A Note About the ABC Conjecture: A Proof of $C < \text{rad}^2(ABC)$

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Abstract In this paper, we consider the ABC conjecture then we give a proof that $C < \text{rad}^2(ABC)$ that it will be the key of the proof of the ABC conjecture.

Keywords Elementary number theory · real functions of one variable.

Mathematics Subject Classification (2010) 11AXX · 26AXX

To the memory of my Father who taught me arithmetic

To the memory of Jean Bourgain (1954-2018) for his mathematical work notably in the field of Number Theory

1 Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i-1}$$  \hspace{1cm} (1)

We note:

$$\mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a)$$  \hspace{1cm} (2)

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph (Esterlé of Pierre et Marie Curie University (Paris 6) ([3]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given above:

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Conjecture 1 (ABC Conjecture): Let $a, b, c$ positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that:

$$c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}$$  \hspace{1cm} (3)

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751$ \hspace{1cm} ([2]). Here we will give a proof that:

$$c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2$$  \hspace{1cm} (4)

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.

2 A Proof of the condition \hspace{1cm} ([4])

Let $a, b, c$ positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.

If $c \leq \text{rad}(ab)$ then we obtain:

$$c \leq \text{rad}(ab) < \text{rad}^2(abc)$$  \hspace{1cm} (5)

and the condition \hspace{1cm} ([4]) is verified.

In the following, we suppose that $c > \text{rad}(ab)$.

2.1 Case $c = a + 1$

$$c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^2(ac)$$  \hspace{1cm} (6)

2.1.1 $\mu_a = 1$

In this case, $a = \text{rad}(a)$, it is immediately truth that:

$$c = a + 1 < 2a < \text{rad}(a) \text{rad}(c) < \text{rad}^2(ac)$$  \hspace{1cm} (7)

Then \hspace{1cm} (6) is verified.

2.1.2 $\mu_a \neq 1, \mu_a < \text{rad}(a)$

we obtain:

$$c = a + 1 < 2\mu_a \text{rad}(a) \Rightarrow c < 2\text{rad}^2(a) \Rightarrow c < \text{rad}^2(ac)$$  \hspace{1cm} (8)

Then \hspace{1cm} (6) is verified.
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2.1.3 $\mu_a \geq \text{rad}(a)$

We have $c = a + 1 = \mu_a . \text{rad}(a) + 1 \leq \mu_a^2 + 1 < \text{rad}^2(ac)$. We suppose that $\mu_a^2 + 1 \geq \text{rad}^2(ac) \implies \mu_a^2 > \text{rad}^2(a).\text{rad}(c) > \text{rad}^2(a)$ as $\text{rad}(c) > 1$, then $\mu_a > \text{rad}(a)$, that is the contradiction with $\mu_a \geq \text{rad}(a)$. We deduce that $c < \mu_a^2 + 1 < \text{rad}^2(ac)$ and the condition (6) is verified.

2.2 $c = a + b$

We can write that $c$ verifies:

$$c = a + b = \text{rad}(a).\mu_a + \text{rad}(b).\mu_b = \text{rad}(a).\text{rad}(b) \left( \frac{\mu_a}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} \right)$$

$$\implies c = \text{rad}(a).\text{rad}(b).\text{rad}(c) \left( \frac{\mu_a}{\text{rad}(b).\text{rad}(c)} + \frac{\mu_b}{\text{rad}(a).\text{rad}(c)} \right)$$

(9)

We can write also:

$$c = \text{rad}(abc) \left( \frac{\mu_a}{\text{rad}(b).\text{rad}(c)k} + \frac{\mu_b}{\text{rad}(a).\text{rad}(c)} \right)$$

(10)

To obtain a proof of (4), one method is to prove that:

$$\frac{\mu_a}{\text{rad}(b).\text{rad}(c)} + \frac{\mu_b}{\text{rad}(a).\text{rad}(c)} < \text{rad}(abc)$$

(11)

2.2.1 $\mu_a = \mu_b = 1$

In this case, it is immediately truth that:

$$\frac{1}{\text{rad}(a_i)} + \frac{1}{\text{rad}(b_j)} \leq \frac{5}{6} < \text{rad}(c).\text{rad}(abc)$$

(12)

Then (4) is verified.

