Definitive Proof of Legendre's Conjecture

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1 Abstract

Legendre's conjecture, states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. In this paper, an equation was derived that accurately determines the number of prime numbers less than n for large values of n. Then it is proven by induction that there is at least one prime number between n^2 and $(n+1)^2$ for all positive integers n thus proving Legendre's conjecture.

2 Functions

Before we get into the proof, let me define a few functions that are necessary.

Let the function $l(x)$ represent the largest prime number less than x. For example, $l(10.5) = 7, l(20) = 19$ and $l(19) = 17$.

Let the function $\lambda(x)$ represent the largest prime number less than or equal to x. For example, $\lambda(10.5) = 7, \lambda(20) = 19$ and $\lambda(23) = 23$.

Let the function $k(n)$ represent the number of composite numbers in the set of odd integers less than or equal to n excluding 1. For example, $k(15) = 2$ since there are two composite numbers 9 and 15 that are less than or equal to 15.

Let the function $\pi(n)$ represent the number of prime numbers in the set of odd integers less than or equal to n. For example, $\pi(15) = 5$ since there are 5 prime numbers {3,5,7,11,13} that are less than or equal to 15.

Therefore, if there are P elements in the set of odd integers less than n, then $\pi(n) = P - k(n)$.

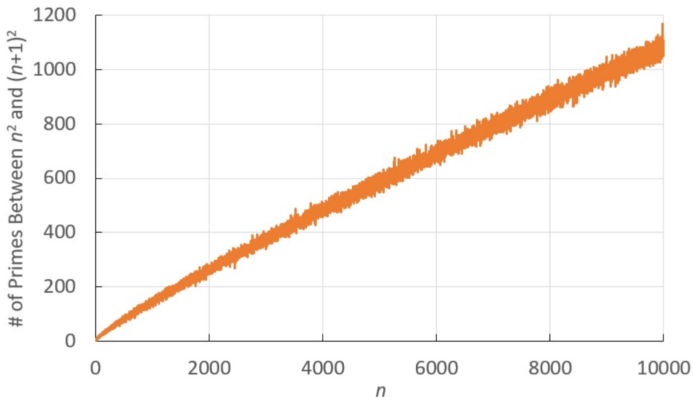


Figure 1: The number of primes between n^2 and $(n+1)^2$ steadily increases with increasing n.

3 Methodology

Legendre's conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. The conjecture is one of Landau's problems (1912) on prime numbers.

A quick look at the prime numbers between n^2 and $(n+1)^2$ shows that validity of Legendre's conjecture is very plausible (Figure 1). The difference between n^2 and $(n+1)^2$ and the number of primes between n^2 and $(n+1)^2$ steadily increase with increasing n. For $n = 999$, the difference between 999^2 and 1000^2 is 1999 and there are 139 primes between them. For $n = 9,999$, the difference between $9,999^2$ and $10,000^2$ is 19,999 with 1077 primes between them.

To calculate the number of primes between n^2 and $(n+1)^2$, we need a formula that accurately predicts the number of primes less than n. Although the prime number theorem states that $\pi(n) \lim_{n\to\infty} = n/\ln(n)$, this equation differs significantly from $\pi(n)$ even for very large values of n. At $n = 1,000,000$, the value of $n/\ln(n)$ underestimates $\pi(n)$ by 7.8%. Even at $n = 100,000,000$, the value of $n/\ln(n)$ underestimates $\pi(n)$ by 5.8%. A better equation for $\pi(n)$ is necessary.

In this paper, an equation is derived that determines the number of prime numbers less than n and, as n increases, the accuracy of the equation increases very rapidly. Then by induction, it is shown that there is at least one prime number between n^2 and $(n+1)^2$ thus proving the Legendre conjecture is true.

