Definitive Prove of Legendre’s conjecture

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1 Abstract

Legendre’s conjecture, states that there is a prime number between $n^2$ and $(n + 1)^2$ for every positive integer $n$. In this paper, an equation was derived that determines the number of prime numbers less than $n$ for large values of $n$. Then it is proven by mathematical induction that there is at least 1 prime number between $n^2$ and $(n + 1)^2$ for all positive integers $n$ thus proving Legendre’s conjecture.

2 Functions

Before we get into the proof, let me define a few functions that are necessary.

Let the function $l(x)$ represent the largest prime number less than $x$. For example, $l(10.5) = 7$, $l(20) = 19$ and $l(19) = 17$.

Let the function $\lambda(x)$ represent the largest prime number less than or equal to $x$. For example, $\lambda(10.5) = 7$, $\lambda(20) = 19$ and $\lambda(23) = 23$.

Let the function $k(n)$ represent the number of composite numbers in the set of odd numbers less than or equal to $n$ excluding 1. For example, $k(15) = 2$ since there are two composite numbers 9 and 15 that are less than or equal to 15.

Let the function $\pi(n)$ represent the number of prime numbers in the set of odd numbers less than or equal to $n$. For example, for $n = 15$, $\pi(n) = 5$ since there are 5 prime numbers $\{3, 5, 7, 11, 13\}$ less than 15.

Let capital $P$ represent the number of all the odd integers less than $n$ excluding 1.

Therefore $\pi(n) = P - k(n)$. 
3 Methodology

Legendre’s conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between \( n^2 \) and \( (n+1)^2 \) for every positive integer \( n \). The conjecture is one of Landau’s problems (1912) on prime numbers. In this paper, an equation is derived to determine the number of prime numbers less than \( n^2 \). Then by mathematical induction, it is shown that there is at least 1 prime between \( n^2 \) and \( (n+1)^2 \) thus proving the Legendre conjecture is true.

Let us start with the list all odd numbers less than \( n \) excluding 1 as shown below. \{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,...n\}
There are \( P = (n-1)/2 \) numbers in the list.
Excluding 3, every third number (highlighted in yellow) beginning with 9 is divisible by 3.
\{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,...n\}
Number of numbers divisible by 3 \( \lim_{n \to \infty} P/3 \)

Excluding 5, every fifth number beginning with 15 is divisible by 5.
\{3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,...n\}
But notice that, of the set of numbers divisible by 5, every third number is also divisible by 3.
\{15,25,35,45,55,65,75,85,95,105,...n\}
So to avoid double counting, we must multiply by (2/3) giving the following:
Number of numbers divisible by 5 and not 3 \( \lim_{n \to \infty} P(2/3)(1/5) \)

Excluding 7, every seventh number beginning with 21 is divisible by 7.
But notice that every 3rd number (yellow) is also divisible by 3 and every 5th number (green) is divisible by 5.
\{21,35,49,63,77,91,105,119,133,147,161,175,...n\}
\{21,35,49,63,77,91,105,119,133,147,161,175,...n\}
So to avoid double counting, we must multiply by (2/3) and (4/5) giving the following:
The number of numbers divisible by 7 and not 5 or 3 \( \lim_{n \to \infty} P(2/3)(4/5)(1/7) \)

The general formula for the number of numbers divisible by prime number \( p \) but not equal to \( p \) as \( n \to \infty \) is as follows:
Number of numbers divisible only by \( p \) \( \lim_{n \to \infty} P(2/3)(4/5)(6/7)(10/11)...((l(p)-1)/l(p))(1/p) \)
Number of numbers divisible only by \( p \) \( \lim_{n \to \infty} = P(1/p) \prod_{q=3}^{l(p)} (q - 1)/q \)

