A Complete Proof of the $ABC$ Conjecture: The End of The Mystery

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Abstract In this paper, we consider the $ABC$ conjecture. Firstly, we give a proof of a the first conjecture that $C < rad^2(ABC)$. It is the key of the proof of the $ABC$ conjecture. Secondly, a proof of the $ABC$ is given for $\epsilon \geq 1$, then for $\epsilon \in [0,1]$ for the two cases: $c \leq rad(abc)$ and $c > rad(abc)$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = 6^{1+\epsilon}e\left(\frac{1}{\epsilon^2} - \epsilon\right)$. Five numerical examples are presented. It is the end of the mystery of the $ABC$ conjecture!

Keywords Elementary number theory · real functions of one variable.

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To the memory of my Father who taught me arithmetic
To the memory of Jean Bourgain (1954-2018) for his mathematical work notably in the field of Number Theory

1 Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $rad(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a) \prod_i a_i^{\alpha_i-1} \quad (1)$$
We note:

\[ \mu_a = \prod_i a_i^{a_i - 1} \implies a = \mu_a \cdot \text{rad}(a) \tag{2} \]

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) (1). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

**Conjecture 1 (ABC Conjecture):** Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then for each \( \epsilon > 0 \), there exists a constant \( K(\epsilon) \) such that:

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \tag{3} \]

\( K(\epsilon) \) depending only of \( \epsilon \).

We know that numerically,\[ \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \tag{2} \]. A conjecture was proposed that \( c < \text{rad}^2(abc) \) (4). Here we will give a proof of it.

**Conjecture 2** Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[ c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2 \tag{4} \]

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.

**2 A Proof of the conjecture (2)**

Let \( a, b, c \) positive integers, relatively prime, with \( c = a + b \). We suppose that \( b < a \).

If \( c < \text{rad}(ab) \) then we obtain:

\[ c < \text{rad}(ab) < \text{rad}^2(abc) \tag{5} \]

and the condition (4) is verified.

In the following, we suppose that \( c \geq \text{rad}(ab) \).

2.1 Case \( c = a + 1 \)

\[ c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^2(ac) \tag{6} \]

2.1.1 \( \mu_a = 1 \)

In this case, \( a = \text{rad}(a) \), it is immediately truth that:

\[ c = a + 1 < 2a < \text{rad}(a)\text{rad}(c) < \text{rad}^2(ac) \tag{7} \]

Then (6) is verified.
2.1.2 $\mu_a \neq 1, \mu_a < \text{rad}(a)$

we obtain:

$$c = a + 1 < 2\mu_a . \text{rad}(a) \Rightarrow c < 2\mu^2_a \Rightarrow c < \text{rad}^2(ac)$$  \hspace{1cm} (8)

Then (6) is verified.

2.1.3 $\mu_a \geq \text{rad}(a)$

We have:

$$c = a + 1 = \mu_a . \text{rad}(a) + 1 \leq \mu_a^2 + 1 < \text{rad}^2(ac)$$

We suppose that $\mu^2_a + 1 \geq \text{rad}^2(ac) \Rightarrow \mu^2_a > \text{rad}^2(a). \text{rad}(c) > \text{rad}^2(a)$ as $\text{rad}(c) > 1$, then $\mu_a > \text{rad}(a)$, that is the contradiction with $\mu_a \geq \text{rad}(a)$. We deduce that $c < \mu^2_a + 1 < \text{rad}^2(ac)$ and the condition (6) is verified.

2.2 $c = a + b$

We can write that $c$ verifies:

$$c = a + b = \text{rad}(a). \mu_a + \text{rad}(b). \mu_b = \text{rad}(a). \text{rad}(b) \left( \frac{\mu_a}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} \right)$$

$$\Rightarrow c = \text{rad}(a). \text{rad}(b). \text{rad}(c) \left( \frac{\mu_a}{\text{rad}(b). \text{rad}(c)} + \frac{\mu_b}{\text{rad}(a). \text{rad}(c)} \right)$$  \hspace{1cm} (9)

We can write also:

$$c = \text{rad}(abc) \left( \frac{\mu_a}{\text{rad}(b). \text{rad}(c)} + \frac{\mu_b}{\text{rad}(a). \text{rad}(c)} \right)$$  \hspace{1cm} (10)

To obtain a proof of (4), one method is to prove that:

$$\frac{\mu_a}{\text{rad}(b). \text{rad}(c)} + \frac{\mu_b}{\text{rad}(a). \text{rad}(c)} < \text{rad}(abc)$$  \hspace{1cm} (11)

2.2.1 $\mu_a = \mu_b = 1$

In this case, it is immediately truth that:

$$\frac{1}{\text{rad}(a)} + \frac{1}{\text{rad}(b)} \leq \frac{5}{6} < \text{rad}(c). \text{rad}(abc)$$  \hspace{1cm} (12)

Then (4) is verified.