To derive an equation to determine the number of prime numbers less than n, we start with the set of all odd numbers less than n. Then all the composite numbers in the set that are evenly divisible by 3 are identified. Then all the composite numbers evenly divisible by 5, 7, 11 ... $\lambda(\sqrt{n})$ are identified where $\lambda(\sqrt{n}$ is the largest prime number less than or equal to n. We only have to go up to $\lambda(\sqrt{n})$ because there are no prime numbers greater than $\sqrt{n}$ that evenly divide n that are not evenly divisible by a lower prime number. By summing up the number of composite numbers in the set of odd numbers less than n and subtracting this from the total number of odd numbers less than n, gives us the number of prime numbers less than n.

Let us start with the set of all odd integers less than or equal to integer n excluding 1 as shown below.
{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,... n}

If n is odd, there are $(n-1)/2$ elements in the list. If n is even, there are $(n-2)/2$ elements in the list with $n-1$ as the largest element. In either case, as $n \to \infty$, the number of elements in the list approaches $n/2$.

Looking at those elements in the set that are divisible by 3, we notice that every third element after 3 (highlighted in yellow) beginning with 9, is divisible by 3.

{3,5,7, 9 ,11,13, 15 ,17,19, 21 ,23,25, 27 ,29,31, 33 ,35,37,... n}

Thus, as $n \to \infty$, the number of elements evenly divisible by 3, approaches the following equation:

$$\text{Number of elements divisible by } 3 \lim_{n\to\infty} = (n/2)(1/3)$$

Looking at those elements in the set that are divisible by 5, we notice that every fifth element after 5 (highlighted in yellow) beginning with 15, is divisible by 5.

{3,5,7,9,11,13, 15 ,17,19,21,23, 25 ,27,29,31,33, 35 ,37,... ,n}

But notice that, of the set of elements divisible by 5, every third element is also divisible by 3.

{ 15 ,25,35, 45 ,55,65, 75 ,85,95, 105 ,... ,n}

So to avoid double counting, we must multiply the number of elements evenly divisible by 5 by (2/3) giving the following equation:

$$\text{Number of elements divisible by 5 and not } 3 \lim_{n\to\infty} = (n/2)(2/3)(1/5)$$

Looking at those elements in the set that are divisible by 7, we notice that every seventh element after 7 (highlighted in yellow) beginning with 21, is divisible by 7.

But notice that every 3rd element (yellow) is also divisible by 3 and every 5th element (green) is divisible by 5.

{21,35,49,63,77,91,105,119,133,147,161,175...n}

{21,35,49,63,77,91,105,119,133,147,161,175...n}

So to avoid double counting, we must multiply the number of elements evenly divisible by 7 by (2/3) and (4/5) giving the following equation:

$$\text{Number of elements divisible by 7 and not 5 or}$$
$$3\lim_{n\to\infty} = (n/2)(2/3)(4/5)(1/7)$$

The general formula for the number of elements in the set of odd numbers less than n that are evenly divisible by prime number p and no lower prime number as $n \to \infty$ is as follows:

$$\text{Number of elements divisible only by}$$
$$p\lim_{n\to\infty} = (n/2)(2/3)(4/5)(6/7)(10/11)\ldots((l(p)-1)/l(p))(1/p)$$
$$\text{or}$$
$$\text{Number of elements divisible only by } p\lim_{n\to\infty} = (n/2)(1/p)\prod_{q=3}^{l(p)}(q-1)/q$$

The total number of composite numbers in the set of odd numbers less than or equal to n, defined as $k(n)$, is thus defined as follows:
$k(n) = (n/2)\{1/3 + (2/3)(1/5) + (2/3)(4/5)(1/7) + (2/3)(4/5)(6/7)(1/11) + \ldots + (2/3)(4/5)(6/7)(10/11)\ldots((l(\lambda(\sqrt{n}))-1)/l(\lambda(\sqrt{n})))(1/\lambda(\sqrt{n}))\}$

This can be written as

$$k(n) = (n/2)\sum_{p=3}^{\lambda(\sqrt{n}))}(1/p)\prod_{q=3}^{l(p)}(q-1)/q$$

Let us define the function $W(x)$ as follows:

$$W(x) = \sum_{p=3}^{x}(1/p)\prod_{q=3}^{l(p)}(q-1)/q$$

where x is a prime number and the sum and products are over prime numbers.