The total number of composite numbers in the set of odd numbers less than or equal to \( n \), defined as \( k(n) \), is thus defined as follows:
\[
k(n) = P\left\{1/3 + (2/3)(1/5) + (2/3)(4/5)(1/7) + (2/3)(4/5)(6/7)(1/11) + \ldots + (2/3)(4/5)(6/7)(10/11) \ldots \right\}
\]
This can be written as
\[
k(n) = P \sum_{p=3}^{\lambda(\sqrt{n})} (1/p) \prod_{q=3}^{l(p)} (q - 1)/q
\]
Let us define the function \( W(x) \) as follows:
\[
W(x) = \sum_{p=3}^{x} (1/p) \prod_{q=3}^{l(p)} (q - 1)/q
\]
where \( x \) is a prime number and the sum and products are over prime numbers.

Then the equation for \( k(n) \) simplifies to the following:
\[
k(n) = PW(\lambda(\sqrt{n}))
\]

The number of primes less than or equal to \( n \lim_{n \to \infty} \) is:
\[
\pi(n) = P - k(n)
\]
\[
= P - PW(\lambda(\sqrt{n}))
\]
\[
= P - P(1 - W(\lambda(\sqrt{n})))
\]
As \( n \) approaches \( \infty \), the value of \( P \) approaches \( (n/2) \). Substituting \( P \) with \( (n/2) \) in the above equation gives the following equation for the number of primes less than \( n \) as \( n \) approaches \( \infty \).

**Equation 1:** \( \pi(n) = (n/2)(1 - W(\lambda(\sqrt{n}))) \)

To verify that no mistakes were made in the derivation of equation 1, I plotted the actual number of primes less than \( n \) (blue) against equation 1 (orange) in Figure 1. Equation 1 slightly underestimated the actual number of primes for \( n \leq 5,000 \), but for \( n \leq 50,000 \), the curves were virtually indistinguishable. The curve for the actual number of primes less than \( n \) was made thicker so it can be viewed since it was completely hidden by the number of primes predicted by equation 1.

4 The Proof of Legendre’s Conjecture

In order to use proof by induction, we must first get \( (1 - 2W(p_{i+1})) \) in terms of \( W(p_i) \). To do this, we must look at the actual values of \( 2W(p_i) \).
Figure 1: The actual number of primes less than \( n \) (blue) is slightly underestimated by equation 1 (orange) for values of \( n \) up to 5,000 (A). But for values of \( n \) up to 50,000, (B) the curves are virtually indistinguishable.

\[
1 - W(3) = 1 - (1/3) = 2/3 \\
1 - W(5) = 1 - (1/3) - (2/3)(1/5) = (2/3)(4/5) \\
1 - W(7) = 1 - (1/3) - (2/3)(1/5) - (2/3)(4/5)(1/7) = (2/3)(4/5)(6/7) \\
\]

Notice the value of \( 1 - W(p_i) \) (yellow) can be substituted into the green part of \( 1 - W(p_{i+1}) \). Therefore, these equations can be simplified to:

Equation 2: \[
1 - W(p_{i+1}) = [(p_{i+1} - 1)/p_{i+1}](1 - W(p_i))
\]

Now that we have a formula for number of primes less than \( n \), we can calculate the number of primes between \( n^2 \) and \((n + 1)^2\).

\[
\pi(n^2) = (n^2/2)(1 - W(\lambda(n))) \\
\pi((n + 1)^2) = ((n + 1)^2/2)(1 - W(\lambda(n + 1)))
\]

There are two cases. The first case is where \( n \neq p_i - 1 \) in which case \( \lambda(n) = \lambda(n+1) \). The second case is where \( n = p_i - 1 \) in which case \( \lambda(n) = p_{i-1} \) and \( \lambda(n + 1) = p_i \).

Case 1: Let us look at the case where \( n \neq p - 1 \).

Let us prove for all \( n \neq p - 1 \), there is at least 1 prime number between \( n^2 \) and \((n + 1)^2\). That means the difference between \( \pi((n + 1)^2) \) and \( \pi(n^2) \) must be greater than 1.

