Definitive Proof of Legendre's Conjecture

Kenneth A. Watanabe, PhD

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1 Abstract

Legendre's conjecture, states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. In this paper, an equation was derived that determines the number of prime numbers less than n for large values of n. Then it is proven by mathematical induction that there is at least one prime number between n^2 and $(n+1)^2$ for all positive integers n thus proving Legendre's conjecture.

2 Functions

Before we get into the proof, let me define a few functions that are necessary.

Let the function $l(x)$ represent the largest prime number less than x. For example, $l(10.5) = 7, l(20) = 19$ and $l(19) = 17$.

Let the function $\lambda(x)$ represent the largest prime number less than or equal to x. For example, $\lambda(10.5) = 7, \lambda(20) = 19$ and $\lambda(23) = 23$.

Let the function $k(n)$ represent the number of composite numbers in the set of odd numbers less than or equal to n excluding 1. For example, $k(15) = 2$ since there are two composite numbers 9 and 15 that are less than or equal to 15.

Let the function $\pi(n)$ represent the number of prime numbers in the set of odd numbers less than or equal to n. For example, $\pi(15) = 5$ since there are 5 prime numbers {3,5,7,11,13} that are less than 15.

Let capital P represent the number of all the odd integers less than n excluding 1.

Therefore $\pi(n) = P - k(n)$.

3 Methodology

Legendre's conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. The conjecture is one of Landau's problems (1912) on prime numbers. In this paper, an equation is derived to determine the number of prime numbers less than n^2. Then by mathematical induction, it is shown that there is at least one prime number between n^2 and $(n+1)^2$ thus proving the Legendre conjecture is true.

To derive an equation to determine the number of prime numbers less than n, we start with the set of all odd numbers less than n. Then all the composite numbers in the set that are evenly divisible by 3 are identified. Then all the composite numbers evenly divisible by 5, 7, 11 ... $\lambda(\sqrt{n})$ are identified where $\lambda(\sqrt{n}$ is the largest prime number less than or equal to n. We only have to go up to $\lambda(\sqrt{n})$ because there are no prime numbers greater than $\sqrt{n}$ that evenly divide n that are not evenly divisible by a lower prime number. By summing up the number of composite numbers in the set of odd numbers less than n and subtracting this from the total number of odd numbers less than n, gives us the number of prime numbers less than n.

Let us start with the set of all odd integers less than integer n excluding 1 as shown below.

{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,... n}

There are P = (n-1)/2 elements in the list.

Looking at those elements in the set that are divisible by 3, we notice that every third element after 3 (highlighted in yellow) beginning with 9, is divisible by 3.

{3,5,7, 9 ,11,13, 15 ,17,19, 21 ,23,25, 27 ,29,31, 33 ,35,37,... n}

Thus, as $n \to \infty$, the number of elements evenly divisible by 3, approaches the following equation:

$$\text{Number of elements divisible by } 3 \lim_{n\to\infty} = P/3$$

Looking at those elements in the set that are divisible by 5, we notice that every fifth element after 5 (highlighted in yellow) beginning with 15, is divisible by 5.

{3,5,7,9,11,13, 15 ,17,19,21,23, 25 ,27,29,31,33, 35 ,37,...,n}

But notice that, of the set of elements divisible by 5, every third element is also divisible by 3.

{ 15 ,25,35, 45 ,55,65, 75 ,85,95, 105 ,...,n}

So to avoid double counting, we must multiply the number of elements evenly divisible by 5 by (2/3) giving the following equation:

$$\text{Number of elements divisible by 5 and not } 3 \lim_{n\to\infty} = P(2/3)(1/5)$$

Looking at those elements in the set that are divisible by 7, we notice that every seventh element after 7 (highlighted in yellow) beginning with 21, is divisible by 7.
But notice that every 3rd element (yellow) is also divisible by 3 and every 5th element (green) is divisible by 5.
{21,35,49,63,77,91,105,119,133,147,161,175...n}
{21,35,49,63,77,91,105,119,133,147,161,175...n}
So to avoid double counting, we must multiply the number of elements evenly divisible by 7 by (2/3) and (4/5) giving the following equation:

$$\text{Number of elements divisible by 7 and not 5 or } 3 \lim_{n\to\infty} = P(2/3)(4/5)(1/7)$$