Then the equation for $k(n)$ simplifies to the following:

$$k(n) = (n/2)W(\lambda(\sqrt{n}))$$

Thus, the number of primes less than or equal to $n \lim_{n\to\infty}$ is the total number of odd numbers less than n minus $k(n)$:
$\pi(n) = (n/2) - k(n)$
$\pi(n) = (n/2) - (n/2)W(\lambda(\sqrt{n}))$
$\pi(n) = (n/2)(1 - W(\lambda(\sqrt{n})))$
The equation for the number of primes less than n as $n \to \infty$ is:

$$\textbf{Equation 1: } \pi(n) = (n/2)(1 - W(\lambda(\sqrt{n})))$$

To verify that no mistakes were made in the derivation of equation 1 and to determine at what point the equation converges to the actual number of prime numbers less than n, the actual number of primes less than n (blue) was plotted against equation 1 (orange) in Figure 2. Equation 1 slightly underestimated the actual number of primes for $n <= 5,000$, but for $n <= 50,000$, the curves were virtually indistinguishable. The curve for the actual number of primes less than n was made thicker so it can be viewed since it was completely obscured by the number of primes predicted by equation 1. The curve for $n/\ln(n)$ (gray) was also included for comparison and grossly underestimates the actual number of prime numbers less than n.

A graph of the absolute difference between equation 1 and the actual number of primes less than n for n = 20 to 50,000, shows that as n increases, the error decreases (Figure 3). As n increases, the difference between equation 1 and the actual number of primes decreases down to 0.291% at $n = 50,000$ (blue line). The difference between the prime number theorem $n/\ln(n)$ and the actual number of primes decreases at a much slower rate and at $n = 50,000$, the percent difference is 10% (orange line). More will be discussed about the error later in this paper.

4 The Proof of Legendre's Conjecture

Now that we have an equation that accurately determines the number of primes less than n for large values of n, we can prove Legendre's conjecture by induction. However, to perform proof by induction, we must first get $(1 - W(p_{i+1}))$ in terms of $W(p_i)$. To do this, we must look at the actual values of $(1 - W(p_i))$.

$1 - W(3) = 1 - (1/3) = 2/3$

$1 - W(5) = 1 - (1/3) - (2/3)(1/5) = (2/3)(4/5)$

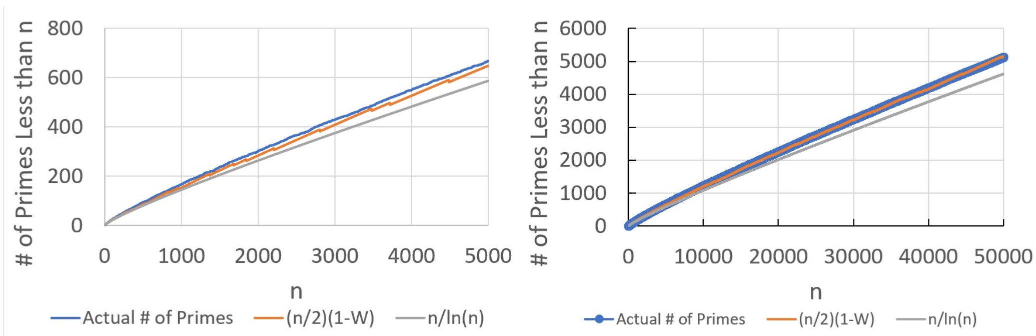


Figure 2: The actual number of primes less than n (blue) is slightly underestimated by equation 1 (orange) for values of n up to 5,000 (A). But for values of n up to 50,000, (B) the curves are virtually indistinguishable. The curve for $n/\ln(n)$ (gray) was also included for comparison.