4
\[ \pi(n^2) = \frac{(n^2/2)(1 - W(\lambda(n)))}{(n + 1)^2} \]

\[ \pi((n + 1)^2) = \frac{((n + 1)^2/2)(1 - W(\lambda(n + 1)))}{(n + 1)^2} = \frac{((n + 1)^2/2)(1 - W(\lambda(n)))}{(n + 1)^2} \]

Let \( \Delta \pi(n^2) \) be the difference between \( \pi((n + 1)^2) \) and \( \pi(n^2) \).

\[ \Delta \pi(n^2) = \pi((n + 1)^2) - \pi(n^2) \]

\[ \Delta \pi(n^2) = ((n + 1)^2/2)(1 - W(\lambda(n))) - (n^2/2)(1 - W(\lambda(n))) \]

\[ \Delta \pi(n^2) = \{(n + 1)^2/2 - n^2/2\}(1 - W(\lambda(n))) \]

\[ \Delta \pi(n^2) = \{(n^2 + 2n + 1 - n^2)/2\}(1 - W(\lambda(n))) \]

\[ \Delta \pi(n^2) = \{(2n + 1)/2\}(1 - W(\lambda(n))) \] Equation 3

To prove \( \Delta \pi(n^2) > 1 \) for all \( n \neq p - 1 \), we will use mathematical induction.

Base case \( n = 3 \). Plugging 3 for \( n \) into equation 3 gives us the following:

\[ \Delta \pi(3^2) = \{(2n + 1)/2\}(1 - W(\lambda(n))) \]

\[ \Delta \pi(3^2) = (2\times 3 + 1)/2 (1 - W(\lambda(3))) \]

\[ \Delta \pi(3^2) = (7/2)(1 - (1/3)) \]

\[ \Delta \pi(3^2) = (7/2)(2/3) \]

\[ \Delta \pi(3^2) = (7/3) > 1 \]

Let’s assume \( \Delta \pi(n^2) = ((2n + 1)/2)(1 - W(\lambda(n))) > 1 \) for all \( n \neq p - 1 \)

Prove that \( \Delta \pi((n + 1)^2) > 1 \)

Plugging \( n + 1 \) for \( n \) in equation 3 gives the following:

\[ \Delta \pi((n + 1)^2) = ((2n + 1)/2)(1 - W(\lambda(n))) \]

\[ \Delta \pi((n + 1)^2) = ((2(n + 1) + 1)/2)(1 - W(\lambda(n + 1))) \]

\[ \Delta \pi((n + 1)^2) = ((2n + 3)/2)(1 - W(\lambda(n))) \]

Taking the ratio of \( \Delta \pi((n + 1)^2)/\Delta \pi(n^2) \) gives:

\[ \Delta \pi((n + 1)^2)/\Delta \pi(n^2) = ((2n + 3)/2)(1 - W(\lambda(n)))/(2n + 1)/2(1 - W(\lambda(n))) \]

\[ \Delta \pi(n^2)/\Delta \pi(n^2) = ((2n + 3)/2)/((2n + 1)/2) \]

\[ \Delta \pi(n^2)/\Delta \pi(n^2) = (2n + 3)/(2n + 1) > 1 \]

This proves that for all \( n \neq p - 1 \) where \( p \) is a prime number, there is at least 1 prime number between \( n^2 \) and \( (n + 1)^2 \).

Case 2: Let us look at the case where \( n = p - 1 \).

\[ \pi(n^2) = (n^2/2)(1 - W(\lambda(n))) \]

\[ \pi((n + 1)^2) = ((n + 1)^2/2)(1 - W(\lambda(n + 1))) \]

Suppose \( n = p_{i+1} - 1 \), then \( \lambda(n) = p_i \) and \( \lambda(n + 1) = p_{i+1} \).