The general formula for the number of elements in the set of odd numbers less than n that are evenly divisible by prime number p and no lower prime number as $n \to \infty$ is as follows:

$$\text{Number of elements divisible only by } p \lim_{n\to\infty} = P(2/3)(4/5)(6/7)(10/11)\ldots((l(p)-1)/l(p))(1/p)$$

or

$$\text{Number of elements divisible only by } p \lim_{n\to\infty} = P(1/p)\prod_{q=3}^{l(p)}(q-1)/q$$

The total number of composite numbers in the set of odd numbers less than or equal to n, defined as $k(n)$, is thus defined as follows:
$k(n) = P\{1/3+(2/3)(1/5)+(2/3)(4/5)(1/7)+(2/3)(4/5)(6/7)(1/11)+\ldots+(2/3)(4/5)(6/7)(10/11)\ldots((l(\lambda(\sqrt{n}))-1)/l(\lambda(\sqrt{n})))(1/\lambda(\sqrt{n}))\}$

This can be written as

$$k(n) = P\sum_{p=3}^{\lambda(\sqrt{n}))}(1/p)\prod_{q=3}^{l(p)}(q-1)/q$$

Let us define the function $W(x)$ as follows:

$$W(x) = \sum_{p=3}^{x}(1/p)\prod_{q=3}^{l(p)}(q-1)/q$$

where x is a prime number and the sum and products are over prime numbers
Then the equation for $k(n)$ simplifies to the following:

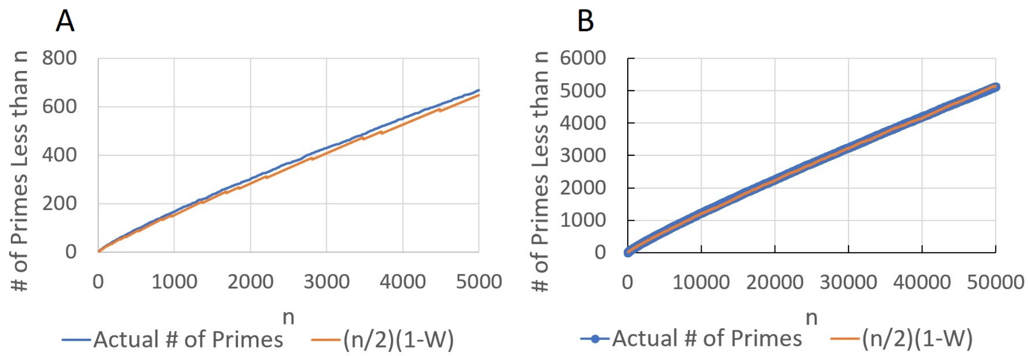

Figure 1: The actual number of primes less than n (blue) is slightly underestimated by equation 1 (orange) for values of n up to 5,000 (A). But for values of n up to 50,000, (B) the curves are virtually indistinguishable.

$$k(n) = PW(\lambda(\sqrt{n}))$$

The number of primes less than or equal to $n \lim_{n\to\infty}$ is:
$\pi(n) = P - k(n)$
$= P - PW(\lambda(\sqrt{n}))$
$= P(1 - W(\lambda(\sqrt{n})))$
As n approaches ∞, the value of P approaches $(n/2)$. Substituting P with $(n/2)$ in the above equation gives the following equation for the number of primes less than n as n approaches ∞.

Equation 1: $\pi(n) = (n/2)(1 - W(\lambda(\sqrt{n})))$

To verify that no mistakes were made in the derivation of equation 1 and to determine at what point the equation converges to the actual number of prime numbers less than n, the actual number of primes less than n (blue) was plotted against equation 1 (orange) in Figure 1. Equation 1 slightly underestimated the actual number of primes for $n <= 5,000$, but for $n <= 50,000$, the curves were virtually indistinguishable. The curve for the actual number of primes less than n was made thicker so it can be viewed since it was completely obscured by the number of primes predicted by equation 1.

4 The Proof of Legendre's Conjecture

In order to use proof by induction, we must first get $(1-2W(p_{i+1}))$ in terms of $W(p_i)$. To do this, we must look at the actual values of $2W(p_i)$.