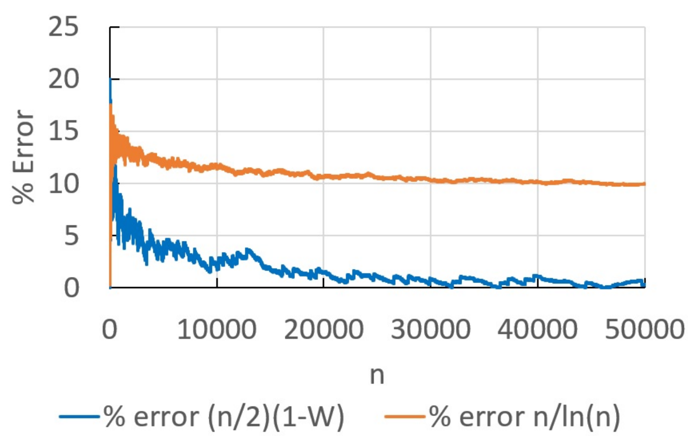


Figure 3: Comparison of equation 1 and $n/\ln(n)$ to the actual number of primes less than n. As n increases, the difference between equation 1 and the actual number of primes rapidly decreases (blue line). The difference between $n/\ln(n)$ and the actual number of primes decreases at a much slower rate (orange line).

$1 - W(7) =$ 1 - (1/3) - (2/3)(1/5) $- (2/3)(4/5)(1/7) = (2/3)(4/5)(6/7)$
$1 - W(11) =$ 1 - (1/3) - (2/3)(1/5) - (2/3)(4/5)(1/7) $-(2/3)(4/5)(6/7)(1/11) = (2/3)(4/5)(6/7)(10/11)$
Notice the value of $1 - W(p_i)$ (yellow) can be substituted into the green part of $1 - W(p_{i+1})$. Therefore, these equations can be simplified to:

Equaton 2: $1 - W(p_{i+1}) = [(p_{i+1} - 1)/p_{i+1}](1 - W(p_i))$

Using equation 1 to determine the number of primes less than n, we can calculate the number of primes between n^2 and $(n + 1)^2$.
$\pi(n^2) = (n^2/2)(1 - W(\lambda(n)))$
$\pi((n + 1)^2) = ((n + 1)^2/2)(1 - W(\lambda(n + 1)))$
There are two cases. The first case is where $p_i \leq n < p_{i+1} - 1$ in which case $\lambda(n) = \lambda(n + 1) = p_i$. The second case is where $n = p_i - 1$ in which case $\lambda(n) = p_{i-1}$ and $\lambda(n + 1) = p_i$.

Case 1: Let us look at the case where $p_i \leq n < p_{i+1} - 1$.
Let us prove for all $p_i \leq n < p_{i+1} - 1$, there is at least 1 prime number between n^2 and $(n+1)^2$. That means the difference between $\pi((n+1)^2)$ and $\pi(n^2)$ must be greater than 1.
$\pi(n^2) = (n^2/2)(1 - W(\lambda(n)))$
$\pi((n + 1)^2) = ((n + 1)^2/2)(1 - W(\lambda(n + 1))) = ((n + 1)^2/2)(1 - W(\lambda(n)))$
Let $\Delta\pi(n^2)$ be the difference between $\pi((n + 1)^2)$ and $\pi(n^2)$.
$\Delta\pi(n^2) = \pi((n + 1)^2) - \pi(n^2)$
$\Delta\pi(n^2) = ((n + 1)^2/2)(1 - W(\lambda(n))) - (n^2/2)(1 - W(\lambda(n)))$
$\Delta\pi(n^2) = \{((n + 1)^2/2) - (n^2/2)\}(1 - W(\lambda(n)))$
$\Delta\pi(n^2) = \{((n + 1)^2 - n^2)/2)\}(1 - W(\lambda(n)))$
$\Delta\pi(n^2) = \{((n^2 + 2n + 1) - n^2)/2)\}(1 - W(\lambda(n)))$

