1. Introduction
In this short note we introduce the blow-up of the Feuerbach’s theorem. By blowing-up, I mean the procedure for replacing certain points with circles (see also [1, Section 8] for more examples of such blow-ups).

2. Formulation of the main fact
Before formulating the main theorem, we formulate the two preceding results.

**Theorem 1** (Feuerbach’s theorem (equivalent version)). Consider any triangle $ABC$ and let three blue circles have chords $BC$, $AC$, $AB$ and pairwise meet on the sides of $ABC$ at $A'$, $B'$, $C'$, respectively. Consider the green circle which is tangent to $BC$, $AC$, $AB$. Then the circumcircle $(A'B'C')$ of $A'B'C'$ is tangent to the green circle.

Feuerbach’s theorem
Theorem 2 (Feuerbach’s theorem on the Hyperbolic plane (Hart’s theorem)). Consider any triangle $ABC$. Let three blue circles have chords $BC$, $AC$, $AB$ and pairwise intersect at $A'$, $B'$, $C'$ (see the picture below). Consider the purple circles $(A'BC)$, $(B'AC)$, $(C'AB)$. Let the green circle is externally tangent to the three purple circles. Then the green circle is tangent to $(A'B'C')$. 

Feuerbach’s theorem in Hyperbolic geometry
Theorem 3 (Blow-up of Feuerbach’s theorem (Main fact)). Consider any three red circles. Let given three blue circles where each of them is internally tangent to the different pairs of red circles (see the picture below). Consider any three orange circles where each one of them is internally tangent to the different pairs of blue circles. Construct the three purple circles where each one of them is internally tangent to the orange circle and is internally tangent to the pair of red circles, see the picture below for more details. Let the green circle is externally tangent to the three purple circles. Then there exits a circle which is externally tangent to the three orange circles and is internally tangent to the green circle.
2.1. **Some tangent theorems as the consequences of the main fact.** Among the Feuerbach's theorem (and the Hart’s theorem) there is another famous tangent theorem which is the consequences of the Theorem 3.

**Theorem 4** (Lev Emelyanov’s theorem [2]). Consider any triangle $ABC$ and a point $P$. Let $A'B'C'$ be the circumcevian triangle of $P$ wrt $(ABC)$ (i.e. $AP$, $BP$, $CP$ meet $(ABC)$ second time at $A'$, $B'$, $C'$). Consider the circle $\omega_a$ which is internally tangent to $(ABC)$ at $A'$ and also is tangent to $BC$. Similarly define $\omega_b$, $\omega_c$. Then there exists a circle which is externally tangent to $\omega_a$, $\omega_b$, $\omega_c$ and is internally tangent to the incircle of $ABC$.

![Lev Emelyanov’s theorem](image1)

**Proof.** Consider the case of Theorem 3 when the three red circles are just points and three purple circles are lines. See the picture below.

![Particular case of Theorem 3](image2)
Now, consider the three radical lines of the three blue circles. These lines are concurrent and in the limiting case when the three blue circles coincide with the circumcircle of $ABC$ (i.e. with $(ABC)$) we get that these three radial lines will degenerate to the three concurrent chords $AA', BB', CC'$ of $(ABC)$. Thus, the three orange circles will degenerate into three circles which are tangent to $BC$, $AC$, $AB$ and to $(ABC)$ at $A'$, $B'$, $C'$, respectively. And now this is exactly Theorem 4.

\[\square\]

3. Unrelated stuff. Upcoming article

The main fact of this paper (Theorem 3) was motivated by Section 8 of the article [1]. In fact, the article [1] and its continuation [3] motivated most of my recent research on plane geometry, and in the near future I am going to write a large unifying article that will include some facts from these two articles. So I will be interested in any feedback about these previous works. Thus, for those who have read [1], [3] and have any remarks, suggestions, comments to them or have any suggestions of facts that might fit in, feel free to communicate me via my special email

plane.alexander@gmail.com

References