$1 - W(3) = 1 - (1/3) = 2/3$
$1 - W(5) = 1 - (1/3) - (2/3)(1/5) = (2/3)(4/5)$
$1 - W(7) = 1 - (1/3) - (2/3)(1/5) - (2/3)(4/5)(1/7) = (2/3)(4/5)(6/7)$
$1 - W(11) = 1 - (1/3) - (2/3)(1/5) - (2/3)(4/5)(1/7) - (2/3)(4/5)(6/7)(1/11) = (2/3)(4/5)(6/7)(10/11)$
Notice the value of $1 - W(p_i)$ (yellow) can be substituted into the green part of $1 - W(p_{i+1})$. Therefore, these equations can be simplified to:

Equaton 2: $1 - W(p_{i+1}) = [(p_{i+1} - 1)/p_{i+1}](1 - W(p_i))$

Now that we have a formula for number of primes less than n, we can calculate the number of primes between n^2 and $(n+1)^2$.
$\pi(n^2) = (n^2/2)(1 - W(\lambda(n)))$
$\pi((n+1)^2) = ((n+1)^2/2)(1 - W(\lambda(n+1)))$
There are two cases. The first case is where $p_i \leq n < p_{i+1} - 1$ in which case $\lambda(n) = \lambda(n+1) = p_i$. The second case is where $n = p_i - 1$ in which case $\lambda(n) = p_{i-1}$ and $\lambda(n+1) = p_i$.

Case 1: Let us look at the case where $p_i \leq n < p_{i+1} - 1$.
Let us prove for all $p_i \leq n < p_{i+1} - 1$, there is at least 1 prime number between n^2 and $(n+1)^2$. That means the difference between $\pi((n+1)^2)$ and $\pi(n^2)$ must be greater than 1.
$\pi(n^2) = (n^2/2)(1 - W(\lambda(n)))$
$\pi((n+1)^2) = ((n+1)^2/2)(1 - W(\lambda(n+1))) = ((n+1)^2/2)(1 - W(\lambda(n)))$
Let $\Delta\pi(n^2)$ be the difference between $\pi((n+1)^2)$ and $\pi(n^2)$.
$\Delta\pi(n^2) = \pi((n+1)^2) - \pi(n^2)$
$\Delta\pi(n^2) = ((n+1)^2/2)(1 - W(\lambda(n))) - (n^2/2)(1 - W(\lambda(n)))$
$\Delta\pi(n^2) = \{((n+1)^2/2) - (n^2/2)\}(1 - W(\lambda(n)))$
$\Delta\pi(n^2) = \{((n+1)^2 - n^2)/2)\}(1 - W(\lambda(n)))$
$\Delta\pi(n^2) = \{((n^2 + 2n + 1) - n^2)/2)\}(1 - W(\lambda(n)))$

Equation 3: $\Delta\pi(n^2) = \{((2n+1)/2)\}(1 - W(\lambda(n)))$

To prove $\Delta\pi(n^2) > 1$ for all $p_i \leq n < p_{i+1} - 1$, we will use mathematical induction.
Base case $n = 3$. Plugging 3 for n into equation 3 gives us the following:
$\Delta\pi(n^2) = \{((2n+1)/2)\}(1 - W(\lambda(n)))$
$\Delta\pi(3^2) = ((2 \times 3 + 1)/2)(1 - W(\lambda(3)))$
$\Delta\pi(3^2) = (7/2)(1 - (1/3))$
$\Delta\pi(3^2) = (7/2)(2/3)$
$\Delta\pi(3^2) = (7/3) > 1$

Let's assume $\Delta\pi(n^2) = ((2n+1)/2)(1 - W(\lambda(n))) > 1$ for all $p_i \leq n < p_{i+1} - 1$
Prove that $\Delta\pi((n+1)^2) > 1$
Plugging $n+1$ for n in equation 3 gives the following:
$\Delta\pi(n^2) = ((2n+1)/2)(1 - W(\lambda(n)))$
$\Delta\pi((n+1)^2) = ((2(n+1)+1)/2)(1 - W(\lambda(n+1)))$
$\Delta\pi((n+1)^2) = ((2n+3)/2)(1 - W(\lambda(n)))$
Taking the ratio of $\Delta\pi((n+1)^2)/\Delta\pi(n^2)$ gives
$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = ((2n+3)/2)(1-W(\lambda(n)))/((2n+1)/2)(1-W(\lambda(n)))$
$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = ((2n+3)/2)/((2n+1)/2)$
$\Delta\pi((n+1)^2)/\Delta\pi(n^2) = (2n+3)/(2n+1) > 1$
This proves that for all $p_i \leq n < p_{i+1} - 1$ where p is a prime number, there is at least 1 prime number between n^2 and $(n+1)^2$.