Equation 3: $\Delta\pi(n^2) = \{((2n + 1)/2)\}(1 - W(\lambda(n)))$

To prove $\Delta\pi(n^2) > 1$ for all $p_i \leq n < p_{i+1} - 1$, we will use induction.
Base case $n = 3$. Plugging 3 for n into equation 3 gives us the following:
$\Delta\pi(n^2) = \{((2n + 1)/2)\}(1 - W(\lambda(n)))$
$\Delta\pi(3^2) = ((2 \times 3 + 1)/2)(1 - W(\lambda(3)))$
$\Delta\pi(3^2) = (7/2)(1 - (1/3))$
$\Delta\pi(3^2) = (7/2)(2/3)$

$\Delta\pi(3^2) = (7/3) > 1$

Let's assume $\Delta\pi(n^2) = ((2n+1)/2)(1 - W(\lambda(n))) > 1$ for all $p_i \leq n < p_{i+1} - 1$
Prove that $\Delta\pi((n+1)^2) > 1$
Plugging $n+1$ for n in equation 3 gives the following:
$\Delta\pi(n^2) = ((2n+1)/2)(1 - W(\lambda(n)))$
$\Delta\pi((n+1)^2) = ((2(n+1)+1)/2)(1 - W(\lambda(n+1)))$
$\Delta\pi((n+1)^2) = ((2n+3)/2)(1 - W(\lambda(n)))$
Taking the ratio of $\Delta\pi((n+1)^2)/\Delta\pi(n^2)$ gives
$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = ((2n+3)/2)(1-W(\lambda(n)))/((2n+1)/2)(1-W(\lambda(n)))$
$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = ((2n+3)/2)/((2n+1)/2)$
$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = (2n+3)/(2n+1) > 1$
This proves that for all $p_i \leq n < p_{i+1} - 1$ where p is a prime number, there is at least 1 prime number between n^2 and $(n+1)^2$. In fact, since $\Delta\pi((n+1)^2) > \Delta\pi(n^2)$, this proves that the number of primes between n^2 and $(n+1)^2$ increases with increasing n, which is corroborated by the data in Figure 1.

Case 2: Let us look at the case where $n = p - 1$.
$\pi(n^2) = (n^2/2)(1 - W(\lambda(n)))$
$\pi((n+1)^2) = ((n+1)^2/2)(1 - W(\lambda(n+1)))$
Suppose $n = p_{i+1} - 1$, then $\lambda(n) = p_i$ and $\lambda(n+1) = p_{i+1}$.
Substituting p_i for $\lambda(n)$ and substituting p_{i+1} for $\lambda(n+1)$ gives the following:
$\pi(n^2) = (n^2/2)(1 - W(p_i))$
$\pi((n+1)^2) = ((n+1)^2/2)(1 - W(p_{i+1}))$
$\pi((n+1)^2) = ((n+1)^2/2)[(p_{i+1} - 1)/p_{i+1}](1 - W(p_i))$ using equation 2
The difference between $\pi(n^2)$ and $\pi((n+1)^2)$ gives:
$\Delta\pi(n^2) = \pi((n+1)^2) - \pi(n^2)$
$\Delta\pi(n^2) = ((n+1)^2/2)[(p_{i+1} - 1)/p_{i+1}](1 - W(p_i)) - [n^2/2](1 - W(p_i))$
$= \{((n+1)^2)(p_{i+1} - 1)/p_{i+1} - n^2\}(1 - W(p_i))/2$
Substituting n with $p_{i+1} - 1$ gives the following:
$= \{p_{i+1}^2(p_{i+1} - 1)/p_{i+1} - (p_{i+1} - 1)^2\}(1 - W(p_i))/2$
$= \{p_{i+1}^2 - p_{i+1} - (p_{i+1}^2 - 2p_{i+1} + 1)\}(1 - W(p_i))/2$
$= \{p_{i+1}^2 - p_{i+1} - p_{i+1}^2 + 2p_{i+1} - 1)\}(1 - W(p_i))/2$
$= \{p_{i+1} - 1\}(1 - W(p_i))/2$