2.2.2 $\mu_a = 1 \text{ and } \mu_b > 1$

As $b < a \Rightarrow \mu_b . \text{rad}(b) < \text{rad}(a) \Rightarrow \frac{\mu_b}{\text{rad}(a)} < \frac{1}{\text{rad}(b)}$, then we deduce that:

$$\frac{1}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} < \frac{2}{\text{rad}(b)} < \text{rad}(c). \text{rad}(abc)$$  \hspace{1cm} (13)

Then (4) is verified.
2.2.3 $\mu_b = 1$ and $\mu_a \leq (b = \text{rad}(b))$

In this case we obtain:

$$\frac{1}{\text{rad}(a)} + \frac{\mu_a}{\text{rad}(b)} \leq \frac{1}{\text{rad}(a)} + 1 < \text{rad}(c) \cdot \text{rad}(abc)$$

(14)

Then (14) is verified.

2.2.4 $\mu_b = 1$ and $\mu_a > (b = \text{rad}(b))$

As $\mu_a > \text{rad}(b)$, we can write $\mu_a = \text{rad}(b) + n$ where $n \geq 1$. We obtain:

$$c = \mu_a \cdot \text{rad}(a) + \text{rad}(b) = (\text{rad}(b) + n) \cdot \text{rad}(a) + \text{rad}(b) = \text{rad}(ab) + n \cdot \text{rad}(a) + \text{rad}(b)$$

(15)

We have $n < b$, if not $n \geq b = \Rightarrow \mu_a \geq 2b = \Rightarrow a \geq 2b \cdot \text{rad}(a) = \Rightarrow a \geq 3b = \Rightarrow c > 3b$, then the contradiction with $c > 2b$. We can write:

$$c < 2 \cdot \text{rad}(ab) + \text{rad}(b) \Rightarrow c < \text{rad}(abc) + \text{rad}(abc) < \text{rad}^2(abc) \Rightarrow c < \text{rad}^2(abc)$$

(16)

2.2.5 $\mu_a, \mu_b \neq 1, \mu_a < \text{rad}(a)$ and $\mu_b < \text{rad}(b)$

We obtain:

$$c = \mu_a \cdot \text{rad}(c) = \mu_a \cdot \text{rad}(a) + \mu_b \cdot \text{rad}(b) < \text{rad}^2(a) + \text{rad}^2(b) < \text{rad}^2(abc)$$

(17)

2.2.6 $\mu_a, \mu_b \neq 1, \mu_a \leq \text{rad}(a)$ and $\mu_b \geq \text{rad}(b)$

We have:

$$c = \mu_a \cdot \text{rad}(a) + \mu_b \cdot \text{rad}(b) < \mu_a \cdot \mu_b \cdot \text{rad}(a) \cdot \text{rad}(b) \leq \mu_b \cdot \text{rad}^2(a) \cdot \text{rad}(b)$$

(18)

Then if we give a proof that $\mu_b < \text{rad}(b) \cdot \text{rad}^2(c)$, we obtain $c < \text{rad}^2(abc)$. As $\mu_b \geq \text{rad}(b) = \Rightarrow \mu_b = \text{rad}(b) + \alpha$ with $\alpha$ a positive integer $\geq 0$. Supposing that $\mu_b \geq \text{rad}(b) \cdot \text{rad}^2(c) = \Rightarrow \mu_b = \text{rad}(b) \cdot \text{rad}^2(c) + \beta$ with $\beta \geq 0$ a positive integer.

We can write:

$$\text{rad}(b) \cdot \text{rad}^2(c) + \beta = \text{rad}(b) + \alpha = \beta < \alpha$$

$$\alpha - \beta = \text{rad}(b) \cdot (\text{rad}^2(c) - 1) > 3 \cdot \text{rad}(b) = \Rightarrow \mu_b = \text{rad}(b) + \alpha > 4 \cdot \text{rad}(b)$$

(19)

Finally, we obtain:

$$\begin{cases} 
\mu_b \geq \text{rad}(b) \\
\mu_b > 4 \cdot \text{rad}(b)
\end{cases}$$

(20)

Then the contradiction and the hypothesis $\mu_b \geq \text{rad}(b) \cdot \text{rad}^2(c)$ is false. Hence:

$$\mu_b < \text{rad}(b) \cdot \text{rad}^2(c) = \Rightarrow c < \text{rad}^2(abc)$$

(21)
2.2.7 \( \mu_a \mu_b \neq 1 \), \( \mu_a \geq \text{rad}(a) \) and \( \mu_b \leq \text{rad}(b) \)

The proof is identical to the case above.