Case 2: Let us look at the case where $n = p - 1$.
$\pi(n^2) = (n^2/2)(1 - W(\lambda(n)))$
$\pi((n+1)^2) = ((n+1)^2/2)(1 - W(\lambda(n+1)))$
Suppose $n = p_{i+1} - 1$, then $\lambda(n) = p_i$ and $\lambda(n+1) = p_{i+1}$.
Substituting p_i for $\lambda(n)$ and substituting p_{i+1} for $\lambda(n+1)$ gives the following:
$\pi(n^2) = (n^2/2)(1 - W(p_i))$
$\pi((n+1)^2) = ((n+1)^2/2)(1 - W(p_{i+1}))$
$\pi((n+1)^2) = ((n+1)^2/2)[(p_{i+1} - 1)/p_{i+1}](1 - W(p_i))$ using equation 2
The difference between $\pi(n^2)$ and $\pi((n+1)^2)$ gives:
$\Delta\pi(n^2) = \pi((n+1)^2) - \pi(n^2)$
$\Delta\pi(n^2) = ((n+1)^2/2)[(p_{i+1} - 1)/p_{i+1}](1 - W(p_i)) - [n^2/2](1 - W(p_i))$
$= \{((n+1)^2)(p_{i+1} - 1)/p_{i+1} - n^2\}(1 - W(p_i))/2$
Substituting n with $p_{i+1} - 1$ gives the following:

$= \{p_{i+1}^2(p_{i+1}-1)/p_{i+1} - (p_{i+1}-1)^2\}(1-W(p_i))/2$
$= \{p_{i+1}^2 - p_{i+1} - (p_{i+1}^2 - 2p_{i+1} + 1)\}(1-W(p_i))/2$
$= \{p_{i+1}^2 - p_{i+1} - p_{i+1}^2 + 2p_{i+1} - 1)\}(1-W(p_i))/2$
$= \{p_{i+1} - 1\}(1-W(p_i))/2$

$$\textbf{Equation 4: } \Delta\pi(n^2) = \{p_{i+1}-1\}(1-W(p_i))/2$$

To prove $\Delta\pi(n^2) > 1$ for all $n = p_{i+1} - 1$, we will use mathematical induction.
Base case $p_{i+1} = 5, p_i = 3$ and $n = p_{i+1} - 1 = 4$.
Plugging 4 for n, and 5 for p_{i+1} and 3 for p_i into equation 4 gives:
$\Delta\pi(4^2) = (5-1)(1-W(3))/2$
$\Delta\pi(4^2) = 4(1-(1/3))/2$
$\Delta\pi(4^2) = 4(2/3)/2$
$\Delta\pi(4^2) = 4/3 > 1$

Assume $\Delta\pi(n^2) > 1$ for all $n = p_{i+1} - 1$
Prove $\Delta\pi(n^2) > 1$ for all $n = p_{i+2} - 1$
$\Delta\pi((p_{i+2}-1)^2) = (p_{i+2}-1)(1-W(p_{i+1}))/2$
$\Delta\pi((p_{i+2}-1)^2) = (p_{i+2}-1)((p_{i+1}-1)/p_{i+1})(1-W(p_i))/2$ Using equation 2
$\Delta\pi((p_{i+2}-1)^2) = \{(p_{i+2}-1)/p_{i+1}\}\{(p_{i+1}-1)(1-W(p_i))/2\}$
Since we know $(p_{i+2}-1)/p_{i+1} > 1$ and we assumed $(p_{i+1}-1)(1-W(p_i))/2 > 1$, the product must be greater than 1. This proves that for all $n = p-1$ where p is a prime number, there is at least 1 prime number between n^2 and $(n+1)^2$.

5 Summary

In summary, I derived the following equation for the number of prime numbers less than n for large values of n.

$$\pi(n) = (n/2)(1 - W(\lambda(\sqrt{n})))$$

where $\lambda(\sqrt{n})$ is the largest prime number less than or equal to $\sqrt{n}$ and $W(x)$ is defined as follows:

$$W(x) = \sum_{p=3}^{x}(1/p)\prod_{q=3}^{l(p))}(q-1)/q$$

where x is a prime number and the sum and products are over prime numbers. It was then proven by mathematical induction, that the number of prime numbers between n^2 and $(n+1)^2$ is greater than 1 for all positive integers n, thus confirming the Legendre Conjecture.

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