Equation 4: $\Delta\pi(n^2) = \{p_{i+1} - 1\}(1 - W(p_i))/2$

To prove $\Delta\pi(n^2) > 1$ for all $n = p_{i+1} - 1$, we will use induction.
Base case $p_{i+1} = 5, p_i = 3$ and $n = p_{i+1} - 1 = 4$.
Plugging 4 for n, and 5 for p_{i+1} and 3 for p_i into equation 4 gives:
$\Delta\pi(4^2) = (5 - 1)(1 - W(3))/2$
$\Delta\pi(4^2) = 4(1 - (1/3))/2$
$\Delta\pi(4^2) = 4(2/3)/2$
$\Delta\pi(4^2) = 4/3 > 1$

Assume $\Delta\pi(n^2) > 1$ for all $n = p_{i+1} - 1$
Prove $\Delta\pi(n^2) > 1$ for all $n = p_{i+2} - 1$
$\Delta\pi((p_{i+2} - 1)^2) = (p_{i+2} - 1)(1 - W(p_{i+1}))/2$
$\Delta\pi((p_{i+2} - 1)^2) = (p_{i+2} - 1)((p_{i+1} - 1)/p_{i+1})(1 - W(p_i))/2$ using equation 2
$\Delta\pi((p_{i+2} - 1)^2) = \{(p_{i+2} - 1)/p_{i+1}\}\{(p_{i+1} - 1)(1 - W(p_i))/2\}$
Since we know $(p_{i+2}-1)/p_{i+1} > 1$ and we assumed $(p_{i+1}-1)(1-W(p_i))/2 > 1$, the product must be greater than 1. This proves that for all $n = p - 1$ where p is a prime number, there is at least 1 prime number between n^2 and $(n+1)^2$ and that the number of prime numbers between n^2 and $(n+1)^2$ also increases with increasing n.

5 Error Analysis

This same analysis could be done with the prime number theorem and it could be proven that $(n + 1)^2/\ln((n + 1)^2) - n^2/\ln(n^2) > 1$ for all n greater than some value. However, the error between $n/\ln(n)$ and $\pi(n)$ is very large even when n is large. Also, it may be difficult to prove that the error decreases as n increases since the error decreases at such a slow rate. These factors would make it difficult for any proof using the prime number theorem to gain acceptance.

However, equation 1 is very accurate at $n = 50,000$ with an error of 0.291%. Figure 3 also shows that the absolute difference between the actual number of primes and the number of primes predicted by equation 1, decreases as n increases. This is expected since the limit $n \to \infty$ was used to estimate number of composite numbers less than n. However, a figure does not make a proof. To prove the error does decreases as n increases, we have to look at each source of error in the derivation of equation 1.

The first source of error was the estimate for the number of elements in the set of odd numbers less than or equal to n. For large values of n, this was estimated as $n/2$ where the actual number of elements is $(n-1)/2$ if n is odd or $(n-2)/2$ if n is even. The difference between $n/2$ and $(n-1)/2$ or $(n-2)/2$ is either 0.5 or 1.0. This is a static error and as n gets very large, this error becomes insignificant since we are only trying to prove that $\pi((n+1)^2)$ exceeds $\pi(n^2)$ by 1. For sufficiently large values of n, for example at $n = 10,000$ there are over 1000 primes between $\pi((n+1)^2)$ and $\pi(n^2)$, an error of 0.5 or 1 will not invalidate the proof.

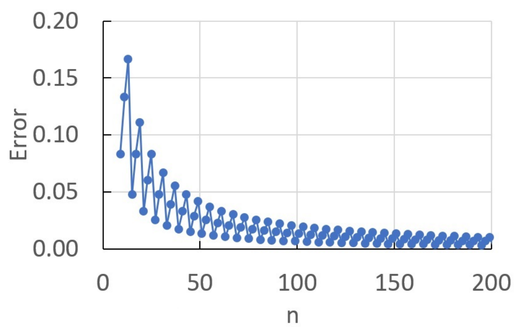
Expanding equation 1 out, we get:

$$\pi(n) = (n/2) - (n/2)\{1/3 + (2/3)(1/5) + (2/3)(4/5)(1/7) + (2/3)(4/5)(6/7)(1/11) + \ldots + (2/3)(4/5)(6/7)(10/11)\ldots((l(\lambda(\sqrt{n})) - 1)/l(\lambda(\sqrt{n})))(1/\lambda(\sqrt{n}))\}$$

