: $A|B|C\rightarrow C|B|A$ (7.5.1)

LET p, q, r: A, B, C
((p<q)<r)>((r<q)<p) ; T**F**TT TTTT T**F**TT TTTT (7.5.2)

Remark 7.5: Eq. 7.5.2 is renamed for 4.4.2.

6. Monotonicity: $A_1,A_2|B|C\rightarrow A_1|B|C$ (7.6.1)

LET p, q, r, s: A_1, A_2, B, C
(((p&q)<r)<s)>((p<r)<s) ; TTTT TTTT TTTT TTTT (7.6.2)

Remark 7.6: Eq. 7.6.2 is renamed for 4.5.1, from left monotonicity.
There is no central monotonicity as Eq. 4.6.1 in the list of 7. Conclusion.

7. Insertion: $A|B_1,I,B_2|C\rightarrow(A|I,C|B_1\rightarrow(B_2|A, I|C\rightarrow A|I|C))$
where $A\cap B_2=B_1\cap C=A\cap C=\varnothing$ (7.7.1)

LET p, q, r, s, t, u: A, I, B, C, B_1, B_2
((((p&u)=(t&s))=(p&s))=(p@p))>((p<((q&s)<t))>((u<((p&q)<s))>(p<~(s>q)))) ;
TTTT TTTT TTTT TTTT(2), FFFF FFFF TFTT TFTT(1),
TTTT TTTT TTTT TTTT(3), FFFF FFFF TFTT TFTT(1),
TTTT TTTT TTTT TTTT(3), FFFF FFFF TFTT TFTT(1),
TTTT TTTT TTTT TTTT(3), FFFF FFFF TFTT TFTT(1),
TTTT TTTT TTTT TTTT(3), FFFF FFFF TFTT TFTT(1),
TTTT TTTT TTTT TTTT(1) (7.7.2)

Remark 7.7: Eq. 7.7.2 differs from 4.7.2.

8. Transitivity: $A|B|C\rightarrow(A|B|d\vee d|B|C$ (7.8.1)

LET p, q, r, s: A, B, C, d
(p<(q<r))>(((p<q)<s)+(s<(q<r))) ; TTTT TTT**F** TTTT TTTT (7.8.2)

Remark 7.8: Eq. 7.8.2 differs from 4.8.2.

Of the 16 equations under sections for Axioms and Conclusion, nine are *not* tautologous. This refutes the axiomatic theory of betweenness.