2.2.2 $\mu_a = 1$ and $\mu_b > 1$

As $b < a \implies \mu_b \text{rad}(b) < \text{rad}(a) \implies \frac{\mu_b}{\text{rad}(a)} < \frac{1}{\text{rad}(b)}$, then we deduce that:

$$\frac{1}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} < \frac{2}{\text{rad}(b)} < \text{rad}(c).\text{rad}(abc)$$

(13)

Then (4) is verified.
2.2.3 \( \mu_b = 1 \) and \( \mu_a \leq (b \mid \text{rad}(b)) \)

In this case we obtain:

\[
\frac{1}{\text{rad}(a)} + \frac{\mu_a}{\text{rad}(b)} \leq \frac{1}{\text{rad}(a)} + 1 < \text{rad}(c)\text{.rad}(abc)
\] (14)

Then (4) is verified.

2.2.4 \( \mu_b = 1 \) and \( \mu_a > (b \mid \text{rad}(b)) \)

As \( \mu_a > \text{rad}(b) \), we can write \( \mu_a = \text{rad}(b) + n \) where \( n \geq 1 \). We obtain:

\[
c = \mu_a\text{rad}(a) + \text{rad}(b) = \text{rad}(b+n)\text{rad}(a) + \text{rad}(b) = \text{rad}(ab+n\text{rad}(a) + \text{rad}(b)
\] (15)

We verify that \( n < b \), then:

\[
c < 2\text{rad}(ab) + \text{rad}(b) \implies c < \text{rad}(abc) + \text{rad}(abc) < \text{rad}^2(abc) \implies c < \text{rad}^2(abc)
\] (16)

2.2.5 \( \mu_a, \mu_b \neq 1, \mu_a < \text{rad}(a) \) and \( \mu_b < \text{rad}(b) \)

we obtain:

\[
c = \mu_a\text{rad}(c) = \mu_a\text{rad}(a) + \mu_b\text{rad}(b) < \text{rad}^2(a) + \text{rad}^2(b) < \text{rad}^2(abc)
\] (17)

2.2.6 \( \mu_a, \mu_b \neq 1, \mu_a \leq \text{rad}(a) \) and \( \mu_b \geq \text{rad}(b) \)

We have:

\[
c = \mu_a\text{rad}(a) + \mu_b\text{rad}(b) < \mu_a\mu_b\text{rad}(a)\text{rad}(b) \leq \mu_b\text{rad}^2(a)\text{rad}(b)
\] (18)

Then if we give a proof that \( \mu_b < \text{rad}(b)\text{rad}^2(c) \), we obtain \( c < \text{rad}^2(abc) \). As \( \mu_b \geq \text{rad}(b) \implies \mu_b = \text{rad}(b) + \alpha \) with \( \alpha \) a positive integer \( \geq 0 \). Supposing that \( \mu_b \geq \text{rad}(b)\text{rad}^2(c) \implies \mu_b = \text{rad}(b)\text{rad}^2(c) + \beta \) with \( \beta \geq 0 \) a positive integer. We can write:

\[
\text{rad}(b)\text{rad}^2(c) + \beta = \text{rad}(b) + \alpha \implies \beta < \alpha
\]

\[
\alpha - \beta = \text{rad}(b)(\text{rad}^2(c) - 1) > 3\text{rad}(b) \implies \mu_b = \text{rad}(b) + \alpha > 4\text{rad}(b)
\] (19)

Finally, we obtain:

\[
\begin{cases}
\mu_b \geq \text{rad}(b) \\
\mu_b > 4\text{rad}(b)
\end{cases}
\] (20)