The sources of error occur at the fractions 1/3, 1/5, 1/7, 1/11 etc... The first fraction 1/3, is an estimate for the number of elements in the set of odd integers less than or equal to n that are evenly divisible by 3. This is not a static error since it depends on n. A graph of difference between the actual fraction of elements evenly divisible by 3 excluding 3, versus 1/3 (Figure 4A) shows that the difference decreases as n gets large. Only odd values of n were plotted since even values of n have the same number of odd integers as $n-1$ and does not add additional information. The graph starts at $n = 9$ since $W(\lambda(\sqrt{n}))$ is not defined for values of n less than 9.

For example, for $n = 9$, there are 4 odd integers less than or equal to n, 1 of which (9) is evenly divisible by 3. So the difference is $(1/3) - (1/4) = 0.08333$.
For $n = 11$, there are 5 odd integers less than or equal to n, 1 of which (9) is evenly divisible by 3. So the difference is $(1/3) - (1/5) = 0.13333$.
For $n = 13$, there are 6 odd integers less than or equal to n, 1 of which (9) is evenly divisible by 3. So the difference is $(1/3) - (1/6)) = 0.16667$.
For $n = 15$, there are 7 odd integers less than or equal to n, 2 of which (9,15) are evenly divisible by 3. So the difference is $(1/3) - (2/7) = 0.04762$.

Though it is quite obvious that Figure 4A is a declining curve, to be rigorous, we must prove that the curve declines. Notice that in Figure 4A, the local maxima occur at $n_i = 7 + 6i$ where i is an integer greater than or equal to 0. The value of i also corresponds to the number of composite integers less than n that are evenly divisible by 3. If we prove that the local maxima will continually decrease $ad\ infinitum$, then this proves the curve in general will



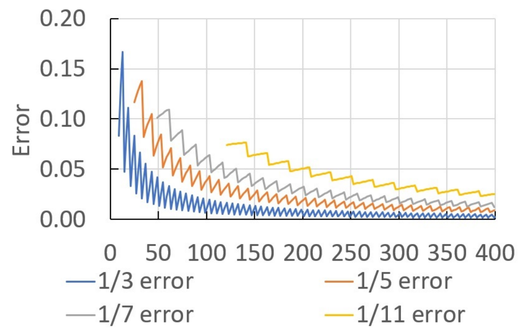


Figure 4: Graph of the error from estimating the fraction of elements evenly divisible by 3 as 1/3 (A) and the fraction of elements evenly divisible by 3, 5, 7 and 11 as 1/3, 1/5, 1/7 and 1/11 respectively (B).

decline. Let the function $f_3(n_i)$ represent the fraction of integers less than n_i that are evenly divisible by 3 excluding 3. Since n_i is odd, the number of odd integers less than n_i is $(n_i - 1)/2$.
$f_3(n_i) = i/((n_i - 1)/2)$
$f_3(n_i) = 2i/(n_i - 1)$

Looking at the local maxima, $n_i = 7 + 6i$, the value of i is $(n_i - 7)/6$. Plugging $(n_i - 7)/6$ into the above equation gives:
$f_3(n_i) = 2[(n_i - 7)/6]/(n_i - 1)$
$f_3(n_i) = (n_i - 7)/3(n_i - 1)$

Let the function $\epsilon_3(n_i)$ equal the difference between $f_3(n_i)$ and $(1/3)$ at the local maxima.
$\epsilon_3(n_i) = ((1/3) - f_3(n_i))$
$\epsilon_3(n_i) = ((1/3) - (n_i - 7)/3(n_1 - 1))$

