Is There a Flaw in the Traditional FLRW Metric?

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Abstract

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric assumes perfect isotropy and homogeneity in its derivation, assumptions that are necessary to provide a reasonably accurate description of the universe as a whole. But there is one other assumption implicit in its derivation, which is the apparently unintentional inclusion of a constant curvature (Ricci) scalar in the term that determines whether the universe is open, closed or flat. Resolving this issue requires an awareness of this difficulty, which is presented as an open problem.

1. Summary of the Friedmann-Lemaître-Robertson-Walker (FLRW) Spacetime

The FLRW walker metric is traditionally presented as

$$ds^2 = c^2 dt^2 - S^2 \left[\frac{dr^2}{1-k^2 r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right] \tag{1.1}$$

where $S = S(t)$ is a time-dependent scale factor (often appearing as $a(t)$ in the literature) that measures the rate of expansion (or contraction) of the universe and k^2 is a constant (usually just expressed as 0 or ± 1) having the dimension length^{-2}. Invariably, the seemingly innocuous k^2 term in the FLRW metric is presented as the main determinator of the fate of the universe: if $k^2 > 0$, the universe will expand forever; if $k^2 < 0$ then the universe will close in on itself; and if $k^2 = 0$ then the universe is Minkowski-flat.

One might logically assume that k^2 should also make its presence known in the scale factor S, since it is also associated with expansion and contraction. That this is indeed the case is shown by the familiar Friedmann equations, which relate the scale factor to the matter density $\rho(t)$ and pressure $P(t)$:

$$\left(\frac{\dot{S}}{S} \right)^2 = \frac{8\pi G}{3c^2} \rho(t) - \frac{k^2}{S^2} \tag{1.2}$$

$$\frac{\ddot{S}}{S} = -\frac{4\pi G}{3c^2} \left[\rho(t) + \frac{3P(t)}{c^2} \right] \tag{1.3}$$

where $\dot{S} = dS/dt$. These equations are exceedingly difficult to solve without specifying certain simplying assumptions, which typically involve specifying a simple cosmological equation of state (such as $P/c^2 = \omega\rho$, where ω is a dimensionless parameter), setting the pressure P to zero or imposing static conditions on the equations. Indeed, the traditional setting of the term $\dot{S}/S$ to the Hubble "constant" H_0 refers to present conditions only. However, assuming an average density of the universe of something like 10^{-30} kg·m^{-3}, the inverse of the Hubble constant gives roughly the correct observed age of the universe. This fact alone provides substantial evidence that the Friedmann metric is a reliable model of the universe.

2. The Problem in Outline

That there is a flaw in the FLRW metric can be shown by considering a detailed derivation of the metric using a constant curvature scalar R, which considerably simplifies the calculations. In a previous paper[3] the writer derived the FLRW metric assuming R to be a non-zero constant, and showed that the metric effectvely describes a de Sitter-like universe:

$$ds^2 = c^2 dt^2 - \cosh^2(\beta c t) \left[\frac{dr^2}{1-\beta^2 r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right] \tag{2.1}$$

where the (constant) parameter β is defined as

$$\beta = \sqrt{\frac{R}{12}} \tag{2.2}$$

and $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, having the dimension of length^{-2}. Comparing this with (1.1), we see that the FLRW parameter k^2 is proportional to R (note that their dimensions are identical). The resulting analysis showed that a universe with constant R would occur only in its very early, radiation-dominated stage (with the cosmological equation of state parameter $\omega = 1/3$) and in a very late, dark-energy stage with $\omega = -1$. In between, then, it must be assumed that the curvature scalar R is time dependent, complicating the analysis. If that is the case, then the k^2 parameter in the traditional FLRW metric cannot be considered a constant, as it must be proportional to R.

The source of the problem can be traced to the fact that the r-dependent term in the denominator of (1.1) results from solving the simple differential equation

$$\frac{d^2 f}{dr^2} - \frac{1}{2}\left(\frac{df}{dr}\right)^2 - \frac{1}{r}\frac{df}{dr} = 0 \tag{2.3}$$

where $f(r)$ is a parameter specifying the FLRW metric's radial dependence (see Adler et al.[1] or Straub for details). The general solution is

$$e^f = \frac{1}{(1 + ar^2)^2} \tag{2.4}$$

where a is a constant *with respect to* r. This does not alleviate its dependence on the time, however, and we are allowed to consider the possibility that $a = a(ct)$. In recognition of (2.1), it is tempting to set

$$a(ct) = \beta^2(ct) = \frac{R(ct)}{12} \tag{2.5}$$

which does not invalidate the general solution (2.4).

Unfortunately, the Friedmann equations associated with the revised metric

$$ds^2 = c^2 dt^2 - S^2 \left[\frac{dr^2}{1 - \beta^2 r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right] \tag{2.6}$$

with $\beta = \beta(ct)$ are now far more difficult to express, much less solve. It is hoped that some valid identification for R will eventually be found, allowing for a more accurate set of Friedmann equations in the future.

3. Comments

In their derivation of the FLRW metric, Adler et al. and others have assumed that a separation of the time and radial parameters in the metric can be expressed

$$ds^2 = c^2 dt^2 - e^g e^f \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \tag{3.1}$$

where the scale parameters e^g and e^f are functions of ct and r, respectively. The proposed revision of the metric would invalidate this separation, resulting in a new scale factor that might appear as $S^2 = e^G$, where $G = G(ct, r)$. This greatly complicates the derivation of the metric, since the Christoffel symbols needed to express $R_{\mu\nu}$ and R in the Einstein field equations are far more complicated. A further complication arises from pure symmetry principles: in a perfectly isotropic and homogeneous universe, two observers A and B separated in space must simultaneously observe the same distribution of matter and energy, so the ratio of e^{f_A} and e^{f_B} should be independent of time. Consequently, the factor e^f must be independent of time as well, in contradiction to the above argument.

References

1. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, 2nd Edition, 1975. Chapter 12 provides a detailed derivation and summary of the Christoffel symbols required for the construction of the Ricci tensors $R_{\mu\nu}$ and the Ricci scalar R. See Eq. (12-45) in that text and its solution for more details.

2. C. Pearson, *Fundamentals of Cosmology*, 2003. There are many excellent texts on physical cosmology available, but this entertaining and informative eight-part presentation provides all the basic information needed for an elementary understanding of the subject. The author uses $R(t)$ as the scale factor, an unfortunate choice considering its possible confusion with the curvature scalar R. The text is out of print, but can be downloaded as a slide presentation from

   http://www.weylmann.com/Pearson.zip

3. W. Straub, *The Friedmann-Lemaître-Robertson-Walker Metric with a Constant Curvature Scalar*, January 11, 2019. Related supplementary information is available from this paper, which can be downloaded from

   http://www.weylmann.com/flrw2.pdf