Substituting \( p_i \) for \( \lambda(n) \) and substituting \( p_{i+1} \) for \( \lambda(n + 1) \) gives the following:

\[ \pi(n^2) = (n^2/2)(1 - W(p_i)) \]

\[ \pi((n + 1)^2) = ((n + 1)^2/2)(1 - W(p_{i+1})) \]

\[ \pi((n + 1)^2) = ((n + 1)^2/2)[(p_{i+1} - 1)/p_{i+1}](1 - W(p_i)) \] using equation 2
The difference between $\pi(n^2)$ and $\pi((n + 1)^2)$ gives:

$$\Delta \pi(n^2) = \pi((n + 1)^2) - \pi(n^2)$$

$$\Delta \pi(n^2) = ((n + 1)^2/2)[(p_{i+1} - 1)/p_{i+1}](1 - W(p_i)) - [n^2/2](1 - W(p_i))$$

$$= \{((n + 1)^2)(p_{i+1} - 1)/p_{i+1} - n^2\}(1 - W(p_i))/2$$

Substituting $n$ with $p_{i+1} - 1$ gives the following:

$$= \{p_{i+1}^2 - p_{i+1}^2 - (p_{i+1} - 1)^2\}(1 - W(p_i))/2$$

$$= \{p_{i+1}^2 - p_{i+1}^2 - (p_{i+1}^2 - 2p_{i+1} + 1)\}(1 - W(p_i))/2$$

$$= \{p_{i+1}^2 - p_{i+1}^2 - p_{i+1}^2 + 2p_{i+1} - 1\}(1 - W(p_i))/2$$

$$= \{p_{i+1}^2 - p_{i+1} - 1\}(1 - W(p_i))/2$$

To prove $\Delta \pi(n^2) > 1$ for all $n = p_{i+1} - 1$, we will use mathematical induction.

Base case $p_{i+1} = 5$, $p_i = 3$ and $n = p_{i+1} - 1 = 4$.

Plugging 4 for $n$, and 5 for $p_{i+1}$ and 3 for $p_i$ gives:

$$\Delta \pi(4^2) = (5 - 1)(1 - W(3))/2$$

$$\Delta \pi(4^2) = 4(1 - (1/3))/2$$

$$\Delta \pi(4^2) = 4(2/3)/2$$

$$\Delta \pi(4^2) = 4/3 > 1$$

Assume $\Delta \pi(n^2) > 1$ for all $n = p_{i+1} - 1$

Prove $\Delta \pi(n^2) > 1$ for all $n = p_{i+2} - 1$

$$\Delta \pi((p_{i+1} - 1)^2) = (p_{i+1} - 1)(1 - W(p_i))/2$$

$$\Delta \pi((p_{i+2} - 1)^2) = \{(p_{i+2} - 1)(1 - W(p_i))(p_{i+1} - 1)/p_{i+1}\}/2$$ Using equation 2

$$\Delta \pi((p_{i+2} - 1)^2) = \{(p_{i+2} - 1)/p_{i+1}\}\{(p_{i+1} - 1)(1 - W(p_i))/2\}$$

Since we know $(p_{i+2} - 1)/p_{i+1} > 1$ and we assumed $(p_{i+1} - 1)(1 - W(p_i))/2 > 1$, the product must be greater than 1. This proves that for all $n = p - 1$ where $p$ is a prime number, there is at least 1 prime number between $n^2$ and $(n + 1)^2$.

5 Summary

In summary, I derived the following equation for the number of prime numbers less than $n$ for large values of $n$.

$$\pi(n) = (n/2)(1 - W(\lambda(\sqrt{n})))$$

where $\lambda(\sqrt{n})$ is the largest prime number less than or equal to $\sqrt{n}$ and $W(x)$ is defined as follows:

$$W(x) = \sum_{p=3}^{x} (1/p) \prod_{q=3}^{t(p)} (q - 1)/q$$

where $x$ is a prime number and the sum and products are over prime numbers.

I have proven by mathematical induction, that the number of prime num-
bers between $n^2$ and $(n+1)^2$ is greater than 1 for all positive integers $n$, thus confirming the Legendre Conjecture.

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