Let's define $\Delta\epsilon_3(n_i)$ as the difference between $\epsilon(n_{i+1})$ and $\epsilon(n_i)$.
$\Delta\epsilon_3(n_i) = ((1/3) - (n_{i+1} - 7)/3(n_{i+1} - 1)) - ((1/3) - (n_i - 7)/3(n_i - 1))$
$\Delta\epsilon_3(n_i) = -(n_i - 7)/3(n_i - 1)) - (n_{i+1} - 7)/3(n_{i+1} - 1))$
Since $n_{i+1} = n_i + 6$, we can substitute for n_{i+1} to get the following:
$\Delta\epsilon_3(n_i) = (n_i - 7)/(n_i - 1)) - (n_i + 6 - 7)/(n_i + 6 - 1))$
$\Delta\epsilon_3(n_i) = (n_i - 7)/(n_i - 1)) - (n_i - 1)/(n_i + 5))$
$\Delta\epsilon_3(n_i) = [(n_i - 7)(n_i + 5) - (n_i - 1)^2]/[(n_i - 1)(n_i + 5)]$
$\Delta\epsilon_3(n_i) = [(n_i^2 - 2n_i - 35) - (n_i^2 - 2n_i + 1)]/[(n_i - 1)(n_i + 5)]$

$\Delta\epsilon_3(n_i) = -36/[(n_i - 1)(n_i + 5)]$
Since $\Delta\epsilon_3(n_i)$ is a negative number for all n_i, the value of $\epsilon_3(n_{i+1}) < \epsilon_3(n_i)$ for all n_i. This means that the error will continually decrease as n increases, therefore, the error due to the approximation of 1/3 as the fraction of elements divisible by 3 decreases.

The next fraction in the equation is 2/3 which represents the number of elements in the set of odd integers less than or equal to n that are not evenly divisible by 3. The error in this estimate has the same magnitude as the error 1/3 and since we have proven the error for 1/3 decreases, the error for 2/3 must also decreases. The next fraction 1/5, is an estimate for the number of elements in the set of odd integers less than or equal to n that are evenly divisible by 5. As can be seen in Figure 4B, this curve also appears to be declining with local maxima at $n_i = 13 + 10i$. The same proof can be performed by defining $f_5(n)$ and $\epsilon_5(n)$ and $\Delta\epsilon_5(n)$. It can proven that $\Delta\epsilon_5(n_i) = -120/[(n_i - 1)(n_i + 9)]$ which is also less than 0.

In fact, for all prime numbers p less than $\lambda(\sqrt{n})$, it can be proven that the difference between $1/p$ and the fraction of odd integes less than or equal to n that are evenly divisible by p, decreases as n increases. The local maxima will occur at $n_i = (3p-2)+2p \times i$. The functions $f_p(n)$ and $\epsilon_p(n)$ and $\Delta\epsilon_p(n)$ can be defined and it can be proven that $\Delta\epsilon_p(n_i) = -6(p^2-p)/[(n_i-1)(n_i+2p-1)]$ which is less than 0 for all p. Therefore, $\epsilon_p(n_{i+1}) < \epsilon_p(n_i)$ for all p.

Since all sources of error decline as n increases, the overall error must decline as n increases. The local maxima may align in some areas and not align in other areas resulting in peaks in Figure 3. However, since local maxima of all the curves in Figure 4B decline with increasing n, subsequent alignments of local maxima will result in peaks with a lower magnitude.

6 Summary

In summary, the following equation was derived that accurately determines the number of prime numbers less than n for large values of n.

$$\pi(n) = (n/2)(1 - W(\lambda(\sqrt{n})))$$

where $\lambda(\sqrt{n})$ is the largest prime number less than or equal to $\sqrt{n}$ and $W(x)$ is defined as follows:

$$W(x) = \sum_{p=3}^{x}(1/p)\prod_{q=3}^{l(p))}(q-1)/q$$

where x is a prime number, $l(p)$ is the largest prime number less than p, and the sum and products are over prime numbers.

It was then proven by induction, that the number of prime numbers between n^2 and $(n+1)^2$ is greater than 1 for all positive integers n, thus confirming the Legendre Conjecture.

It was also shown that the error between equation 1 and the actual number of primes less than n is very small ($\epsilon = 0.291\%$ for $n = 50,000$) and it was proven that as n increases, the error decreases.

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