2.2.8 \( \mu_a \mu_b \neq 1 \), \( \mu_a \geq \text{rad}(a) \) and \( \mu_b \geq \text{rad}(b) \)

We write:

\[
c = \mu_a \text{rad}(a) + \mu_b \text{rad}(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2 \mu_b^2 < \text{rad}^2(a) \cdot \text{rad}^2(b) \cdot \text{rad}^2(c) = \text{rad}^2(abc)
\]

(22)

Supposing that \( \mu_a \mu_b \geq \text{rad}(abc) \), we obtain:

\[
\mu_a \mu_b \geq \text{rad}(abc) \Rightarrow \text{rad}(a) \cdot \text{rad}(b) \cdot \mu_a \mu_b \geq \text{rad}^2(ab) \cdot \text{rad}(c) \Rightarrow \\
ab \geq \text{rad}^2(ab) \cdot \text{rad}(c) \Rightarrow a^2 > ab \geq \text{rad}^2(ab) \cdot \text{rad}(c) \\
\Rightarrow a > \sqrt{\text{rad}(ab) \cdot \text{rad}(c)} \geq \sqrt{\text{rad}(ab)} \sqrt{\text{rad}(c)} \\
\begin{cases} 
  c > \sqrt{7} \text{rad}(ab) \geq 3 \text{rad}(ab) \\
  c \geq \text{rad}(ab)
\end{cases}
\]

(23)

The inequality \( c \geq 3 \text{rad}(ab) \) gives the contradiction with the condition \( c \geq \text{rad}(ab) \) supposed at the beginning of this section. Then we obtain \( \mu_a \mu_b - \text{rad}(abc) < 0 \Rightarrow c < \text{rad}^2(abc) \).

We announce the theorem:

**Theorem 1 (Abdelmajid Ben Hadj Salem, 2019)** Let \( a, b, c \) positive integers relatively prime with \( c = a + b \) and \( 1 \leq b < a \), then \( c < \text{rad}^2(abc) \).

3 The Proof of The \( ABC \) Conjecture

We denote \( R = \text{rad}(abc) \).

3.1 Case: \( \epsilon \geq 1 \)

Using the result of the theorem above, we have \( \forall \epsilon \geq 1 \):

\[
c < R^2 \leq R^{1+\epsilon} < K(\epsilon) \cdot R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} \left( \frac{1}{\epsilon^2} - \epsilon \right), \quad \epsilon \geq 1
\]

(24)
3.2 Case: $\epsilon < 1$

3.2.1 Case: $c \leq R$

In this case, we can write:

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon}e\left(\frac{1}{\epsilon^2} - \epsilon\right), \quad \epsilon < 1$$

(25)

and the ABC conjecture is true.

3.2.2 Case: $c > R$

In this case, we confirm that:

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon}e\left(\frac{1}{\epsilon^2} - \epsilon\right), \quad 0 < \epsilon < 1$$

(26)

If not, then $\exists \epsilon_0 \in ]0, 1[$, so that the triplets $(a, b, c)$ checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0)$$

(27)

are in finite number. We have:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \implies R^{1-\epsilon_0} > K(\epsilon_0)$$

(28)

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > K(\epsilon_0) \implies c > K(\epsilon_0)\left(\frac{1}{1-\epsilon_0}\right)$$

(29)

We deduce that it exists an infinity of triples $(a, b, c)$ verifying [27], hence the contradiction. Then the proof of the ABC conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with $a, b, c$ relatively coprime:

$$c < K(\epsilon).\mathrm{rad}(abc)^{1+\epsilon} \quad \text{with} \quad K(\epsilon) = 6^{1+\epsilon}e\left(\frac{1}{\epsilon^2} - \epsilon\right)$$

(30)

