We can divide the numbers and analytic functions by zero with a natural sense

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Abstract: As a very famous common sense, we know that we can not divide the numbers and functions by zero. For this, we will give a very natural new meaning that we can divide the numbers and analytic functions by zero with many evidences.

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1 Division by zero calculus

The division by zero has the long and mysterious story over the world (see, for example, C.V. Boyer [1], H.G. Romig [16] and Google site with the division by zero) with its physical viewpoints since the document of zero in India in AD 628. In particular, note that Brahmagupta (598 -668?) established four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brāhmasphuṭasiddhānta (648). Our world history, however, stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we will see that his definition is right and suitable.

For some recent situation on the division by zero, see [13].

The division by zero was indeed trivial and clear as in the followings:

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $a/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [2] for example.

Division by zero is trivial and clear from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [2, 20].

The simple field structure containing division by zero was established by M. Yamada ([5]).

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [8, 9, 10, 11, 12, 13] for example.

The division by zero opens a new world since Aristotelēs-Euclid. See the references for recent related results. See [3, 4, 6, 15] for example.

As the number system containing the division by zero, the Yamada field structure is completed. However, for applications of the division by zero to **functions**, we need the **concept of the division by zero calculus** for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z - i}{z + i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z - (-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z - i) \cdot \frac{1}{z + i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n, \tag{1.5}$$

we **define** the identity, by the division by zero

$$f(a) = C_0. \tag{1.6}$$

Note that here, there is no problem on any convergence of the expansion (1.5) at the point $z = a$, because all the terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$.

For the correspondence (1.6) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (1.5), we can **define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n! C_n.$$

Apart from the motivation, we define the division by zero calculus by (1.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

As many important applications of the division by zero calculus, we can see the references, however, as important examples, we state here only three examples.

1: For the function $f(z) = 1/z$, $f(0) = 0$. Note that the point at infinity is represented by zero.

2: For the function $f(z) = \exp(1/z)$, $f(0) = 1$. Note that the value 1 is the Picard's exceptional value at the origin $z = 0$.

3: For the function $f(z) = \tan z$, $f(\pi/2) = 0$. Note that the gradient of the y axis is zero.

It is a famous word that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero. For this old and general concept, we will give a simple and affirmative answer. In particular, certainly we gave several generalizations of division as in referred in the above, however, we will wish to understand with some good feelings for **the division by zero**. We wish to know the division by zero with some good feelings. We wish to give clearly a good meaning for the division by zero in this paper.

2 Conclusion

We can divide the numbers and analytic functions by zero.

We will give a precise meaning for the above statement that is the main purpose of this short paper.

Theorem: *For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus*

$$\frac{f(z)}{z^n} \tag{2.1}$$

at the point $z = 0$, for any positive integer n.

This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \mid_{z=0} . \tag{2.2}$$

For example,

$$\frac{e^x}{x^n} \mid_{x=0} = \frac{1}{n!}.$$

In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

We gave many applications of the theorem, see the references. However, we will state one typical example.

For example, for the simple example for the line equation on the x, y plane

$$ax + by + c = 0$$

we have, formally

$$x + \frac{by + c}{a} = 0,$$

and so, by the division by zero, we have, for $a = 0$, the reasonable result

$$x = 0.$$

This case may be looked as the cases of $a \to 0$.

For the equation $y = mx$, from

$$\frac{y}{m} = x,$$

we have, by the division by zero, $x = 0$ for $m = 0$. This gives the case $m = \pm\infty$ of the gradient of the line. – This will mean that the equation $y = mx$ represents the general line through the origin containing the line $x = 0$ in this sense. – This method was applied in many cases, for example see also [8, 9].

However, from

$$\frac{ax + by}{c} + 1 = 0,$$

for $c = 0$, we have the contradiction, by the division by zero

$$1 = 0.$$

For this case, we should consider that

$$\frac{ax + by}{c} + \frac{c}{c} = 0,$$

that is always valid. **In this sense, we can divide an equation by zero.**

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