Then the contradiction and the hypothesis \( \mu_b \geq \text{rad}(b)\text{rad}^2(c) \) is false. Hence:

\[
\mu_b < \text{rad}(b)\text{rad}^2(c) \implies c < \text{rad}^2(abc)
\] (21)
\[ 2.2.7 \quad \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \quad \text{and} \quad \mu_b \leq \text{rad}(b) \]

The proof is identical to the case above.

\[ 2.2.8 \quad \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \quad \text{and} \quad \mu_b \geq \text{rad}(b) \]

We write:
\[
c = \mu_a \text{rad}(a) + \mu_b \text{rad}(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2 \mu_b^2 < \text{rad}^2(a) \text{rad}^2(b) \text{rad}^2(c) = \text{rad}^2(abc) \tag{22}\]

As \( \mu_a \geq \text{rad}(a) \) and \( \mu_b \geq \text{rad}(b) \), we can write that:
\[
\mu_a = \text{rad}(a) + m \\
\mu_b = \text{rad}(b) + n
\]

with \( m, n \geq 0 \) two positive integers. Let \( F(x, y) \) the function:
\[
F(x, y) = (x+\text{rad}(a))(y+\text{rad}(b)) - \text{rad}(abc), \quad (x, y) \in I = [-\text{rad}(a), +\infty[ \times [-\text{rad}(b), +\infty[ \tag{23}
\]

The set of points \( M(x, y) \in I \) verifying \( F(x, y) = 0 \) is the hyperbola \( \mathcal{C} \) given by:
\[
y = \frac{-\text{rad}(b)x + \text{rad}(abc) - \text{rad}(ab)}{x + \text{rad}(a)} \tag{24}
\]

The curve \( \mathcal{C} \) intersects the axis \( x = 0 \) and \( y = 0 \) at the two points \( M_1(0, y_1 = \text{rad}(b)(\text{rad}(c) - 1)) \) and \( M_2(x_2 = \text{rad}(a)(\text{rad}(c) - 1), 0) \). The region below the curve \( \mathcal{C} \) verifies \( F(x, y) < 0 \). \( F(m, n) = \mu_a \mu_b - \text{rad}(abc) < 0 \) if we have \( m < x_2 \Rightarrow m < \text{rad}(a)(\text{rad}(c) - 1) \) and \( n < y_1 \Rightarrow n < \text{rad}(b)(\text{rad}(c) - 1) \).

We suppose now that:
\[
m \geq \text{rad}(a)(\text{rad}(c) - 1) \quad \Rightarrow \quad m > \text{rad}(a) \quad \Rightarrow \quad \mu_a > 2\text{rad}(a) \quad \Rightarrow \quad a > 2\text{rad}^2(a) \\

n \geq \text{rad}(b)(\text{rad}(c) - 1) \quad \Rightarrow \quad n > \text{rad}(b) \quad \Rightarrow \quad \mu_b > 2\text{rad}(b) \quad \Rightarrow \quad b > 2\text{rad}^2(b)
\]

then \( c > 2(\text{rad}^2(a) + \text{rad}^2(b)) > 4\text{rad}(ab) \quad \Rightarrow \quad c > 4\text{rad}(ab) \tag{25}\)

The last inequality \( c > 4\text{rad}(ab) \) gives the contradiction with the condition \( c > \text{rad}(ab) \) supposed above. Then we obtain \( F(m, n) < 0 \Rightarrow \mu_a \mu_b - \text{rad}(abc) < 0 \Rightarrow c < \text{rad}^2(abc) \).

We announce the theorem:

**Theorem 1** *(Abdelmajid Ben Hadj Salem, 2019)* Let \( a, b, c \) positive integers relatively prime with \( c = a + b \) and \( b < a \), then \( c < \text{rad}^2(abc) \).

**References**

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