Q.E.D

4. Examples

In this section, we are going to verify some numerical examples.
4.1 Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

\[ 3^{10} \times 109 + 2 = 23^5 = 6436343 \]  
(31)

\[ a = 3^{10}, 109 \Rightarrow \mu_a = 3^9 = 19683 \text{ and } \text{rad}(a) = 3 \times 109, \]
\[ b = 2 \Rightarrow \mu_b = 1 \text{ and } \text{rad}(b) = 2, \]
\[ c = 23^5 = 6436343 \Rightarrow \text{rad}(c) = 23. \]

Then \[ \text{rad}(abc) = 2 \times 3 \times 109 \times 23 = 15042. \]

For example, we take \( \epsilon = 0.01 \), the expression of \( K(\epsilon) \) becomes:

\[ K(\epsilon) = 6^{1.01},_{9999.99} = 1.8884880155640644914779227374022 e + 4343 \]  
(32)

Let us verify (30):

\[ c < K(\epsilon) \text{rad}(abc)^{1+\epsilon} \implies c = 6436343 < K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \implies 6436343 \ll K(0.01) \times 15042 \]  
(33)

Hence (30) is verified.

4.2 Example of A. Nitaj

4.2.1 Case 1

The example of Nitaj about the ABC conjecture [1] is:

\[ a = 11^{16}, 13^2, 79 = 613474843408551921511 \Rightarrow \text{rad}(a) = 11.13.79 \]  
(34)

\[ b = 7^2.41^2.311^3 = 2477678547239 \Rightarrow \text{rad}(b) = 7.41.311 \]  
(35)

\[ c = 2.3^3.5^{23}.953 = 613474845886230468750 \Rightarrow \text{rad}(c) = 2.3.5.953 \]  
(36)

\[ \text{rad}(abc) = 2.3.5.7.11.13.41.79.311.953 = 28828335646110 \]  
(37)

we take \( \epsilon = 100 \) we have:

\[ c < K(\epsilon) \text{rad}(abc)^{1+\epsilon} \implies 613474845886230468750 < 6^{101},e^{-99.9999}(2.3.5.7.11.13.41.79.311.953)^{101} \implies 613474845886230468750 < 8255869305610435609546415285004e + 48 \]

then (30) is verified.

4.2.2 Case 2

We take \( \epsilon = 0.5, \) then:

\[ c < K(\epsilon) \text{rad}(abc)^{1+\epsilon} \implies 613474845886230468750 < 6^{1.5},e^{3.5}(2.3.5.7.11.13.41.79.311.953)^{1.5} \implies 613474845886230468750 < 75333109597556257182261.66 \]  
(39)

We obtain that (30) is verified.
4.2.3 Case 3

We take \(\epsilon = 1\), then

\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies 613\,474\,845\,886\,230\,468\,750 < 6^2(2.3.5.7.11.13.41.79.311.953)^2 \implies 613\,474\,845\,886\,230\,468\,750 < 29\,918\,625\,700\,491\,952\,961\,692\,755\,600 \] (40)

We obtain that (30) is verified.

4.3 Example of Ralf Bonse

The example of Ralf Bonse about the ABC conjecture [2] is:

\[ 2543^4.182587.2802983.85813163 + 2^{15}.377.11.173 = 5^{56}.245983 \] (41)

\[ a = 2543^4.182587.2802983.85813163 \]

\[ b = 2^{15}.377.11.173 \]

\[ c = 5^{56}.245983 \]

\[ \text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163 \]

\[ \text{rad}(abc) = 1.5683959920004546031461002610848e + 33 \] (42)

For example, we take \(\epsilon = 0.01\), the expression of \(K(\epsilon)\) becomes:

\[ K(\epsilon) = 6^{1.01}.e^{9999.99} = 5.2903884296336672264108948608106e + 4343 \]

Let us verify (30):

\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies c = 5^{56}.245983 < 6^{1.01}.e^{9999.99}(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01} \]
\[ \implies 3.4136998783296235160378273576498e + 44 < 1.7819595478010681971905561514574e + 4377 \] (43)

The equation (30) is verified. Ouf, end of the mystery!

5 Conclusion

This is an elementary proof of the ABC conjecture, confirmed by four numerical examples. We can announce the important theorem:

**Theorem 2** (David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) Let \(a, b, c\) positive integers relatively prime with \(c = a + b\), then for each \(\epsilon > 0\), there exists \(K(\epsilon)\) such that :

\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \] (44)

where \(K(\epsilon)\) is a constant depending of \(\epsilon\) equal to \(6^{1+\epsilon}\epsilon \left(1 - \frac{\epsilon^2}{2}\right)\).
References

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